Remarks on the metric on 4D QFTs

Tom Hartman
Stanford University LITP

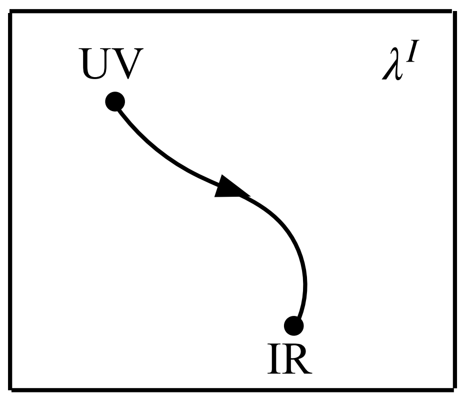
Based on work to appear with G. Mathys and A. Stergiou

Consider a CFT deformed by marginally relevant operators,

$$S = S_{\text{CFT}} + \lambda^I \int \mathcal{O}_I$$

This defines a space of QFT's with coordinates λ^I.

RG flows:



In two dimensions, A. B. Zamolodchikov showed that this space of QFT's has a natural metric,

$$G_{IJ} = \langle \mathcal{O}_I(0) \mathcal{O}_J(1) \rangle$$

The Zamolodchikov metric appears in the RG flow of the c-function,

$$\mu \frac{\partial}{\partial \mu} \tilde{c} = G_{IJ} \beta^I \beta^J$$

$\tilde{c}$ agrees with the central charges at UV/IR fixed points, and β^I is the beta function for coupling λ^I.

The Zamolodchikov metric is manifestly positive by unitarity.

This proves the c-theorem $c_{UV} > c_{IR}$.

Therefore the RG is irreversible in 2d.

The Zamolodchikov metric can also be written down in higher dimensions,
but it does not control RG flows.

In four dimensions, [Jack and Osborn '90] derived a similar flow equation for the a-function:

$$\mu \frac{\partial}{\partial \mu} \tilde{a} = \frac{1}{8} H_{IJ} \beta^I \beta^J$$

However, H_{IJ} is not the Zamolodchikov metric, and it is not manifestly positive.

H_{IJ} "Jack-Osborn Metric"

At leading order in perturbation theory, they are proportional,

$$H_{IJ} \propto G_{IJ} + \mathcal{O}(\beta, \partial\beta)$$

This established the *perturbative a*-theorem in 1990.

We now have a *non-perturbative* a-theorem:

Komargodski and Schwimmer '11 — Dilaton amplitudes

Casini, Teste, Torroba '17 — Quantum information

TH, Mathys '23 — averaged null energy condition (ANEC)

None of these approaches make use of the Jack-Osborn metric, so the interpretation and positivity of this metric was still mysterious.

The goal of this talk is to

- Demystify the Jack-Osborn metric
- Explain why it's positive
- Connect it to the non-perturbative approaches.

Outline

I. The Local RG [Osborn ’89, Jack and Osborn ’90s]

II. RG equations for light-ray operators

III. Connecting to dilaton scattering

The Local RG Formalism

Set d=4.

In a CFT,

$$T = 0$$

$$\langle T \rangle_g = -a(\text{Euler}) + c(\text{Weyl})^2$$

Varying the anomaly gives correlation functions involving the trace,

$$\langle T_{\mu\nu}(x_1) T_{\alpha\beta}(x_2) \ldots T(x_0) \rangle \sim \frac{\delta}{\delta g^{\mu\nu}(x_1)} \frac{\delta}{\delta g^{\alpha\beta}(x_2)} \cdots \langle T(x_0) \rangle$$

$$\sim (\text{delta functions})$$

(All correlation functions are Euclidean or Lorentzian time-ordered.)

In a renormalizable QFT, there is a similar equation that is almost as powerful:

$$T = \beta^I \mathcal{O}_I$$

In a background with metric $g_{\mu\nu}$ and spacetime dependent couplings $\lambda^I(x)$,

$$\begin{aligned}\langle T - \beta^I \mathcal{O}_I \rangle_{g,\lambda} = &-a(\lambda)(\text{Euler}) + c(\lambda)(\text{Weyl})^2 \\ &+ G_{IJ} \nabla^2 \lambda^I \nabla^2 \lambda^J \\ &- H_{IJ} G^{\mu\nu} \nabla_\mu \lambda^I \nabla_\nu \lambda^J \\ &+ (8 \text{ more terms})\end{aligned}$$

The **16 "anomaly coefficients"** depend on the running couplings.

