The two-flavor Schwinger model at 50: Solving Coleman's puzzles

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Based on `arXiv:2305.04437` with R. Dempsey, I. Klebanov, B. Zan and `arXiv:2605.08042` with G. Cuomo, R. Dempsey, A. Katsevich, I. Klebanov, I. Kochergin, B. Søgaard

London, August 14, 2026

Some history

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  - $\Longleftrightarrow$ free boson w/ $M = g/\sqrt{\pi}$ (where $g > 0$ is elementary charge)
  - Example where gauge-invariance $\not\Longrightarrow$ massless particle.
  - Toy model for strong interactions
- Years that followed: massive model was studied [Lowenstein, Swieca, Coleman, Jackiw, Susskind, Casher, Kogut, ...]
- 1976: Coleman introduced **the 2-flavor massive Schwinger model** [Coleman '76]

$$\mathcal{L} = \sum_{\alpha=1}^{2} \left(i\bar{\psi}^\alpha (\partial\!\!\!/ - iA\!\!\!/)\psi_\alpha - m_\alpha \bar{\psi}^\alpha \psi_\alpha \right) - \frac{1}{4g^2} F_{\mu\nu}^2 - \frac{\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu} \,.$$

  - Toy model for QCD with two light quark flavors.
  - 3 puzzles left unsolved.

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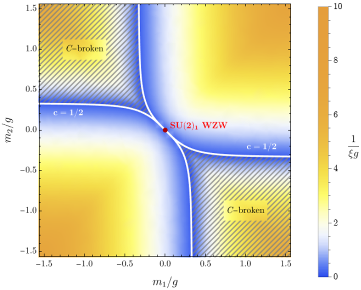
- Phase diagram similar to QCD w/ 2 light flavors [Creutz '18]



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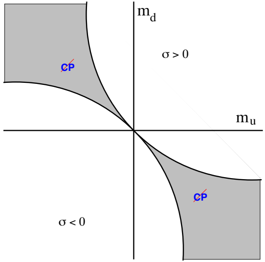

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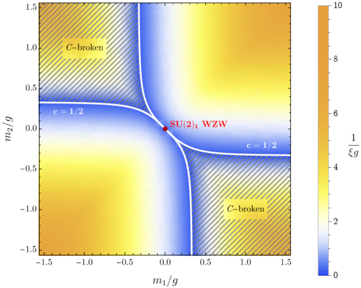

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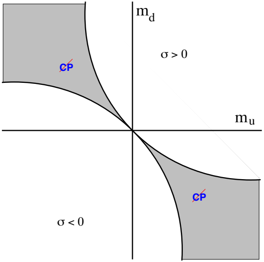

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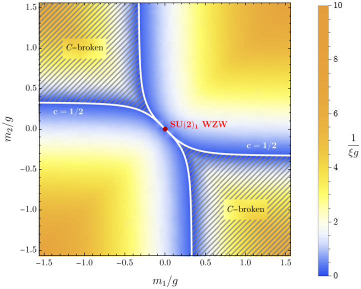

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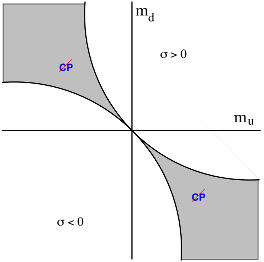

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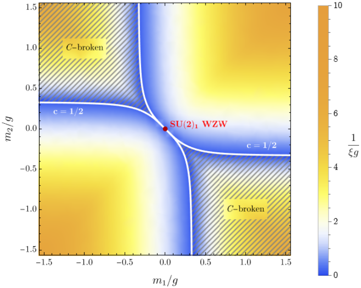

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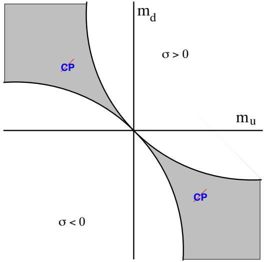

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Parameters of the two-flavor Schwinger model

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  has 3 dim'less params: m_1/g, m_2/g, θ.

