Existence of extremal solutions in 1d CFT

To appear with Filip Herček

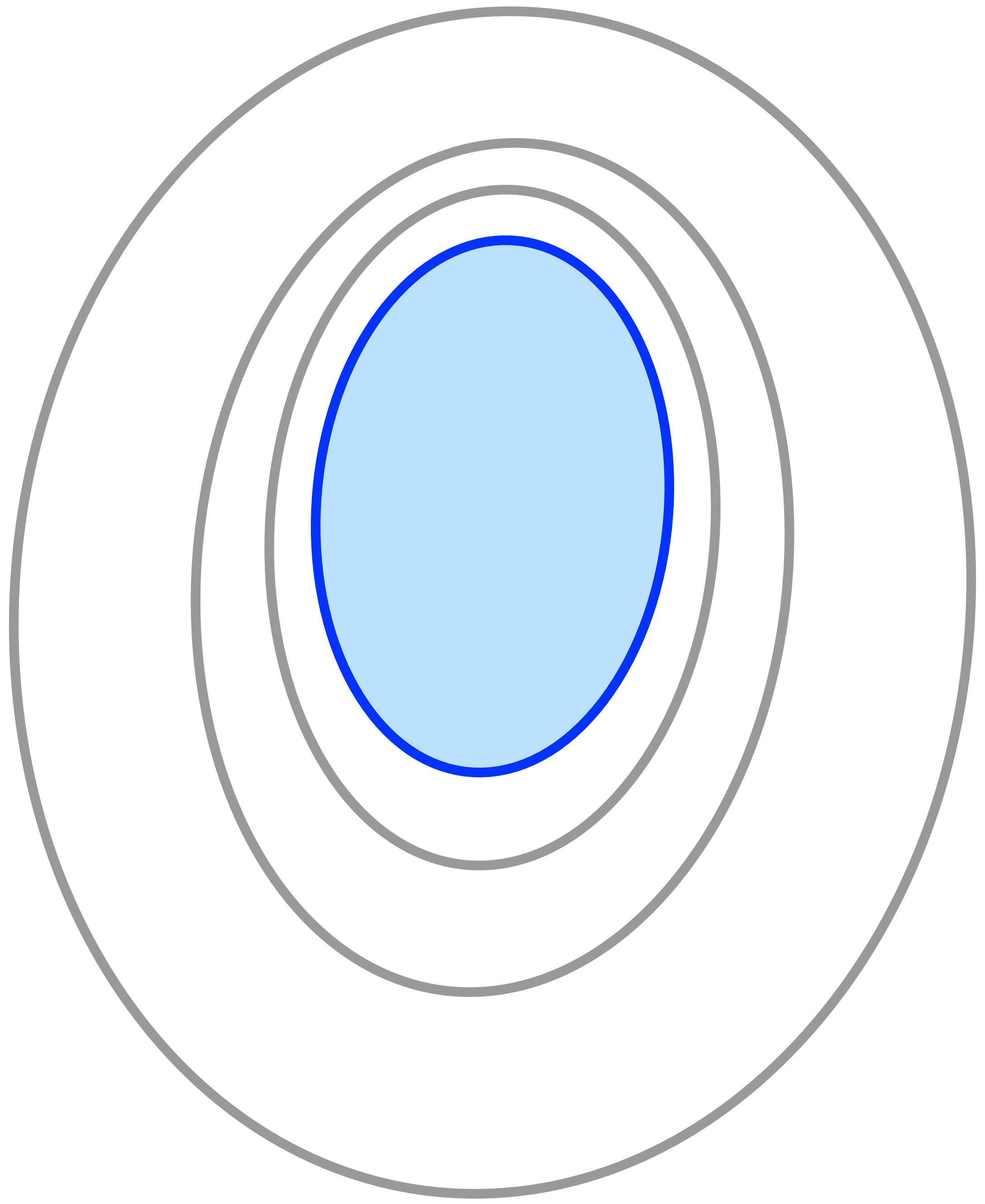
Bootstrap 2026

Nat Levine

University of Amsterdam

Extremal solutions and functionals

$\Lambda \to \infty$



Extremal solutions and functionals

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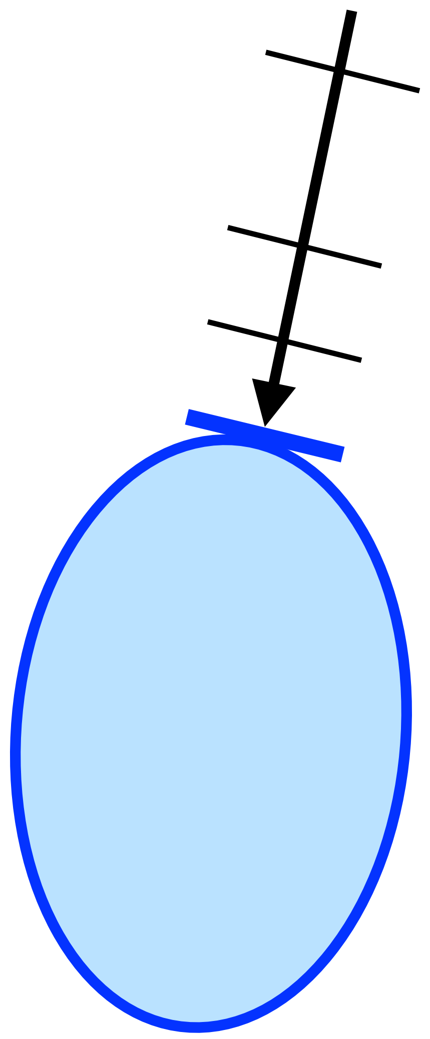

