Worldsheet bootstrap for strings on AdS [1]

Luis Fernando Alday

University of Oxford

Bootstrap 2026 - London

[1] In collaboration with Chester, Hansen, Naderi, Pitombo, Sangare, Zhong.

What will this talk be about?

A set of tools to compute String Theory amplitudes on AdS

More specifically: scattering of four massless strings

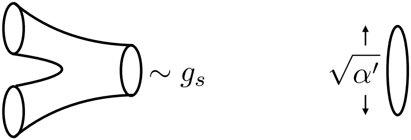
- Closed strings: gravitons on $AdS_5 \times S^5$ - AdS Virasoro-Shapiro.
- Open strings: gluons on $AdS_5 \times S^3$ - AdS Veneziano.

We will start with the story in flat space.

Four-graviton amplitude - Flat space

4pt graviton amplitude in flat space

- The parameters of the theory are g_s and α'.



- The amplitude depends on the momenta p_i and polarisations ϵ_i of the (external) gravitons.
- SUSY fixes the dependence on the polarisations:

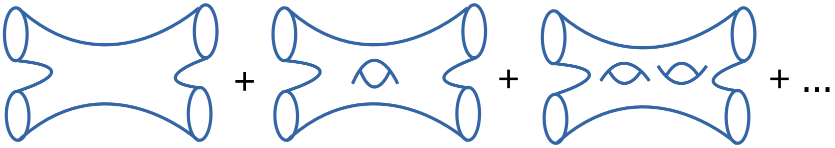
$$A(g_s, \alpha', p_i, \epsilon_i) = \underbrace{pref(\epsilon_i, p_i)}_{\text{simple prefactor}} \times \underbrace{A(g_s, \alpha', s, t, u)}_{\text{we focus on this}}$$

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

Massless scattering: $s + t + u = 0$

String theory scattering amplitudes

- The computation organises in a genus expansion



$$g_s^2 A_{closed}^{(tree)}(\alpha', s, t) + g_s^4 A_{closed}^{(gen\ 1)}(\alpha', s, t) + g_s^6 A_{closed}^{(gen\ 2)}(\alpha', s, t) + \cdots$$

- In flat space we can use the world-sheet theory to compute these amplitudes. At genus zero:

$$A_{closed}(\alpha', s, t) \sim \int_{\mathbb{C}} \frac{|z|^{-2\alpha' s}|1-z|^{-2\alpha' t}}{|z|^2|1-z|^2} d^2 z$$

Four-graviton amplitude - Flat space

Leading order in g_s: Virasoro-Shapiro amplitude

$$A_{closed}(\alpha', s, t) = \alpha'^3 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

- Crossing symmetric in s, t, u $(s+t+u=0)$
- Poles due to the exchange of particles (of mass $m = 2\sqrt{n/\alpha'}$ and spin ℓ)

$$A_{closed}(\alpha', s, t) \sim \frac{P_\ell(t)}{\alpha' s - n}$$

- Regge behaviour

$$A_{closed}(\alpha', s, t) \sim s^{-2+\alpha' \frac{t}{2}}, \quad \text{for large } |s|$$

- α' expansion

$$A_{closed}(\alpha', s, t) \sim \frac{1}{\underbrace{s\,t\,u}_{\text{sugra}}} + \underbrace{2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots}_{\text{stringy corrections}}$$

Four-graviton amplitude - Flat space

Less appreciated property... only odd ζ-values appear in the expansion!

- Zeta-values arise from polylogarithms evaluated at $z = 1$

$$Li_n(z) = \sum_{k=1} \frac{z^k}{k^n} \to Li_n(1) = \zeta(n)$$

- Polylogarithms are not single-valued over the complex plane.
- Map from multi-valued to single-valued polylogarithms

$$Li_n(z) \to \mathcal{L}_n(z, \bar{z})$$

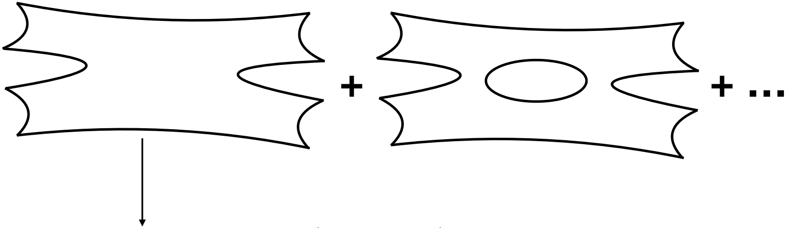
- These evaluated at $z = 1$ generate a subset of all the zeta values, called single-valued zeta values

$$Li_n(1) \to \zeta(2), \zeta(3), \zeta(4), \zeta(5), \cdots, \quad \mathcal{L}_n(1) \to \xcancel{\zeta(2)}, \zeta(3), \xcancel{\zeta(4)}, \zeta(5), \cdots$$

The α' expansion of the VS amplitude contains only single-valued zetas.