- $\frac{\theta}{2\pi}$ = background electric field. Periodicity: $\theta \sim \theta + 2\pi$.

$$H = \int dx \left[\frac{g^2}{2} \underbrace{\left(E + \frac{\theta}{2\pi} \right)}_{E_{\text{eff}}}^2 + \sum_{\alpha=1}^{2} \left(-i\psi_{\alpha}^{\dagger} \gamma^5 (\partial_1 + iA_1) \psi_{\alpha} + m_{\alpha} \psi_{\alpha}^{\dagger} \gamma^0 \psi_{\alpha} \right) \right]$$

- $\psi_1 \to \gamma^5 \psi_1$ (implemented by unitary op. V) has the effect $(m_1, m_2) \to (-m_1, m_2)$ and $\theta \to \theta + \pi$ (due to axial anomaly):

$$V H_{m_1, m_2, \theta} V^{-1} = H_{-m_1, m_2, \theta + \pi}$$

- $\implies$ Can restrict to $m_1 \geq 0$ and $m_2 \geq 0$.

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Approaches

- Three main approaches:
  - Large mass / weak coupling ($m/g \gg 1$):
    Non-relativistic QM with linear potential $+$ perturbations.
  - Small mass / strong coupling ($m/g \ll 1$):
    ($c = 1$ CFT $+$ massive boson) $+$ perturbations
    - Bosonization $\Longrightarrow$ massless theory is equivalent to $SU(2)_1$ WZW model $+$ massive boson of mass $\mu = \frac{\sqrt{2}g}{\sqrt{\pi}}$.
  - Numerics: We use Hamiltonian LGT w/ staggered fermions and MPS on an infinite lattice.
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Hamiltonian lattice gauge theory

- Fermionic d.o.f.'s $c_{n,\alpha}$, $c^\dagger_{n,\alpha}$, gauge d.o.f.'s U_n, L_n:

$$H = \frac{g^2 a}{2}\sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2 - \frac{i}{2a}\sum_{n,\alpha}\left(c^\dagger_{n,\alpha}U_n c_{n+1,\alpha} - c^\dagger_{n+1,\alpha}U^\dagger_n c_{n,\alpha}\right)$$
$$+ \sum_{n,\alpha}\left(m_\alpha - \frac{g^2 a}{4}\right)(-1)^n c^\dagger_{n,\alpha}c_{n,\alpha}\,.$$

(mass shift required by axial symmetry of massless theory)

- Commutation reln's:

$$\{c_{n,\alpha}, c^\dagger_{m,\beta}\} = \delta_{nm}\delta_{\alpha\beta}\,, \qquad [L_n, U_m] = \delta_{nm}U_m\,.$$

- Gauss law:

$$(L_n - L_{n-1})|\text{phys}\rangle = Q_n|\text{phys}\rangle\,, \qquad Q_n = \sum_\alpha c^\dagger_{n,\alpha}c_{n,\alpha} + \begin{cases} 0 & \text{for even } n\,, \\ -2 & \text{for odd } n\,. \end{cases}$$

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  - Two puzzles for $m_1 = m_2$.
  - One puzzle for $m_1 \neq m_2$ (if time).
- Start with $m_1 = m_2 = m > 0$.
  - Model is invariant under $SO(3) = SU(2)/\mathbb{Z}_2$ "isospin" symmetry under which ψ_α form a doublet.