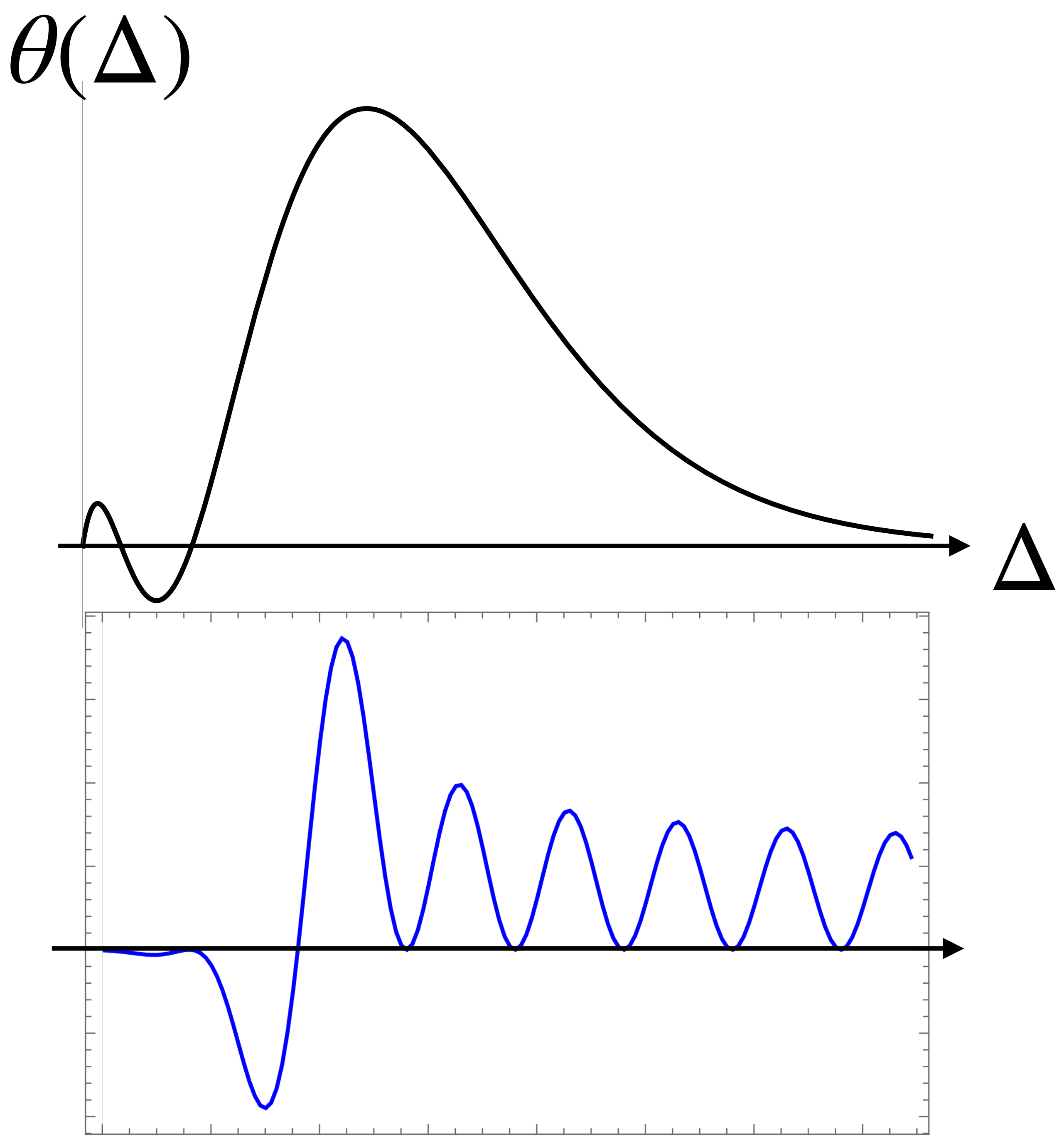

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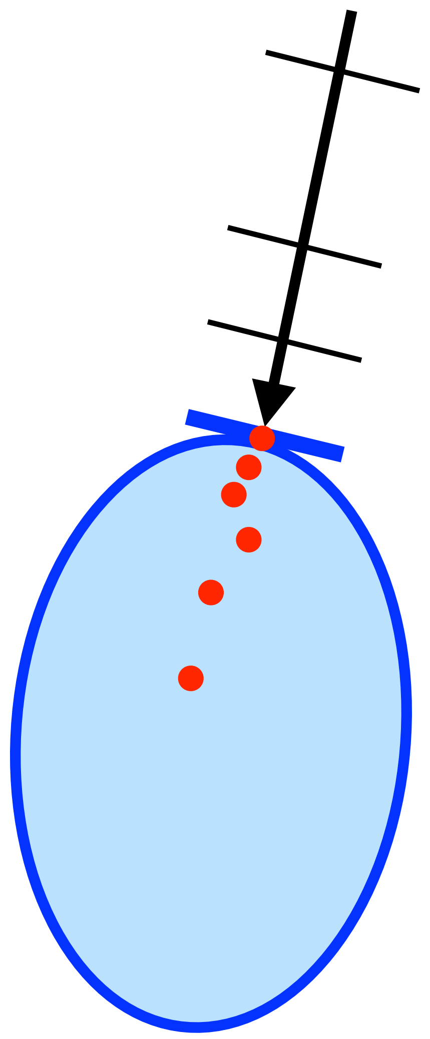
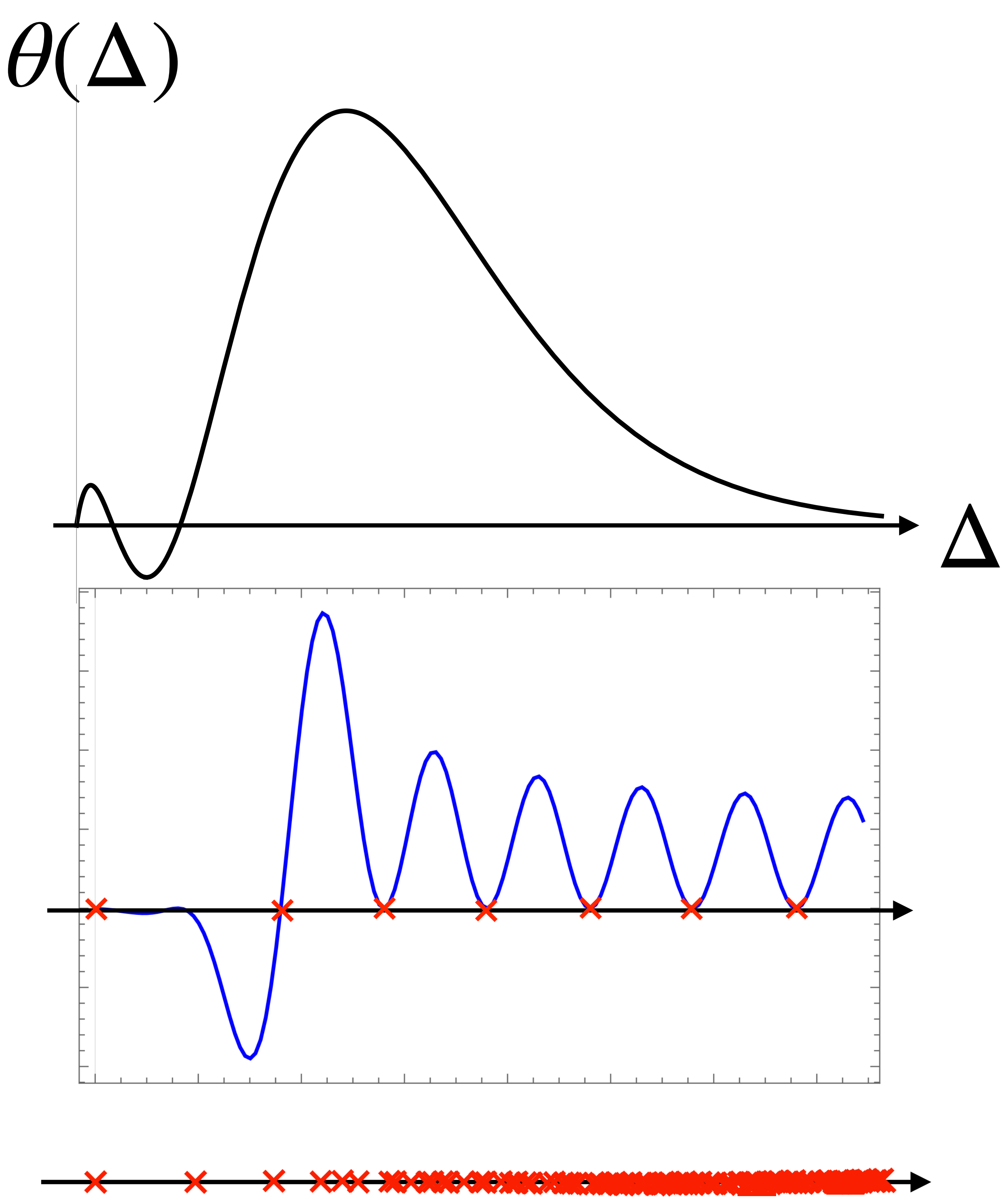
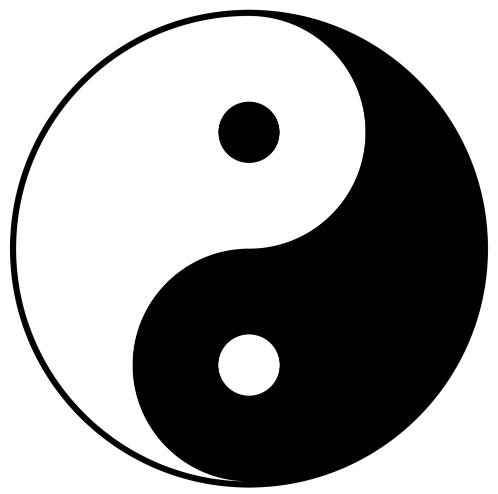






Lore: extremal solutions and functionals…

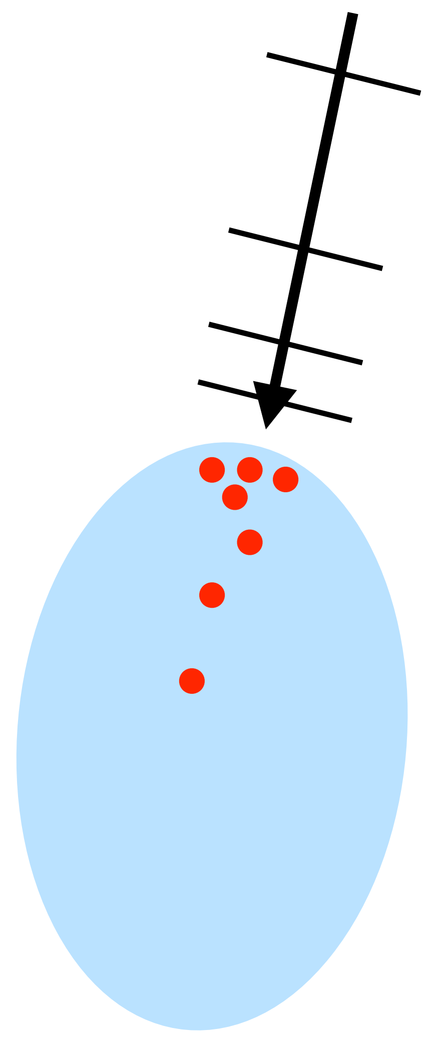
- Exist
- Are unique
- Are dual to each other

~ convex optimisation theory

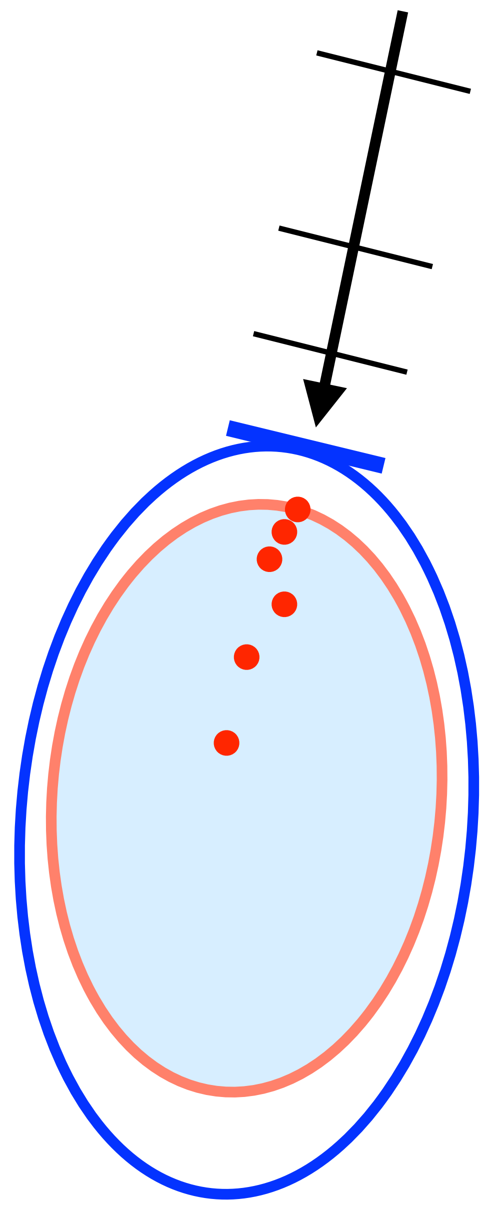
Is this surprising?