This relation *defines* G_{IJ} and H_{IJ}, but they will end up equal to the two different metrics,

G_{IJ} Zamolodchikov metric

H_{IJ} Jack-Osborn metric

The 16 anomaly coefficients are not independent.

Wess-Zumino consistency:

$$\langle [T(x), T(y)] \rangle_{g,\lambda} \sim \left(\frac{\delta}{\delta\sigma(x)} \frac{\delta}{\delta\sigma(y)} - \frac{\delta}{\delta\sigma(y)} \frac{\delta}{\delta\sigma(x)} \right) \log Z$$
$$= 0$$

This gives many relations. One of them is

$$\partial_I a = \frac{1}{8} H_{IJ} \beta^J - \beta^J \partial_J w_I - \partial_I \beta^J w_J$$

$$\Longrightarrow$$

$$\boxed{\mu \frac{\partial}{\partial\mu} \tilde{a} = \frac{1}{8} H_{IJ} \beta^I \beta^J ,} \qquad \tilde{a} = a + \frac{1}{8} \beta^I w_I$$

$H_{IJ} \propto G_{IJ}$ to leading order in perturbation theory, so this proves the perturbative a-theorem.

We are working *d=4*. The same formalism in *d=2* gives Zamolodchikov's equation, $\mu \frac{d}{d\mu} \tilde{c} = G_{IJ} \beta^I \beta^J$.

But let's go back to the definition of the Jack-Osborn metric,

$$\langle T - \beta^I \mathcal{O}_I \rangle_{g,\lambda} = \cdots - H_{IJ} G^{\mu\nu} \nabla_\mu \lambda^I \nabla_\nu \lambda^J + \cdots$$

What's special about this term in the anomaly?

Why should it be a positive metric, even when the beta functions become large?

Outline

I. The Local RG

II. RG equations for light-ray operators

III. Connecting to dilaton scattering

✦ The Local RG equations simplify dramatically when rewritten in terms of light-ray operators.

✦ Positivity is related to the ANEC,

$$\langle \mathcal{O}_I \int du T_{uu} \mathcal{O}_J \rangle \succeq 0$$

The relation $T = \beta^I \mathcal{O}_I$ is closely related to the Callan-Symanzik equation.

To see this, evaluate it at zero momentum:

$$T(0) \sim \mu \frac{\partial}{\partial \mu} \qquad \text{(dilatation)}$$

$$\beta^I \mathcal{O}_I(0) \sim \beta^I \frac{\partial}{\partial \lambda^I}$$

And the total RG derivative is

$$T(0) - \beta^I \mathcal{O}_I(0) \sim \mu \frac{d}{d\mu}$$

The usual Callan-Symanzik equations for correlators are derived by varying the anomaly relation.

For example,

$$\mu \frac{d}{d\mu} \langle \mathcal{O}_I \mathcal{O}_J \rangle \sim \frac{\delta}{\delta \lambda^I} \frac{\delta}{\delta \lambda^J} \langle T - \beta^K \mathcal{O}_K \rangle$$
$$= \text{(purely local terms)}$$

Thus each term in the anomaly can be translated into RG equations.

Let's look again the Jack-Osborn metric:

$$\langle T - \beta^I \mathcal{O}_I \rangle_{g,\lambda} = \cdots - H_{IJ} G^{\mu\nu} \nabla_\mu \lambda^I \nabla_\nu \lambda^J + \cdots$$

A nice way to interpret the Einstein tensor is to turn on a shockwave / ANEC operator,

$$ds^2 = -du dv + d\vec{x}^2 + h(\vec{x})\delta(v) dv^2$$

$$G^{uu} = -2\delta(v)\vec{\partial}^2 h$$

Varying the anomaly gives the RG equation for the ANEC operator,

$$\boxed{\mu \frac{d}{d\mu} \left\langle \mathcal{O}_I(k_1) \int du T_{uu} \mathcal{O}_J(k_2) \right\rangle \sim (\vec{k}_1 \cdot \vec{k}_2) k_{1u}^2 H_{IJ}}$$

dots drop out!