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Spectrum at weak coupling $m/g \gg 1$

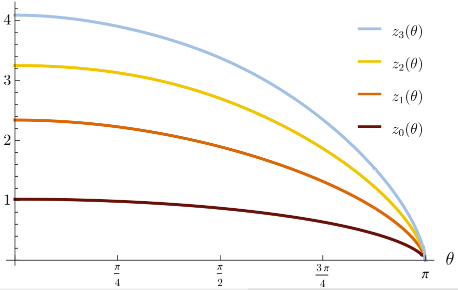
- Spectrum for $\theta \neq \pi$: e^+e^- bound states. At leading order [Coleman '76] :

$$H_{\rm NR} = \frac{p^2}{m} + \frac{g^2}{2}\left(|x| - \frac{\theta}{\pi}x\right) .$$

- Infinite number of energy eigenstates:

$$M^{(n)} = 2m + \frac{g^{4/3}}{(4m)^{1/3}} z_n(\theta) + O(g^{8/3}/m^{5/3}) , \qquad n = 0, 1, 2, \ldots$$

4-fold degeneracy at leading order!



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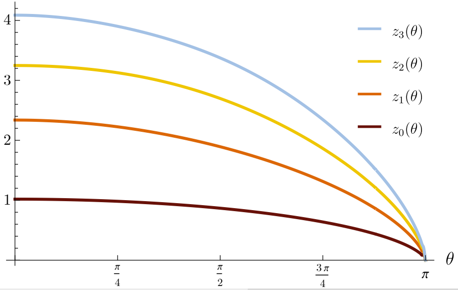
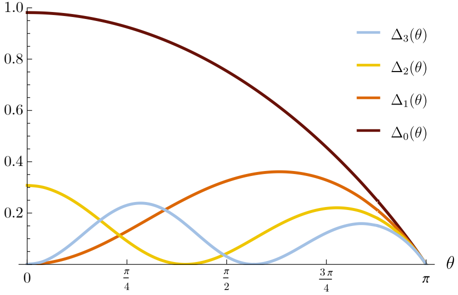


Energy splitting for $m/g \gg 1$

- 4-fold degeneracy broken [Coleman '76] by $H_{\text{singlet}} = \frac{g^2}{2m^2}\delta(x)$:

$$M_{\text{singlet}}^{(n)} - M_{\text{triplet}}^{(n)} = \frac{g^{8/3}}{2^{7/3}m^{5/3}}\Delta_n(\theta) + O(g^4/m^3)$$



Spectrum for $m/g \gg 1$ and $\theta = 0$

- At $\theta = 0$, label states as I^{PG} where $I =$ isospin, $P =$ is parity, $G = Ce^{i\pi Q_2}$ is G-parity ($C =$ charge conjugation):
  - 1^{-+} (π),
  - 0^{--} (η),
  - 0^{++} (σ) and 1^{+-},
  - 1^{-+},
  - 0^{--},
  - 0^{++} and 1^{+-},
  - ...

Spectrum for $m/g \gg 1$ and $\theta = \pi$

- At $\theta = \pi$, the e^+e^- spectrum becomes continuous starting at $2m$!
- **Two degenerate vacua**: $E_{\text{eff}} = E + \frac{\theta}{2\pi} = \pm 1/2$.
- e^+ and e^- are solitons (domain walls) w/ E_{eff} vs. x profile

- e^+ and e^- are "half-asymptotic particles":

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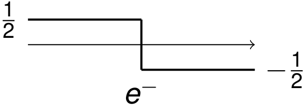

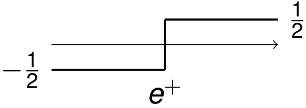

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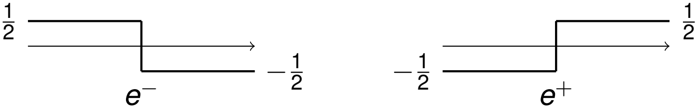
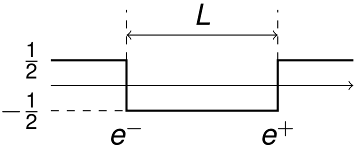
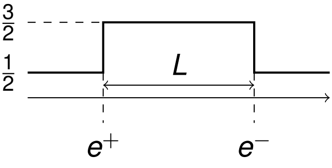
- At $\theta = \pi$, the e^+e^- spectrum becomes continuous starting at $2m$!
- **Two degenerate vacua**: $E_{\text{eff}} = E + \frac{\theta}{2\pi} = \pm 1/2$.
- e^+ and e^- are solitons (domain walls) w/ E_{eff} vs. x profile



- e^+ and e^- are “half-asymptotic particles”:



vs.