What could have happened at $\Lambda = \infty$ ...

Extremal solutions or functionals don't actually exist



Non-zero duality gap



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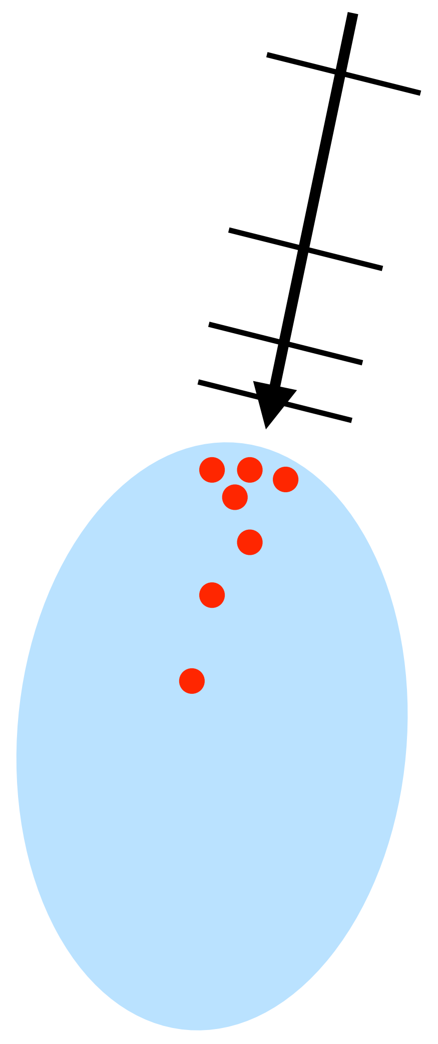
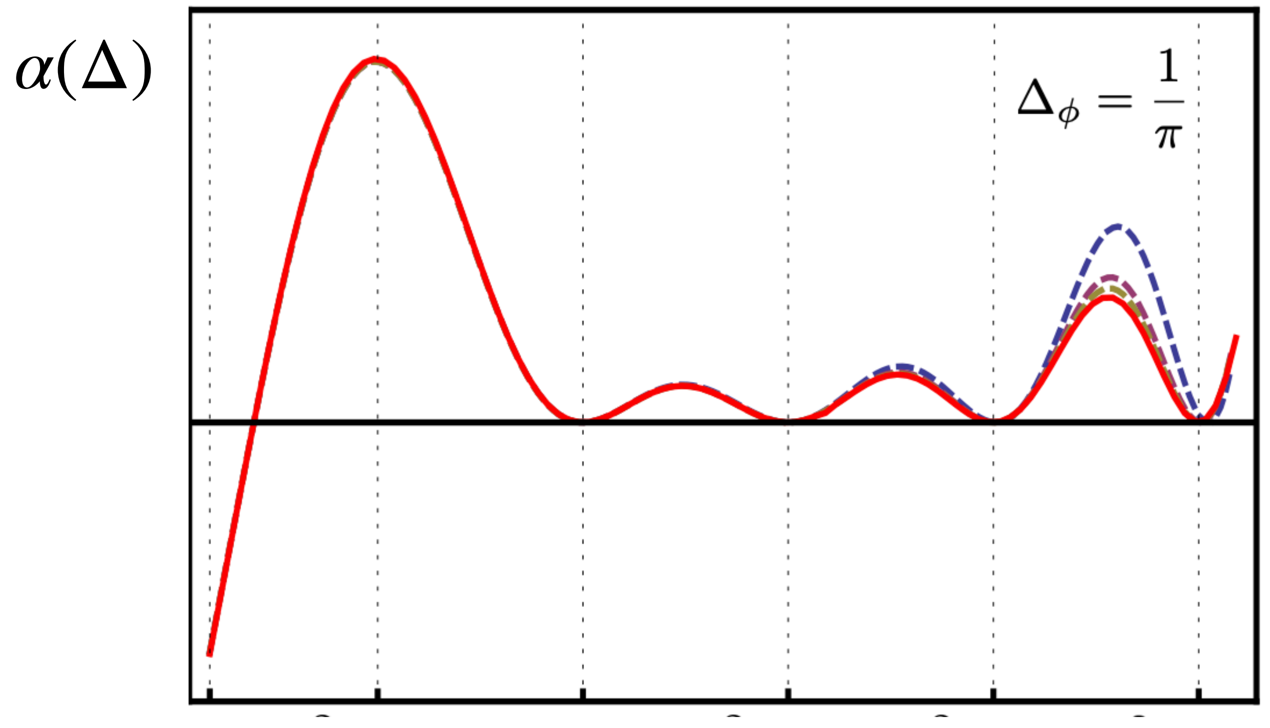
Extremal solutions or functionals don’t actually exist

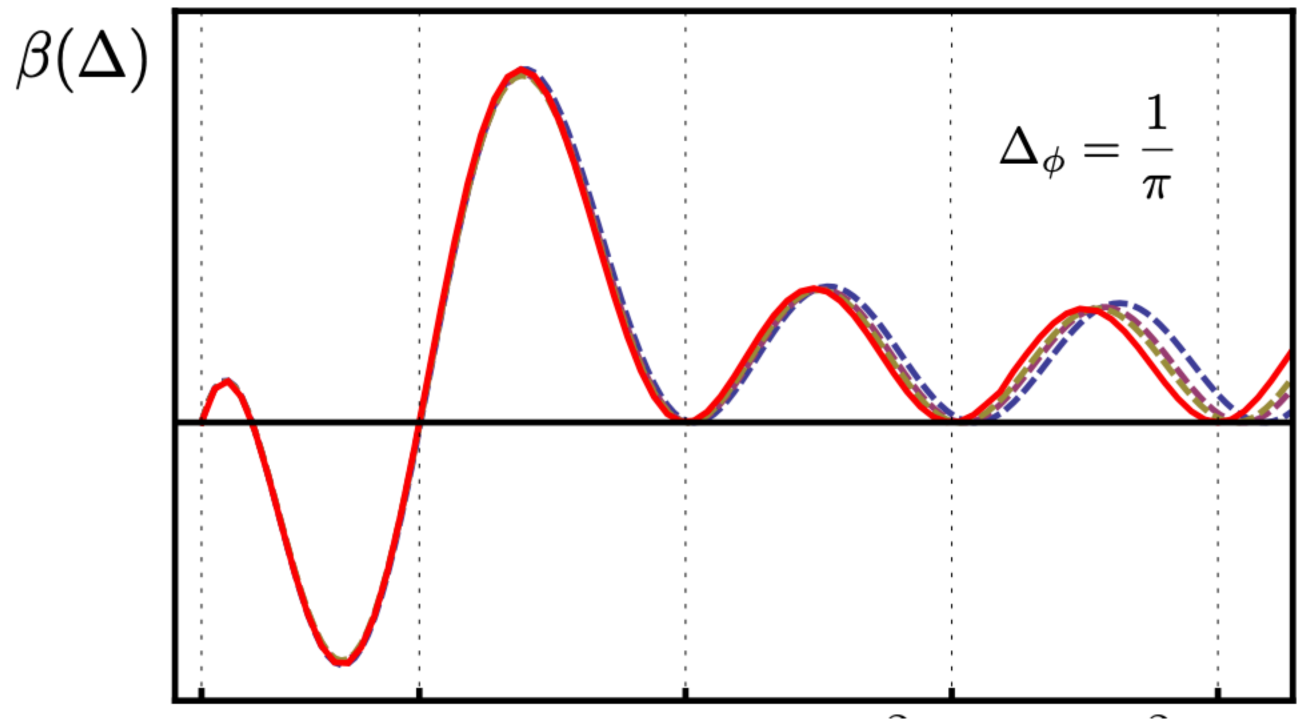


Non-zero duality gap

Reasons to believe

- Robust empirical finding
- Extremal Functional Method [El-Showk Paulos] (arguments at $\Lambda < \infty$)
- Free case: analytic functionals dual to GFF solutions [Mazac] [Mazac Paulos]





from [Mazac Paulos]

See also

[Paulos]
[Mazac Rastelli Zhou]
[Caron-Huot Mazac Rastelli DSD]

[Zechuan's talk]

Question

Can we rigorously prove existence of extremal solutions / functionals?

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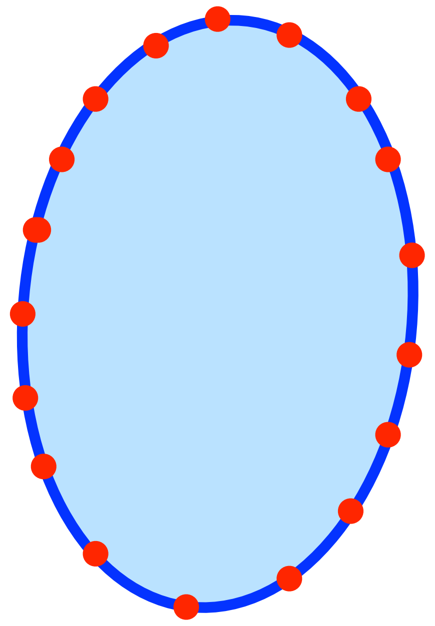
Can we ***rigorously*** prove existence of ***extremal solutions*** / functionals?