Veneziano Amplitude

- We can also consider the scattering of open strings in flat space



$$A_{open}(\alpha', s, t) = \int_0^1 \frac{x^{-\alpha' s}(1-x)^{-\alpha' t}}{x(1-x)} dx = \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(-\alpha' s - \alpha' t)}$$

- Low energy expansion

$$A_{open}(\alpha', s, t) = \left(\frac{1}{s} + \frac{1}{t}\right)\left(1 - \zeta(2)\alpha'^2\, s\, t + \zeta(3)\alpha'^3\, s\, t\, u + \cdots\right)$$

- Contains both $\zeta(2n+1)$ and $\zeta(2n)$, but!

SV map

- Single value map

$$Li_n(z) \xrightarrow{\text{SV}} \mathcal{L}_n(z,\bar{z})$$

- At the level of zeta-values

$$\zeta(2n) \xrightarrow{\text{SV}} 0, \quad \zeta(2n+1) \xrightarrow{\text{SV}} 2\zeta(2n+1)$$

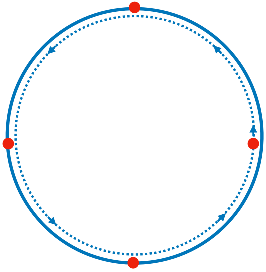
- Takes the open string amplitude to the closed string amplitude!

$$\frac{1}{s+t} A_{open}(\alpha', s, t) \xrightarrow{\text{SV}} (s+t) A_{closed}(\alpha', s, t)$$

- The KLT relations are actually an avatar of these relations!

Monodromy relations

- The Veneziano amplitude is not crossing symmetric, but it satisfies monodromy relations



$$= 0$$

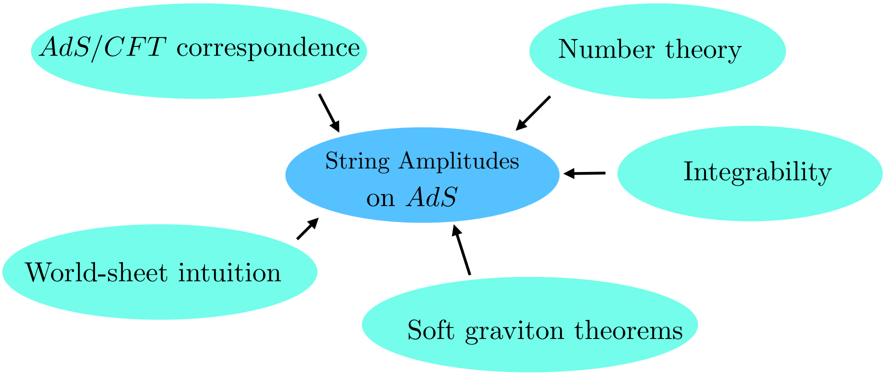
$$\underbrace{e^{\pm i\pi\alpha' s}} A_{open}(s,u) + A_{open}(s,t) + \underbrace{e^{\mp i\pi\alpha' t}} A_{open}(u,t) = 0$$

Monodromy factors, as vertex operators cross each other

How much of all this story generalises to AdS?

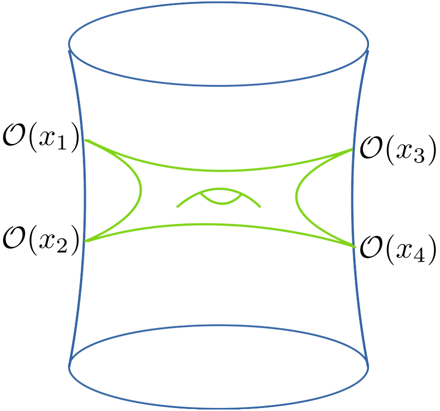
VS in AdS

In curved backgrounds we don't have a world-sheet theory...but for the special case of $AdS_5 \times S^5$ we can make a lot of progress!