$\Delta\text{Energy} = 0$ $\Delta\text{Energy} = g^2L$

Spectrum at strong coupling $m/g \ll 1$

- At small mass, use bosonization: $\psi_\alpha \to \phi_\alpha$. Define $\phi_\pm = \frac{\phi_1 \pm \phi_2}{\sqrt{2}}$.
- After integrating out the gauge field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_+\partial^\mu\phi_+ - \frac{\mu^2}{2}\min_{n\in\mathbb{Z}}\left(\phi_+ + \frac{\theta+2\pi n}{2\sqrt{2\pi}}\right)^2 + \frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- + \frac{e^{\gamma_E}}{\pi}m\sqrt{\mu\mu_-}N_\mu\cos\left(\sqrt{2\pi}\phi_+\right)N_{\mu_-}\cos\left(\sqrt{2\pi}\phi_-\right)$$

w/ $\mu \equiv \sqrt{\frac{2}{\pi}}g$ and μ_- is arbitrary.

- Compact scalars: $\phi_+ \sim \phi_+ + \sqrt{2\pi}$ and $\phi_- \sim \phi_- + \sqrt{2\pi}$ and $(\phi_+, \phi_-) \sim (\phi_+ + \sqrt{\frac{\pi}{2}}, \phi_- + \sqrt{\frac{\pi}{2}})$.
- At $m = 0$: massless ϕ_- **at self-dual radius** ($\Longleftrightarrow SU(2)_1$ WZW) + massive ϕ_+ w/ mass μ.

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Effective field theory for ϕ_-

- Small m: also integrate out $\phi_+ \implies SO(3)$-preserving **EFT for ϕ_-**

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- + \cos\frac{\theta}{2}\frac{e^{\gamma_E}}{\pi} m\sqrt{\mu\mu_-} N_{\mu_-}\cos\left(\sqrt{2\pi}\phi_-\right)$$
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where $I_s(\theta) \approx 5.55 - 4.53\cos\theta$ and N_{μ_-} denotes normal ordering.

- Leading deformation for $\theta \neq \pi$: $\cos\left(\sqrt{2\pi}\phi_-\right)$ w/ dim $\Delta = 1/2 \longrightarrow$ relevant deformation.

- Leading deformation at $\theta = \pi$: $\partial_\mu\phi_-\partial^\mu\phi_-$ and $\cos\left(\sqrt{8\pi}\phi_-\right)$ have $\Delta = 2 \longrightarrow$ BKT RG flow.

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- For $\boxed{\theta \neq \pi}$: truncate EFT after first two terms

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- This is a sine-Gordon theory w/ $SO(3)$ symm. **Integrable!**
- Spectrum: isotriplet $\vec{\pi}$ and isosinglet σ.

$$M_\pi = \frac{2^{\frac{7}{6}}e^{\frac{\gamma_E}{3}}\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})^{\frac{4}{3}}} g^{\frac{1}{3}} m^{\frac{2}{3}} \cos^{\frac{2}{3}}\frac{\theta}{2} \approx 2.008 g^{\frac{1}{3}} m^{\frac{2}{3}} \cos^{\frac{2}{3}}\frac{\theta}{2},$$