Motivations

1. How bootstrap works?

2. ‘Hybrid’ bootstrap : search directly for optimal bounds?

Fantasy



Direct numerical search for extremal solutions
$\rightarrow$ optimal bounds?

1d study [Ghosh Paulos Suchel]

Cf. [Gliozzi] approach and ‘fast bootstrap’ of [Afkhami-Jeddi Hartman Tajdini]

3. Old question: Can 3d Ising be extremal?

Slater condition in SDP

Cf. discussion of Slater condition in SDP at $\Lambda < \infty$

SDPB [Simmons-Duffin]
Skydiving [Liu et al]
Navigator [Reehorst et al]

Here: single correlator, i.e. linear programming. ($\Lambda < \infty$ questions are trivial)

$$\Lambda = \infty$$

This talk: extremal solutions exist for 1d CFT OPE-max

$$\langle \phi\phi\phi\phi \rangle \sim G(z)$$

$$= \sum_{\Delta} a_\Delta \, G_\Delta(z) \qquad\qquad G_\Delta = \mathcal{N}_\Delta \, z^{\Delta - 2\Delta_\phi} \, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

Dynamical data $\{\Delta \;, \;\; a_\Delta = \lambda^2_{\phi\phi\Delta}\}$

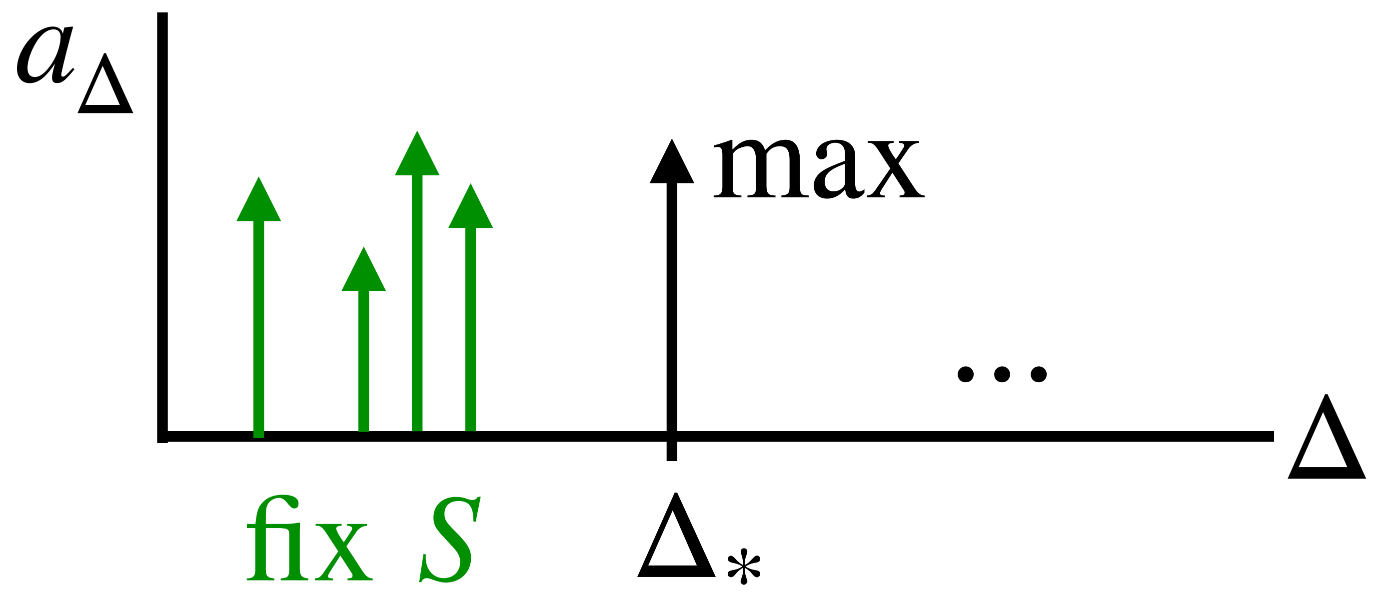
Crossing $\quad F(z) := G(z) - G(1-z) = 0$

Unitarity $\quad a_\Delta \geq 0$

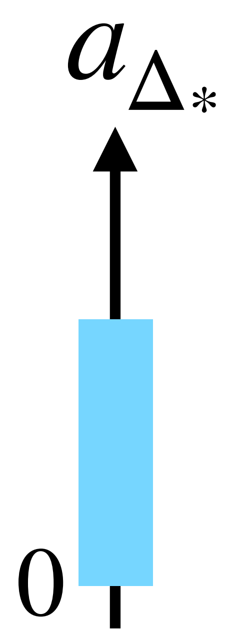
OPE maximisation (primal)

Fix gap $\Delta_* \geq \sqrt{2}\Delta_\phi$ and operators $S = \sum_{0 \leq \Delta < \Delta_*} a_\Delta^{\text{fixed}} G_\Delta$ below the gap,

i.e. $G = S + \sum_{\Delta \geq \Delta_*} a_\Delta G_\Delta$.