For comparison, the full RG equation for $\frac{d}{d\mu}\langle \mathcal{O}_I T_{\mu\nu} \mathcal{O}_J \rangle$ has 207 terms.

This is the key observation:

The Jack-Osborn metric is the RG flow of the ANEC operator.

$$\mu\frac{d}{d\mu}\langle\mathcal{O}_I\mathcal{O}_J\rangle \quad \to \quad G_{IJ} \qquad \text{Zamolodchikov metric}$$

$$\mu\frac{d}{d\mu}\langle\mathcal{O}_I\int duT_{uu}\mathcal{O}_J\rangle \quad \to \quad H_{IJ} \qquad \text{Jack-Osborn metric}$$

Now we will connect this to the *a*-theorem.

Varying the anomaly equation also leads to

$$\mu\frac{d}{d\mu}\langle T\int duT_{uu}T\rangle = 0$$

The second key point: **This RG equation is the Jack-Osborn flow equation for the *a*-function.**

$$\mu\frac{d}{d\mu}\Big[\ \langle T\int duT_{uu}T\rangle\Big] = \mu\frac{d}{d\mu}\Big[\beta^I\beta^J\langle\mathcal{O}_I\int duT_{uu}\mathcal{O}_J\rangle + 8\tilde{a}(k_1\cdot k_2)^2k_{1u}^2\Big]$$

Putting these all together gives the flow equation for the anomaly,

$$
\begin{aligned}
\mu \frac{d}{d\mu} \tilde{a} &= \frac{1}{8} H_{IJ} \beta^I \beta^J \\
&= \mu \frac{d}{d\mu} \left[\beta^I \beta^J \langle \mathcal{O}_I \int du T_{uu} \mathcal{O}_J \rangle_{k_1 . k_2 k_{1u}^2} \right]
\end{aligned}
$$

Integrating over scales gives the sum rule

$$
\boxed{\Delta a = \beta^I \beta^J \langle \mathcal{O}_I \int du T_{uu} \mathcal{O}_J \rangle_{(k_1 . k_2) k_{1u}^2}}
$$

This is positive by the ANEC.

This sum rule is identical to the one derived by other methods in [TH, Mathys '23].

Outline

I. The Local RG

II. RG equations for light-ray operators

III. Connecting to dilaton scattering

So far, we found positivity of Δa but not of the metric itself.

We have not shown the a-function is monotonic.

This seems to require the 4-point function / dilaton amplitude.

[Baume, Keren-Zur, Rattazzi, Vitale ’14]

In the ANEC discussion, we took advantage of the fact that a shows up as a contact term in the 3-point functions of the QFT (not CFT!):

$$\tilde{a} = \frac{1}{8}\langle T \int du T_{uu} T\rangle - \frac{1}{8}\beta^I \beta^J \langle \mathcal{O}_I \int du T_{uu} \mathcal{O}_J\rangle \Big|_{k_1.k_2 k_{1u}^2}$$

It is well known from [Komargodski and Schwimmer '11] that it also shows up in the 4-point function,

$$\langle T(k_1)T(k_2)T(-k_2)T(-k_1)\rangle \quad \sim \quad a(k_1.k_2)^2 + \cdots$$

We will use this to "rediscover" the dilaton scattering amplitude from the correlation functions and the local RG.

In this context this will not give us anything truly new because there are no new positive quantities to be discovered.

But this is a general procedure to uncover positive sum rules from correlators, which could perhaps be stronger than positivity from scattering amplitudes.