$$M_\sigma = \sqrt{3}M_\pi$$

- $\sqrt{3}$ factor also predicted from semiclassical analysis.
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- At $\theta = 0$: spectrum at strong cpling: $I^{PG} = 1^{-+}, 0^{++}$.
  spectrum at weak cpling: $I^{PG} = 1^{-+}, 0^{--}, 0^{++}, 1^{+-}, \ldots$
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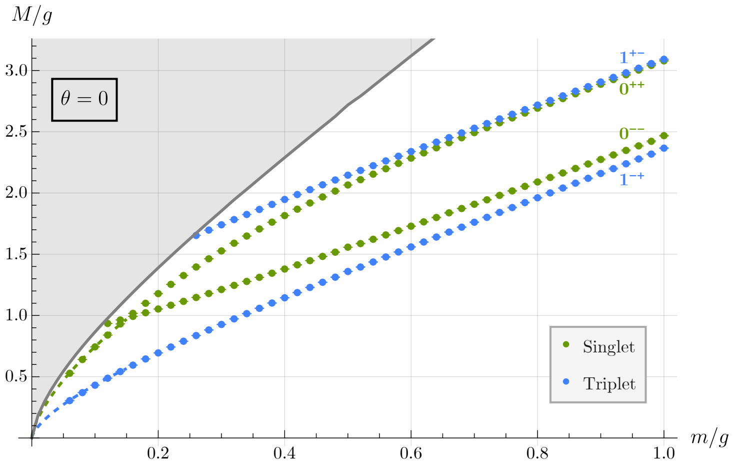
Resolution of Coleman's 2nd puzzle

- **Resolution:** Level-crossing!!



Strong coupling spectrum for $\theta = \pi$

- At $\boxed{\theta = \pi}$, the leading perturbation has $\Delta = 2$:

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- First, there is no phase transition b/w weak and strong coupling
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- C-broken region has 2 degenerate vacua and extends to $m = 0$.

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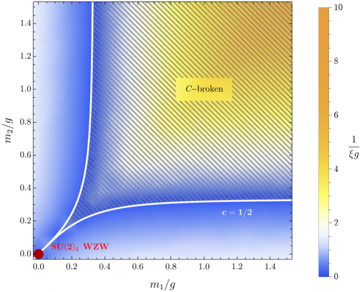

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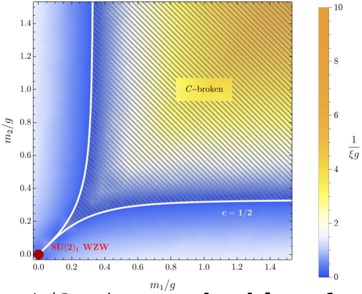

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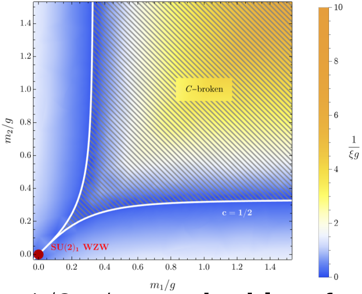

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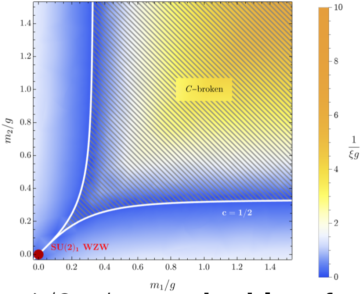

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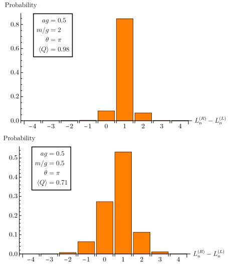

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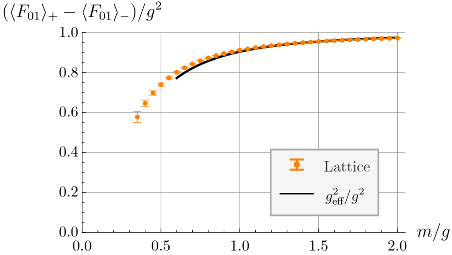

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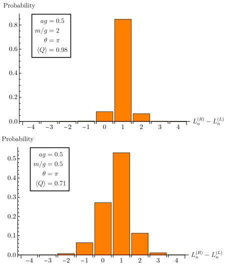