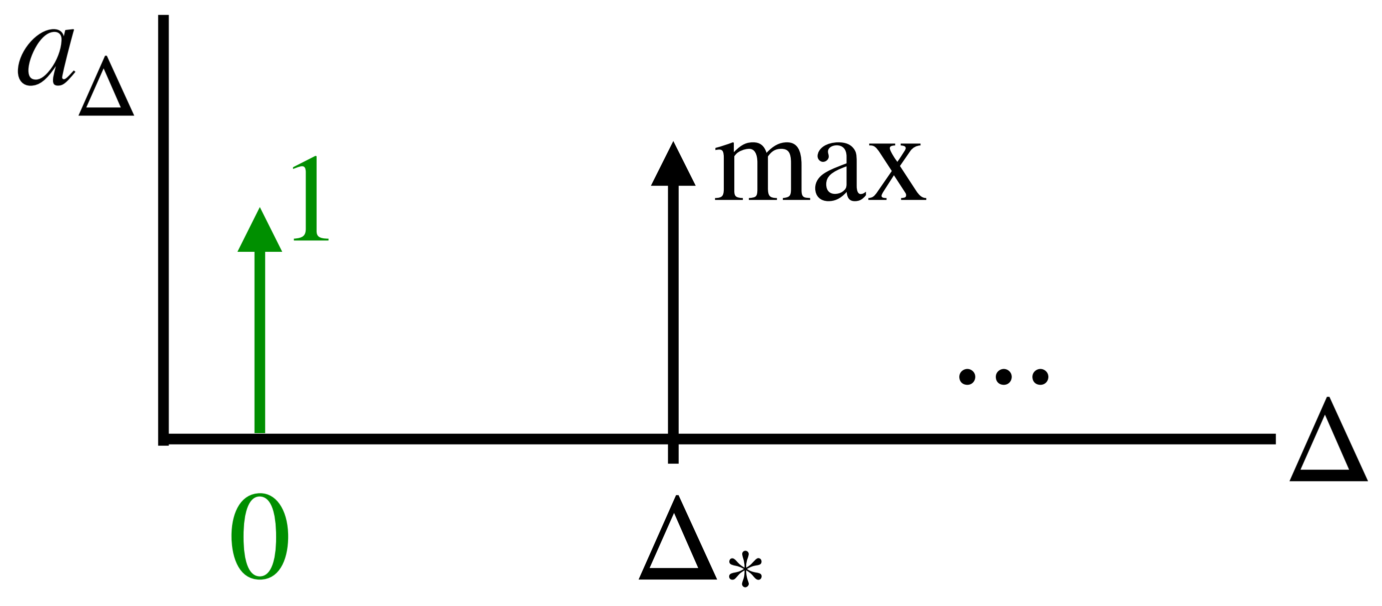


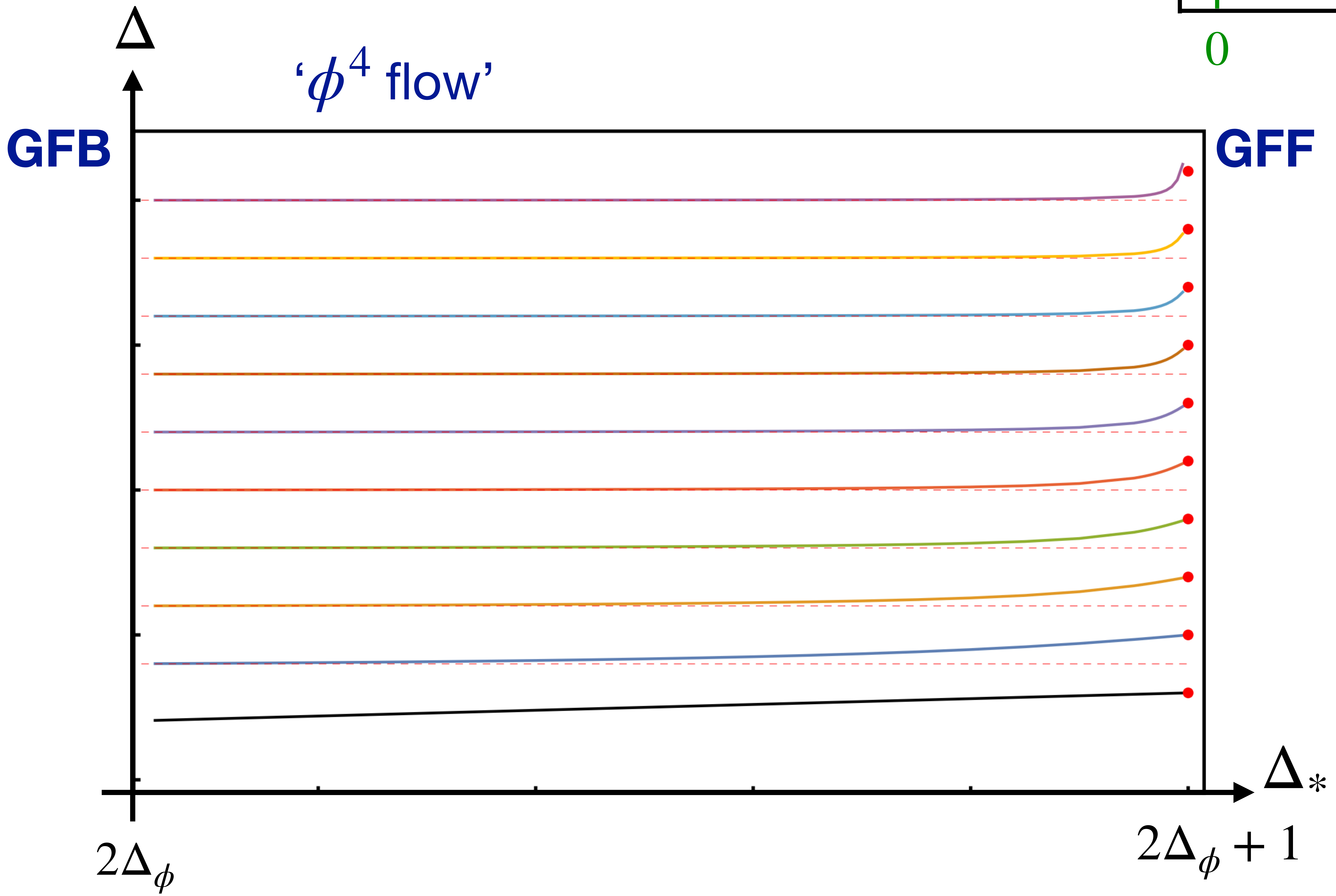
Max	a_{Δ_*}
subject to	crossing $F(z) := G(z) - G(1-z) = 0$
	unitarity $a_\Delta \geq 0$



Example

$$S(z) = G_0(z)$$





[Paulos Zan] numerics

c.f. [Antonio's talk]

OPE density formalised

OPE density $a(\Delta)$ $=$ ***Radon measure*** on $[0,\infty)$

$:=$ locally integrable measure

E.g. discrete continuous

OPE maximisation (primal) — ***with Radon measures***

Fix gap $\Delta_* \geq \sqrt{2}\Delta_\phi$ and operators $S = \int_{[0,\Delta_*)} \mathrm{d}\Delta \; a(\Delta)^{\text{fixed}} G_\Delta$ below the gap,

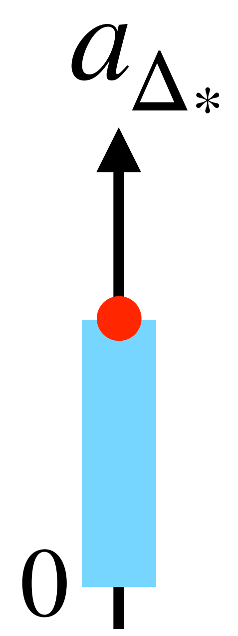
i.e. $G = S + \int_{[\Delta_*,\infty)} \mathrm{d}\Delta \; a(\Delta)\, G_\Delta$.

Max	$a_{\Delta_*} \equiv a_{\{\Delta_*\}} = \lim_{\epsilon\to 0} \int_{\Delta_*-\epsilon}^{\Delta_*+\epsilon} \mathrm{d}\Delta \; a(\Delta)$
subject to	crossing $F(z) := G(z) - G(1-z) = 0$
	unitarity $a(\Delta) \geq 0$ i.e. $\int \mathrm{d}\Delta \; a(\Delta)\, f(\Delta) \geq 0$ if $f(\Delta) \geq 0$

Result

Theorem: *If the primal problem is feasible, then an **extremal solution exists** and the **duality gap is zero**.*

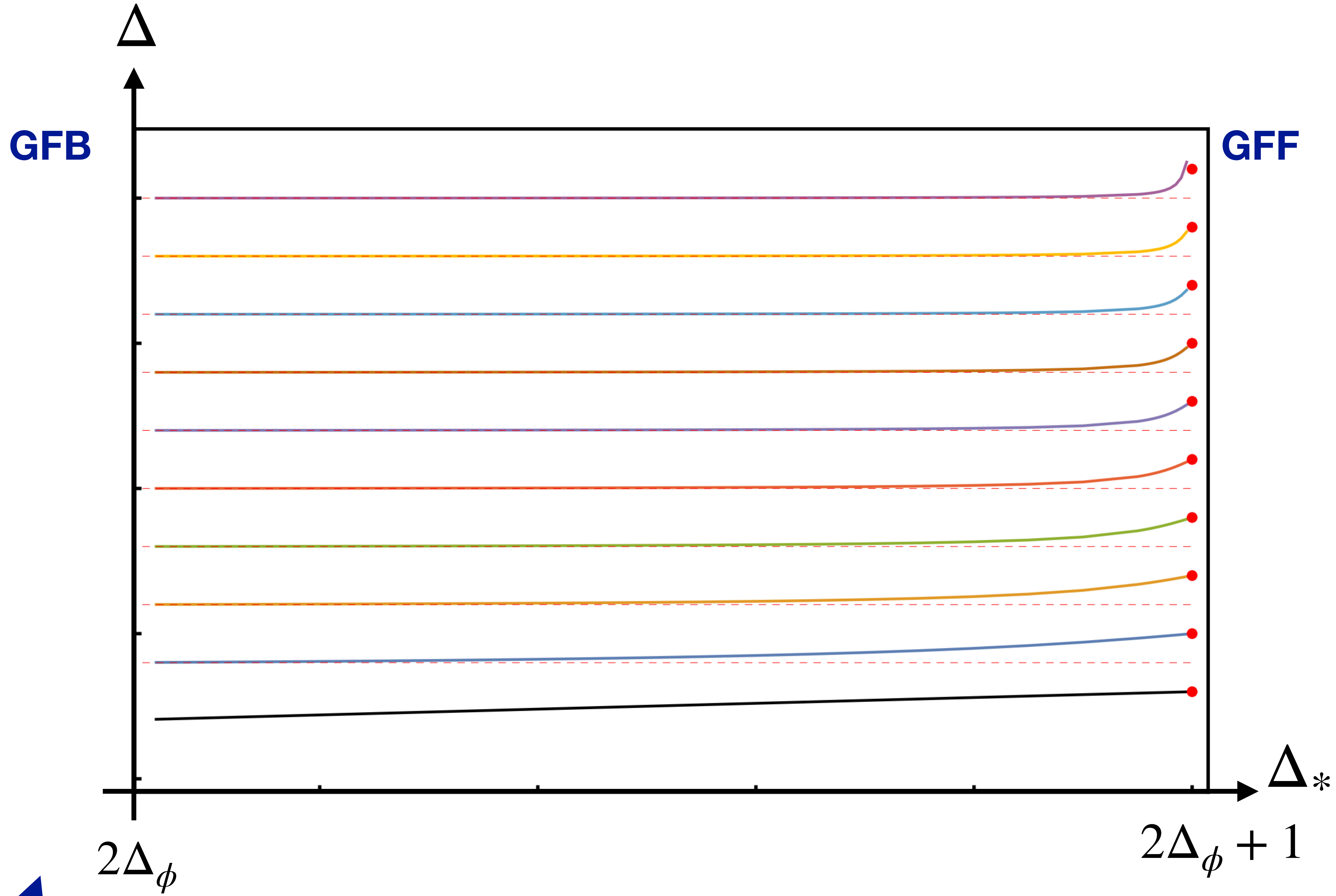
$$\max a_{\Delta_*} = \inf\left(\text{bounds}\right)$$