(Work in progress…)

The most general contact term in the forward limit:

$$\begin{aligned}(\text{local}) = \langle TTTT \rangle &\\ &- \lambda_1 \beta^I \langle \mathcal{O}_I TTT \rangle \\ &- \lambda_2 \beta^I \beta^J \langle \mathcal{O}_I \mathcal{O}_J TT \rangle + \cdots \\ &+ \gamma_1 \langle TTT \rangle - \gamma_2 \beta^I \langle \mathcal{O}_I TT \rangle + \cdots \\ &+ \alpha_1 \langle TT \rangle - \alpha_2 \beta^I \langle \mathcal{O}_I T \rangle - \alpha_3 \beta^I \beta^J \langle \mathcal{O}_I \mathcal{O}_J \rangle \end{aligned}$$

$$\equiv G_{\text{contact}}$$

This is calculated explicitly from the Jack-Osborn anomaly $\langle T - \beta^I \mathcal{O}_I \rangle_{g,\lambda}$

Now turn this into a forward dispersion relation in terms of $s = -2k_1 \cdot k_2$

$$(\text{local}) = \oint \frac{ds}{s^m} G_{\text{contact}} = \int \frac{ds}{s^m} \text{Im } G_{\text{contact}} + \text{crossed}$$

Meanwhile, we can write the most general *positive* combination of 2, 3, and 4-point functions as

$$\sum_X |\langle X|T(k_1)T(k_2) + \alpha T(k_1+k_2)|0\rangle|^2$$

one free coefficient

Look at the most general local combination, and impose:

- **Universality:** Does not depend on local counterterms
- **Positivity:** The dispersion relation has a positive integrand

The unique* solution is the on-shell dilaton amplitude.

$$
\begin{aligned}
-32\tilde{a} =& \langle\!\langle T(-k_1)T(-k_2)T(k_2)T(k_1)\rangle\!\rangle \\
& - \Big[\langle\!\langle T(0)T(k_1)T(-k_1)\rangle\!\rangle + \langle\!\langle T(k_1+k_2)T(-k_1)T(-k_2)\rangle\!\rangle \\
& \quad + \langle\!\langle T(k_1-k_2)T(-k_1)T(k_2)\rangle\!\rangle + \langle\!\langle T(-k_1-k_2)T(k_1)T(k_2)\rangle\!\rangle \\
& \quad + \langle\!\langle T(0)T(k_2)T(-k_2)\rangle\!\rangle + \langle\!\langle T(-k_1+k_2)T(-k_2)T(k_1)\rangle\!\rangle\Big] \\
& \quad + \Big[\langle\!\langle T(-k_1-k_2)T(k_1+k_2)\rangle\!\rangle + \langle\!\langle T(-k_1+k_2)T(k_1-k_2)\rangle\!\rangle\Big] \\
& - \beta^I\beta^J\langle\!\langle T(k_4)\mathcal{O}^I(k_3)\mathcal{O}^J(k_2)T(k_1)\rangle\!\rangle \\
& + 2\beta^I\beta^J\langle\!\langle \mathcal{O}_I(-k_1-k_2)\mathcal{O}_J(k_1)T(k_2)\rangle\!\rangle \\
& + 2\beta^I\beta^J\langle\!\langle \mathcal{O}_I(-k_1+k_2)\mathcal{O}_J(k_1)T(-k_2)\rangle\!\rangle \\
& - \beta^I\beta^J\langle\!\langle \mathcal{O}^I(k_1+k_2)\mathcal{O}^J(-k_1-k_2)\rangle\!\rangle \\
& - \beta^I\beta^J\langle\!\langle \mathcal{O}^I(k_1-k_2)\mathcal{O}^J(-k_1+k_2)\rangle\!\rangle
\end{aligned}
$$

This recovers the Komargodski-Schwimmer sum rule for Δa and gives a manifestly positive formula for the Jack-Osborn metric,

$$H_{IJ} \sim \sum_X \langle 0|\mathcal{H}_I^\dagger|X\rangle\langle X|\mathcal{H}_J|0\rangle \Big|_{(k_1.k_2)^2}$$

$$\mathcal{H}_I = T(k_1)\mathcal{O}_I(k_2) + 2i\mathcal{O}_I(k_1+k_2)$$

[Baume, Keren-Zur, Rattazzi, Vitale ’14]

Comments

We’ve recovered a result that you already knew in a more complicated way.

Why bother?

It is unclear *why* the dilaton scattering amplitude is so useful, and why it gives the *a*-theorem.
Why should background fields be on shell?
In principle, correlation functions of T contain much more information.

What this argument showed is that in four dimensions, a systematic study of correlation functions lands on the same result as the dilaton amplitude.

Other applications? ~~6d a-theorem~~

Thank you.