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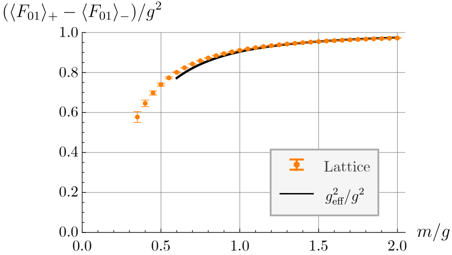

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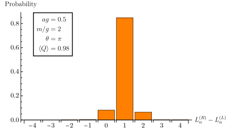

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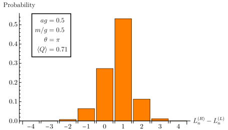

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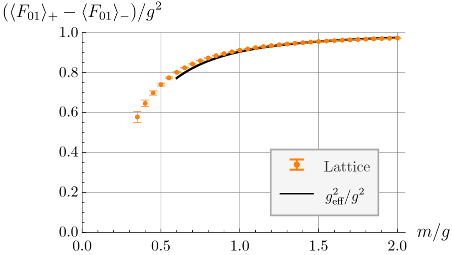
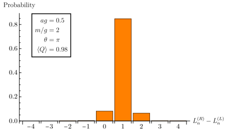


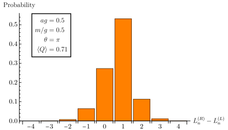
- **Resolution:** $\langle Q_{\text{gauge}} \rangle$ changes gradually and $\rightarrow \begin{cases} 1\,, & \text{as } \frac{m}{g} \to \infty \\ 0\,, & \text{as } \frac{m}{g} \to 0 \end{cases}$
- Weak coupling: $\langle Q_{\text{gauge}} \rangle < 1$ explained by vacuum polarization!
- Strong coupling: how does $\langle Q_{\text{gauge}} \rangle \to 0$??

Resolution of Coleman's 3rd puzzle

- Observation: the two C-breaking vacua are not e'states of $E(x)$!
- $\Longrightarrow$ distribution of Q_{gauge} for solitons.



$\frac{m}{g} = 2$



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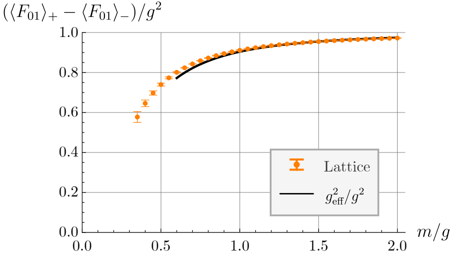

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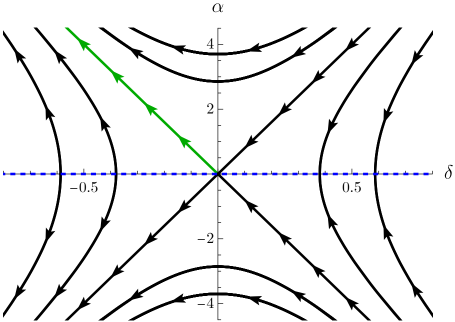
BKT flow

- More generally, for

$$\mathcal{L}_{\text{EFT}} = \frac{1-\delta}{2}\partial_\mu\phi_-\partial^\mu\phi_- + \alpha\frac{e^{2\gamma_E}}{32\pi}\mu_-^2 N_{\mu_-}\cos\left(\sqrt{8\pi}\phi_-\right)$$

we have the BKT RG flow diagram



- Our EFT preserves $SO(3)$ and has $\alpha = -8\delta + O(\delta^2)$ (green line).