Example $S(z) = G_0(z)$

Feasible for any $\Delta_* \in [\sqrt{2}\Delta_\phi,\ 2\Delta_\phi + 1]$

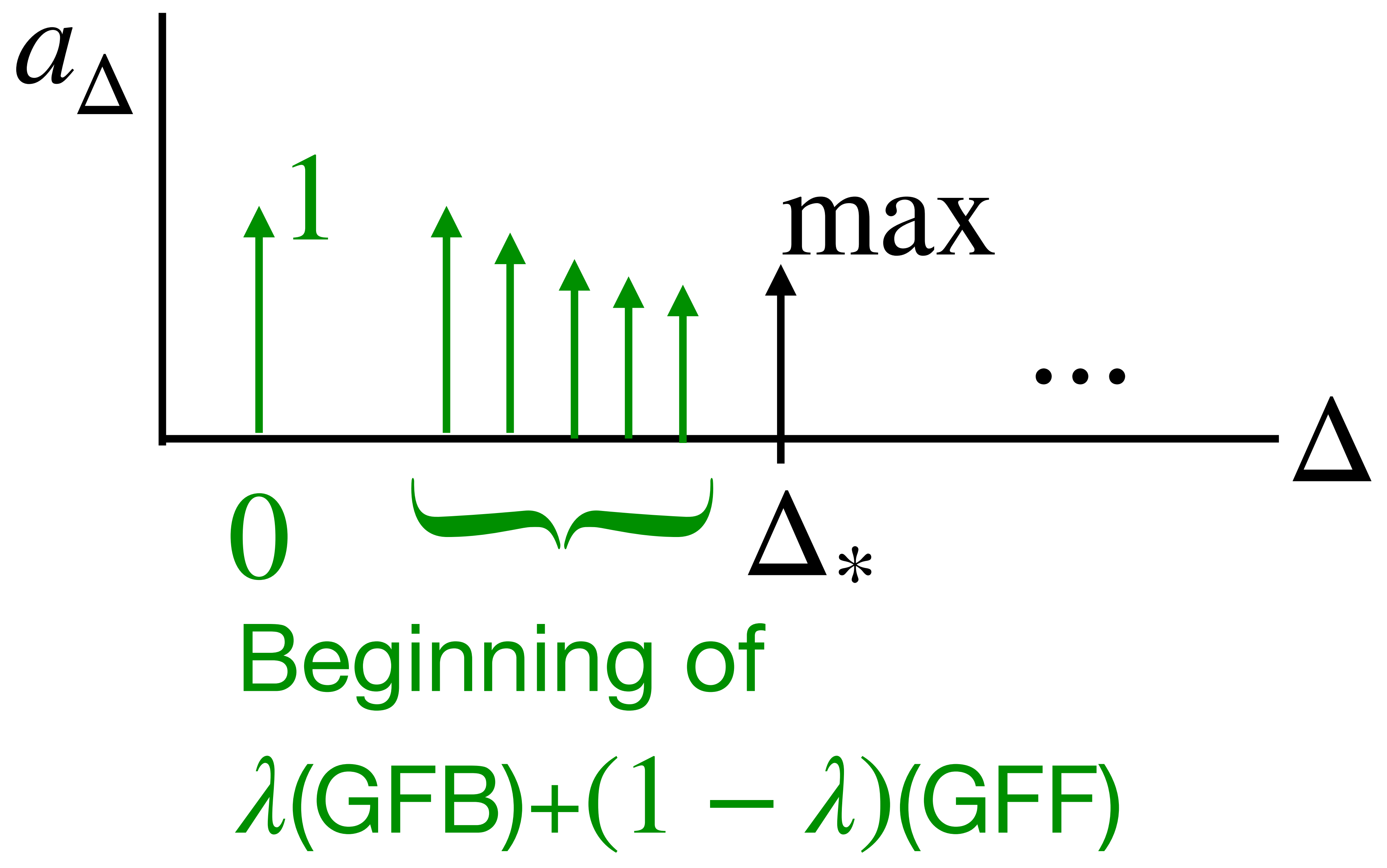
$\xrightarrow{\text{theorem}}$ **extremal solution exists**



Can also go lower, down to $\sqrt{2}\Delta_\phi$

More examples

GFB / GFF + anomalous gap



Interacting choices of S

See explorations of [Ghosh Paulos Suchel]

Structure

1) Formalism

2) Optimisation

1) Convex optimisation in Hilbert space

Setup

X a Hilbert space

$$(P) = \inf_{x \in X} \big(f(x) + g(x) \big)$$

$f, g : X \to \mathbb{R} \cup \{+\infty\}$ convex and lower semi-continuous (lsc)

continuous functions are lsc

$$1_S(x) = \begin{cases} 0 \,, & x \in S \\ +\infty \,, & x \notin S \end{cases}$$

lsc $\iff$ S closed

Convex conjugate

$$f^*(x) := \sup_{y \in X} \big(\langle x \mid y \rangle - f(y) \big)$$

Dual problem

$$(D) = \sup_{x \in X} \big(-f^*(x) - g^*(-x) \big)$$

Theorem (Fenchel-Rockafellar):

[Fenchel 51]
[Rockafellar 66,74]

Suppose (P) and (D) are feasible. Then $(P) \geq (D)$.

If also $0 \in \operatorname{int}\left(\operatorname{dom} f^* - \operatorname{dom} \mathrm{g}^*\right)$, then $(P) = (D)$ and (P) is attained.