BKT flow

- Along $SO(3)$-preserving line

$$\Lambda \frac{d\delta}{d\Lambda} = \beta_\delta = 2\delta^2 + 2\delta^3 + O(\delta^4)$$

- Integrating along RG trajectory: $\Lambda_{\text{IR}} = \Lambda_{\text{UV}} \sqrt{|\delta_{\text{UV}}|} e^{\frac{1}{2\delta_{\text{UV}}}}$. In UV, $\delta_{\text{UV}} = -\frac{1}{2A_s}\frac{m^2}{g^2}$ and take $\Lambda_{\text{UV}} = \sqrt{\frac{2}{\pi}} g$,

$$\Lambda_{\text{IR}} \approx \frac{m}{\sqrt{\pi A_s}} e^{-A_s \frac{g^2}{m^2}}, \qquad A_s \approx 0.111\,.$$

- Λ_{IR} is the only mass scale; observables $\propto$ powers of Λ_{IR} determined by scaling dims in $SU(2)_1$ WZW CFT:

$$\boxed{M_{\text{sol}} \sim \Lambda_{\text{IR}}}\,, \qquad \langle E(x)\rangle_\pm \sim \Lambda_{\text{IR}}^{1/2} \Longrightarrow \boxed{\langle Q_{\text{gauge}}\rangle \sim \Lambda_{\text{IR}}^{1/2}}\,,$$

$$\langle \bar\psi_\alpha \psi_\alpha(x)\rangle \sim \Lambda_{\text{IR}}^2\,, \qquad \text{width of } C\text{-broken region} \sim \Lambda_{\text{IR}}^{3/2}\,, \qquad \text{etc.}$$

- Relation to $SU(2)$ Thirring model, which is integrable.

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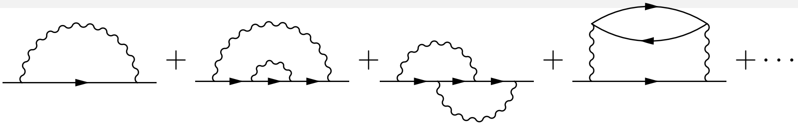
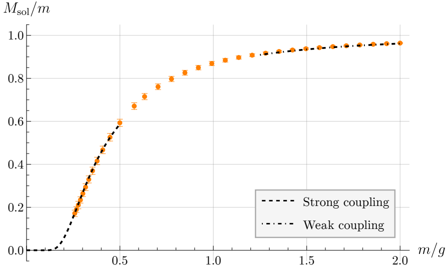
Soliton mass

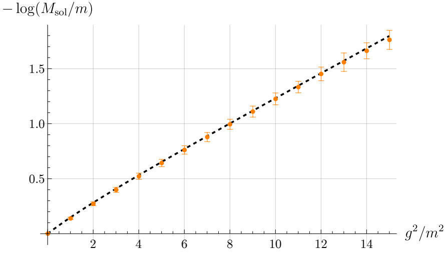


- Weak coupling: self-energy diags give

$$M_{\rm sol} = m\left(1 - \frac{g^2}{2\pi m^2} + \frac{(3\pi^2-4)g^4}{96\pi^2 m^4} + \cdots\right) .$$

- Strong coupling: $M_{\rm sol} \approx K\, m\, e^{-A_s \frac{g^2}{m^2}}$, $K \approx 0.91$

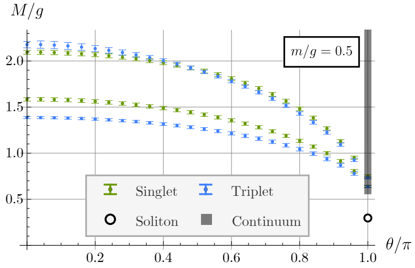




Charge confinement vs. deconfinement

Summary of spectrum (for any *m*):

- $\theta \neq \pi$: singlets & triplets of e^+e^- bound states
- $\theta \to \pi$: singlets & triplets become degenerate and fall apart into soliton-anti-soliton pairs at $\theta = \pi$.
- Charge confinement at $\theta \neq \pi \longrightarrow$ Charge deconfinement at $\theta = \pi$.



Spectrum when $m_1 \neq m_2$

- Coleman found that for $m_1 \neq m_2$ and $\theta = 0$ (for simplicity), the leading order EFT is still [Coleman '76]

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}\partial_\mu \phi_- \partial^\mu \phi_- + \frac{e^{\gamma_E}}{\pi} m \sqrt{\mu \mu_-} N_{\mu_-} \cos\left(\sqrt{2\pi}\phi_-\right)$$

with

$$m = \frac{m_1 + m_2}{2} \,.$$

- This theory has $SO(3)$ symmetry!!
- **Coleman's 1st puzzle:** Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other?
- How does the spectrum depend on $\delta m = (m_1 - m_2)/2$?

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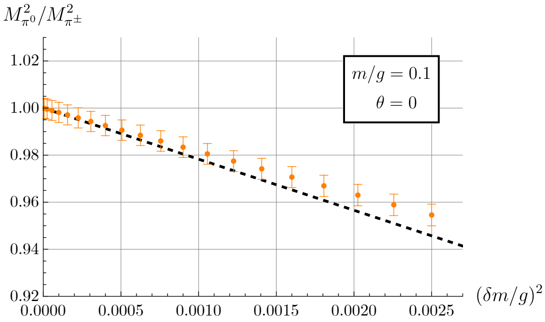
Resolution of Coleman's 1st puzzle

- From correction to EFT, we computed:

$$\frac{M^2_{\pi^0}}{M^2_{\pi^\pm}} = 1 - \frac{2\pi e^{\gamma_E}(\delta m)^2}{3\sqrt{3}g^2} I_s(\pi) + \cdots .$$



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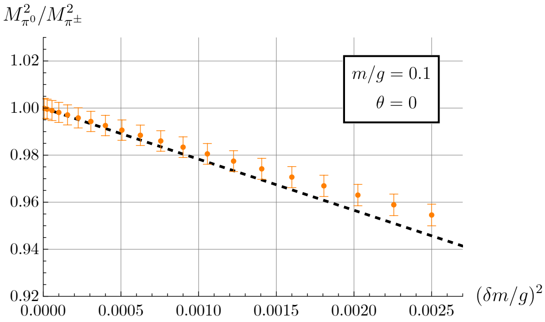

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Conclusion

- The 2-flavor Schwinger model is surprisingly complex! (Spontaneous C-breaking, exponential hierarchy of scales, confinement/deconfinement, connections to integrability, etc.)
- Numerical Hamiltonian lattice results have been crucial for elucidating the physics and checking analytical arguments.

Things I didn't discuss / future directions:

- Numerical implementation using matrix product states and link-enhanced matrix product operators (LEMPOs) [Dempsey, Glück, SSP, Søgaard '25] .
- Simulations of real time scattering of particles and/or solitons.
- Bootstrapping the lattice model, S-matrix bootstrap.
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Extra Slides

Resolution of Coleman's 1st puzzle

- *Massless* theory: CFT w/ $O(4) = \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \rtimes (\mathbb{Z}_2)_P$

$$\psi_{L\alpha} \to (g_L)_\alpha{}^\beta \psi_{L\beta}\,, \qquad \psi_{R\alpha} \to (g_R)_\alpha{}^\beta \psi_{R\beta}\,, \qquad g_L, g_R \in SU(2)\,.$$

- Most general mass term: $m_\alpha{}^\beta \psi_R^{\dagger\alpha} \psi_{L\beta} + (m^\dagger)_\alpha{}^\beta \psi_L^{\dagger\alpha} \psi_{R\beta}$

  $\Longrightarrow \mathrm{Re}\,\vec{m} =$ vector, $\mathrm{Im}\,\vec{m} =$ pseudovector of $O(4)$.

- Lowest CFT op. is a $\Delta = 1/2$ vector of $O(4) \Longrightarrow$ its coeff. prop. to $\mathrm{Re}\,\vec{m} \Longrightarrow O(4)$ broken to $O(3)$ (symm of leading order EFT!).

- We constructed the leading $SO(3)$-breaking term using non-Abelian bosonization [Witten '86] and argued that

$$\frac{M_{\pi^0}^2}{M_{\pi^\pm}^2} = 1 + O(\mathrm{mass}^2/g^2)$$