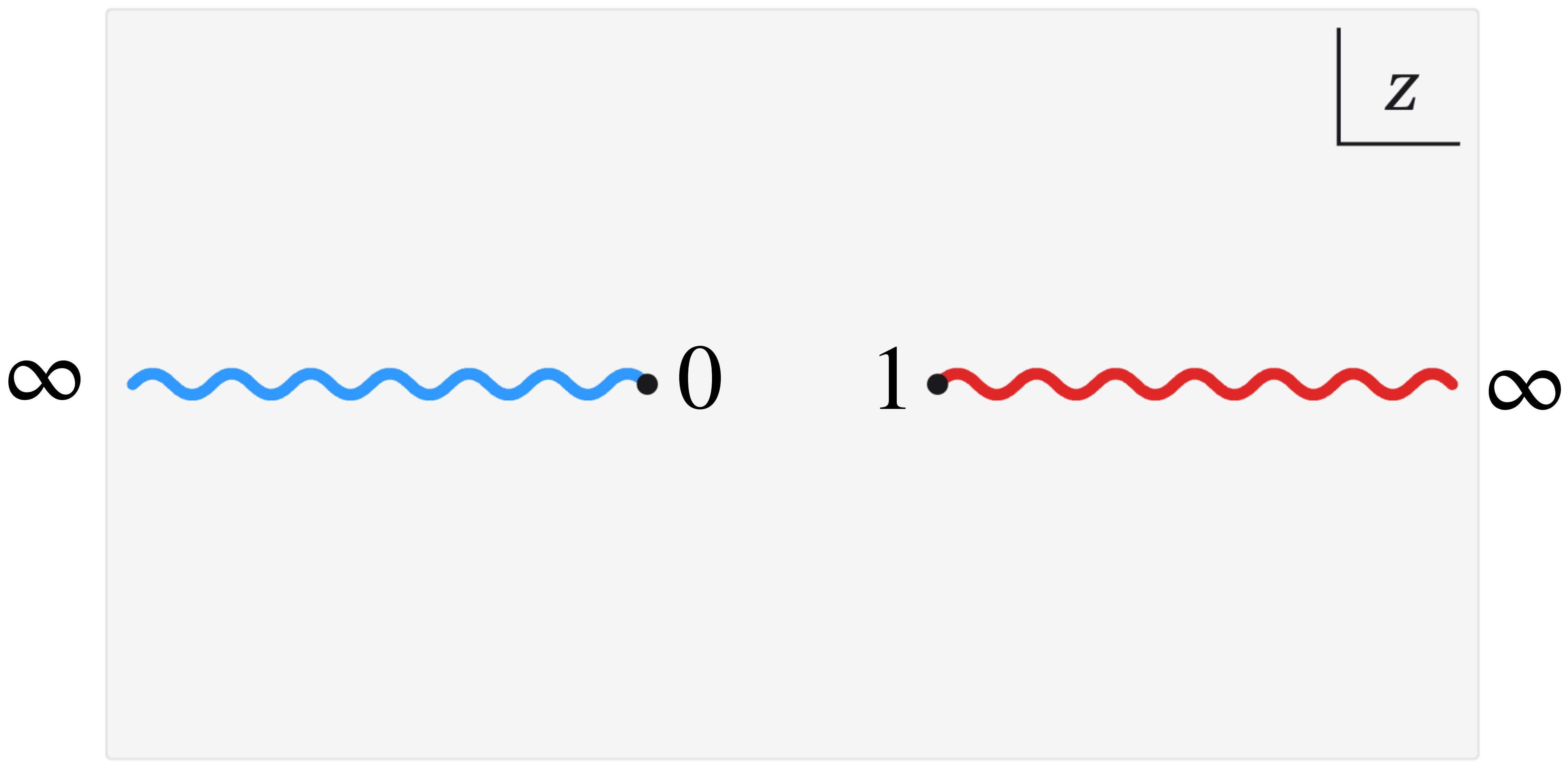
generalised Slater condition

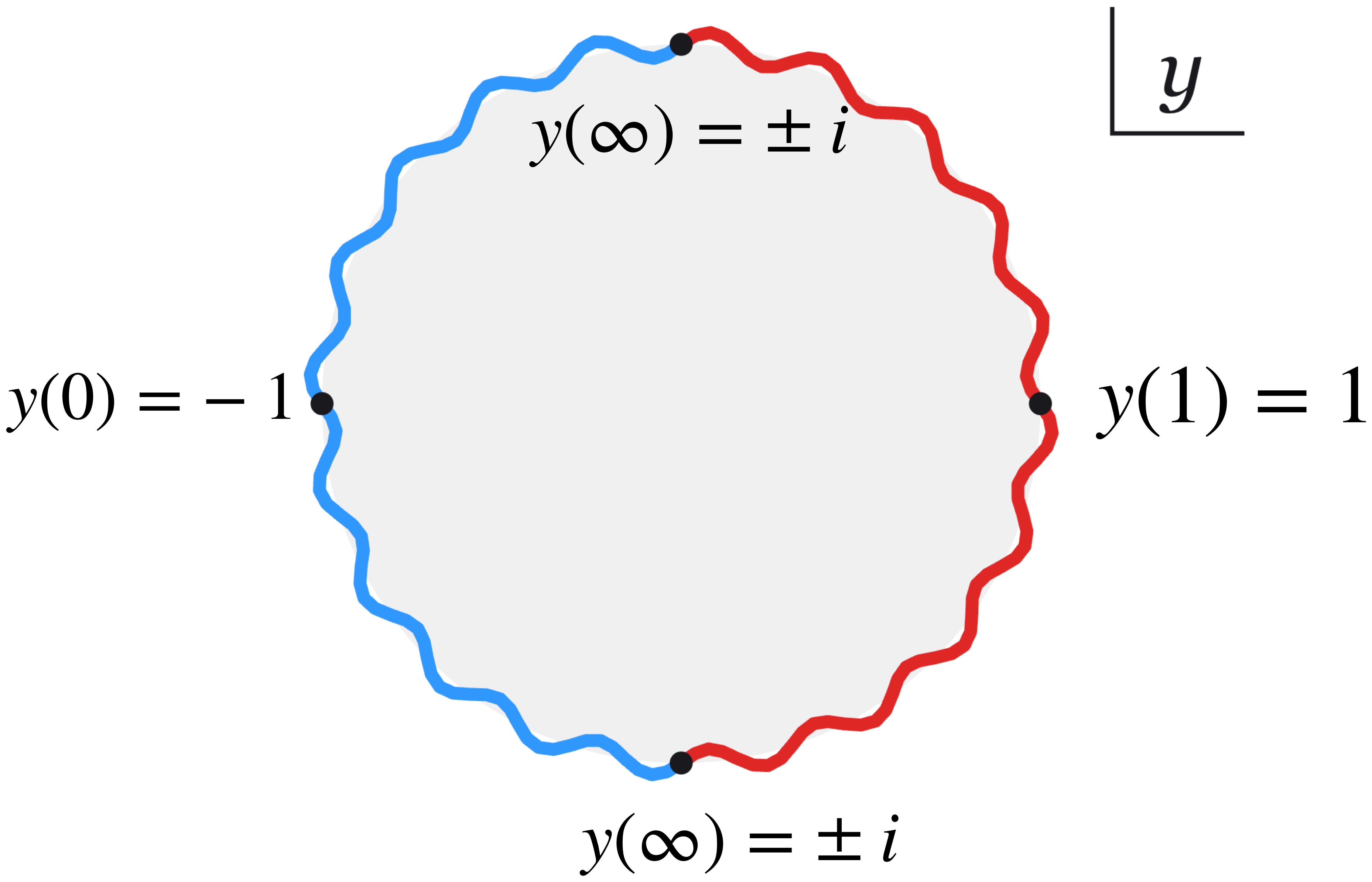
Hilbert spaces are **reflexive**: $(P) \leftrightarrow (D)$ interchangeable

Hilbert space of correlators

Zhukovsky variable

$$z \in \mathbb{C} \backslash (-\infty, 0] \cup [1, \infty) \quad \to \quad y \in \mathbb{D}$$



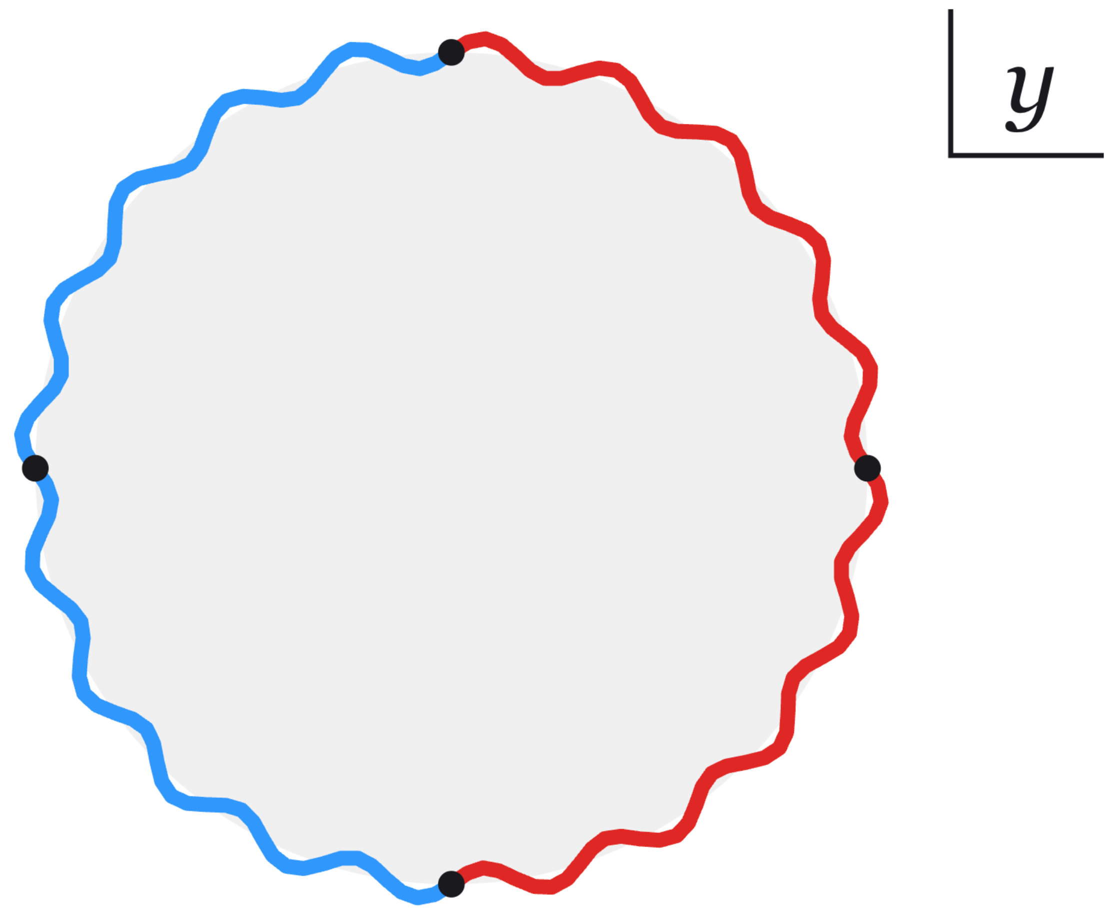


Hilbert space of correlators

Bergman space $$A^2 := \left\{ G(y) \text{ analytic on } \mathbb{D} \;\middle|\; \|G\| < \infty \right\}$$

$$\|G\|^2 = \langle G \,|\, G \rangle$$

$$\langle G \,|\, H \rangle := \int_{\mathbb{D}} \mathrm{d}^2 y \; (1 - |\, y \,|)^{\gamma} \; G(y) \; \overline{H(y)}$$



$$\gamma > 8\Delta_\phi - 1$$

- Separable Hilbert space

OPE converges in A^2

Lemma: *Any unitary solution of crossing* $G(y) = \int \mathrm{d}\Delta\ a(\Delta)\ G_\Delta(y)$

has OPE converging in A^2 :

$$G = \lim_{N\to\infty} \int_0^N \mathrm{d}\Delta\ a(\Delta)\ G_\Delta \quad \in A^2$$

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Uniform bound at boundary $|y| \to 1$:

$$\int \mathrm{d}\Delta\ |a(\Delta)\ G_\Delta(y)| \leq C\,(1-|y|)^{-4\Delta_\phi}$$

à la [Kravchuk Qiao Rychkov]

Use Dominated Convergence

A^2 functionals

$f \in A^2$ $\qquad$ $F = \int \mathrm{d}\Delta \; a(\Delta) \; F_\Delta \in A^2$

$$\theta_f[F] = \langle f | F \rangle$$

- {All functionals known to me} $\subset A^2$ (Including the class from [NL Paulos])

  including derivative functionals

- E.g. evaluation is continuous $F \mapsto F(y) = \langle e_y | F \rangle \,, \; y \in \mathbb{D}$

2) Optimisation

$$F(z) = G(z) - G(1-z) = \int \mathrm{d}\Delta\ a(\Delta)\ F_\Delta(z) \in A^2$$

Recall $$(P) = \inf_{x \in X} \big(f(x) + g(x)\big)$$

Want: $$'' f[F] = -a_{\Delta_*} + 1_{\{a(\Delta) \geq 0\}} '' \qquad g[F] = 1_{\{F=0\}}$$

?

How to… Positivity?

Positive cone above gap $C_+ = \left\{ G = \lim_{N\to\infty} \int_{\Delta_*}^{N} \mathrm{d}\Delta \; a(\Delta) \; F_\Delta \;\middle|\; a(\Delta) \geq 0 \text{ Radon} \right\}$

Need closed constraints: **take closure** $\overline{C_+}$

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Lemma: *For any $F \in \overline{C_+}$, there is a positive Radon measure $a(\Delta) \geq 0$ s.t.*

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converges at each $y \in \mathbb{D}$.

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Use property of **Radon measures**: if a sequence $a_n(\Delta)$ has uniform bound $\int |a_n| \leq C$

then it has a convergent subsequence

How to… Objective function? $a_{\Delta_*}[F]$ not continuous

$$X = \mathbb{R} \oplus A^2$$

(t, F) Constraint $t \leq a_{\Delta_*}$ **closed**

Define allowed set
$$\mathrm{A} := \left\{ (t, F) \in X \;:\; \begin{array}{c} F = S^{(-)} + a_* F_{\Delta_*} + F_+ \\ \\ 0 \leq t \leq a_* \;,\;\; F_+ \in \overline{C_+} \end{array} \right\}$$

- A is convex and closed in X

$t \leq a_* \leq a_{\Delta_*}$

'Best' choice is always $t = a_{\Delta_*}$

OPE-max in X

$$f(t, F) = -t + 1_{\mathrm{A}}(t, F) \qquad g(t, F) = 1_{\{F=0\}}$$

$$(P) = \inf(f + g)$$

OPE-max in X

Theorem: *OPE-max in X is equivalent to physical OPE-max*

There is a map between the values of the objective functions such that for any solution of one problem, there is a solution of the other problem that is at least as good.

In particular, if one is problem is attained, then so is the other.

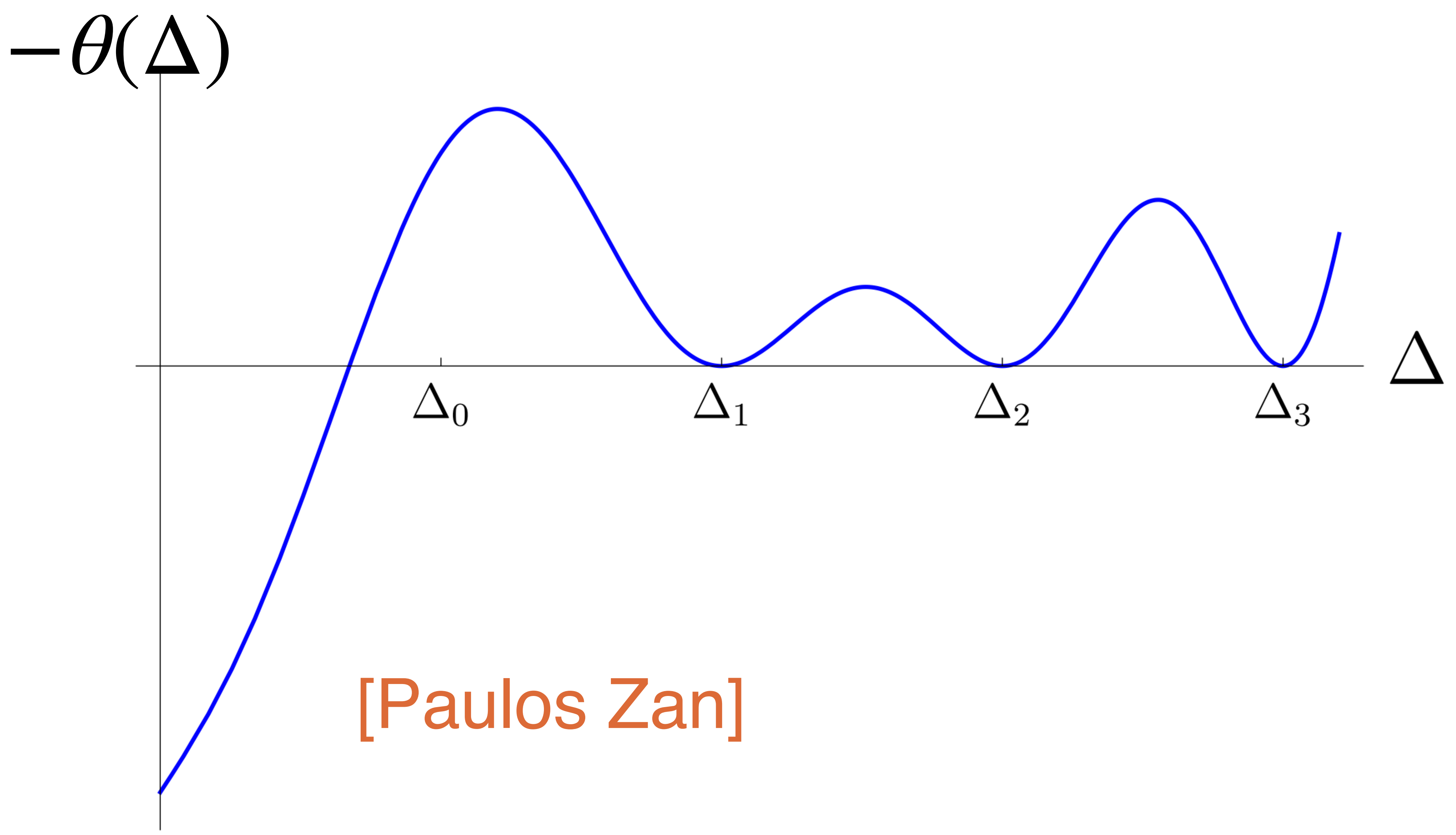
Dual problem in X

$$f^*(t,\theta) = \theta[S^{(-)}] + 1_{\{\theta(\Delta \geq \Delta_*) \leq 0,\ \theta(\Delta_*) \leq -1-t\}}$$

$$g^*(t,\theta) = 1_{t=0}$$

Positive functionals $\rightarrow$ bounds $a_{\Delta_*} \leq -\theta[S^{(-)}]$

Min	$-\theta[S^{(-)}]$
subject to	positivity

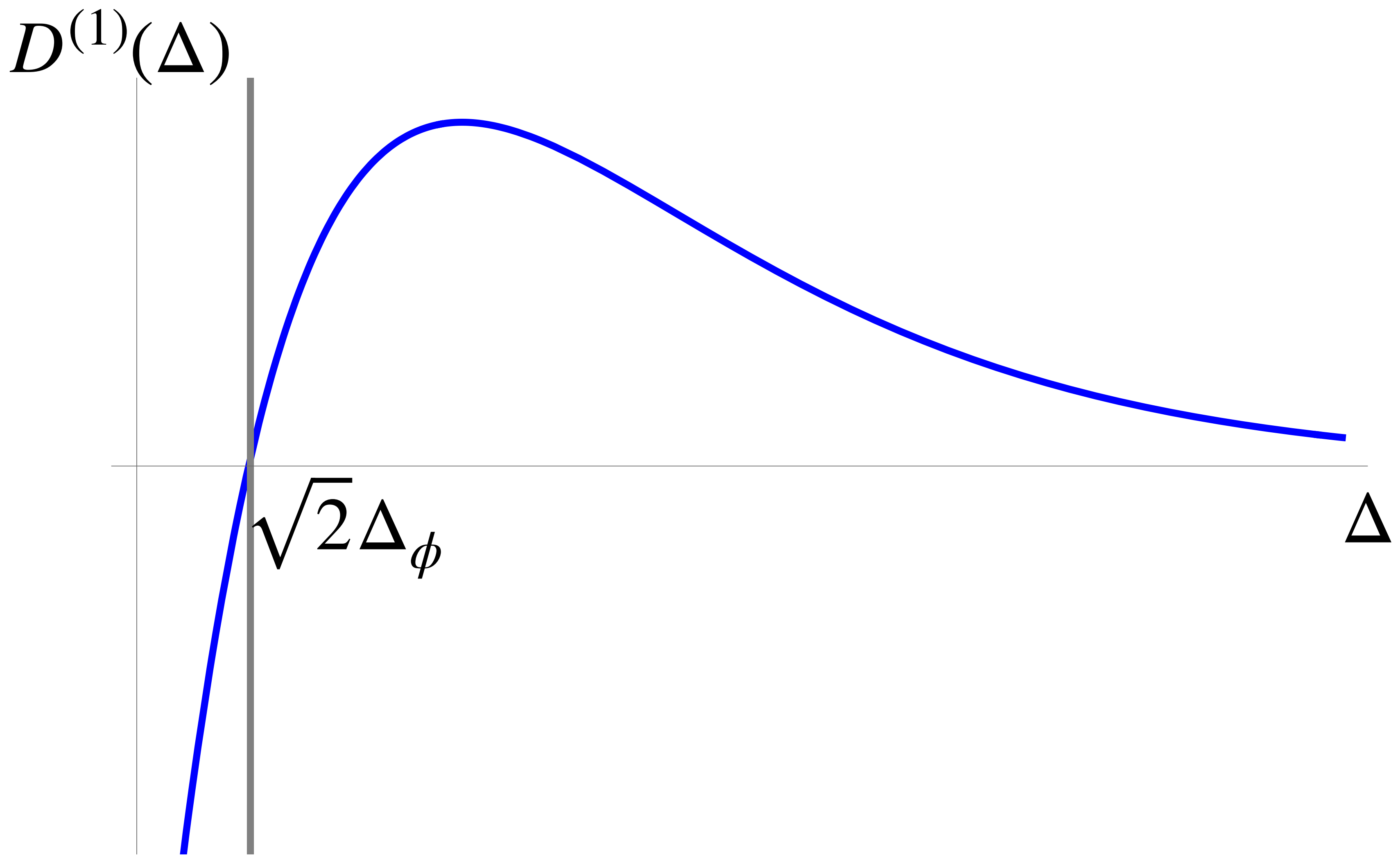


Slater condition for the dual in X

Sufficient condition: existence of a **strictly positive functional**

First derivative $D^{(1)} = \partial_y|_{y=0}$

$D^{(1)}[F_\Delta] = F'_\Delta(0) > 0$ for $\Delta \geq \Delta_* \geq \sqrt{2}\Delta_\phi$



$\Longrightarrow$ Slater for (D)

Apply FR theorem:

- *Zero duality gap*
- *Primal attainment* ■

Summary

- Extremal solutions exist for 1d OPE-max and have zero duality gap

  Interacting, exact solutions

E.g. ϕ^4 flow

Huge class of more general examples

$S(z) \in A^2$ can be anything (even non unitary)

- Convex optimisation in infinite dimensions. Fenchel-Rockafellar theorem
- Bergman space of correlators (Hilbert spaces are nice)

Outlook

- Discreteness & sparseness of extremal solution?
- Existence of extremal functionals? (Dual attainment)
- Uniqueness?
- $\Lambda \to \infty$ convergence?
- Other problems: Gap-max , $d > 1$, multiple correlators

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(In)consistency with
3d Ising being extremal
for a finite system of
correlators?

Hybrid bootstrap!
Solve directly for islands?

Thank you!