AdS/CFT

String amplitudes on *AdS* $\leftrightarrow$ Correlators of local operators in the CFT at the boundary.



$$\mathcal{A}(g_s, \alpha', s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

AdS/CFT

String theory on $AdS_5 \times S^5$ $\leftrightarrow$ 4d $\mathcal{N} = 4$ SYM

(g_s, R) $\qquad$ (g_{YM}, N)

$$g_s \approx \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$$

String amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N} = 4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: Scalar operator of dim. 2 in the stress-tensor multiplet

Consider $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ in a $1/N$ expansion.

4d $\mathcal{N} = 4$ SYM - Kinematics

The observable

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle, \quad \mathcal{O}^{IJ}(x) = Tr\varphi^{(I}\varphi^{J)}$$

- Fixed by symmetries up to a function of two variables

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle = pref(I_i, J_i, x_i) \times \underbrace{\mathcal{G}(U,V)}_{\text{we focus on this}}$$

$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

- Very similar to the structure in flat space.
- $\mathcal{G}(U, V)$ is a highly-non trivial function of N and λ!

$1/N$ expansion and Mellin amplitude

- Tree level $\rightarrow$ leading non-trivial term in the $1/N$ expansion:

$$\mathcal{G}(U,V) = \underbrace{\mathcal{G}_{disc}(U,V)}_{disconnected} + \frac{1}{N^2}\underbrace{\mathcal{G}_{tree}(U,V)}_{tree-level} + \cdots$$

- $\mathcal{G}_{tree}(U,V)$ still a highly non-trivial function of λ...

Mellin space

$\mathcal{G}_{tree}(U,V) \rightarrow \mathcal{M}_{tree}(s,t,u)$, with $s+t+u=0$.

$$\boxed{\mathcal{G}_{tree}(U,V) = \int_{-i\infty}^{i\infty} dsdt U^s V^t \underbrace{\Gamma(s,t,u)}_{\text{prefactor}} \quad \underbrace{\mathcal{M}_{tree}(s,t,u)}_{\text{VS amplitude in } AdS_5 \times S^5}}$$

Formulation where the correlator looks much more like an amplitude.

AdS VS amplitude

$\underline{\mathcal{M}_{tree}(s,t)}$

1. Crossing symmetric in s, t, u.
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{tree}(s,t) = C^2_{\Delta,\ell} \sum_{m=0}^{\infty} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular}$$

3. Regge limit

$$\mathcal{M}_{tree}(s,t) \sim \frac{1}{s^2}, \quad \text{for large } |s| \text{ and } Re(t) < 2$$

4. Admits an expansion around flat space

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{VS in flat space}} + \underbrace{\frac{\alpha'}{R^2} A^{(1)}(s,t) + \frac{\alpha'^2}{R^4} A^{(2)}(s,t) + \cdots}_{\text{curvature corrections}}$$

Can we use these constraints to fix $\mathcal{M}_{tree}(s,t)$? yes! with one more assumption.

AdS Virasoro-Shapiro around flat space

- Low energy expansion of curvature corrections:

$$A^{(0)}(s,t) = \frac{1}{s\,t\,u} + 2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots$$

$$A^{(1)}(s,t) = \underbrace{\frac{s^2+t^2+u^2}{(s\,t\,u)^2}}_{\text{gravity on } AdS} + \underbrace{\alpha_1\alpha'^4 + \alpha_2\alpha'^6(s^2+t^2+u^2) + \cdots}_{\text{unknown coefficients}}$$

<u>Assumption</u>: these unknown coefficients are also single-valued zetas!

AdS Virasoro-Shapiro around flat space

Very powerful when supplemented with the correct structure of poles!

- While $A^{(0)}(s,t)$ has single poles, corrections are more complicated:

$$A^{(1)}(s,t) \sim \frac{r_n^{(0)}(t)}{(\alpha' s - n)^4} + \frac{r_n^{(1)}(t)}{(\alpha' s - n)^3} + \cdots$$

- This follows from (2). In general

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{simple poles}} + \frac{\alpha'}{R^2} \underbrace{A^{(1)}(s,t)}_{\text{quartic poles}} + \frac{\alpha'^2}{R^4} \underbrace{A^{(2)}(s,t)}_{\text{seventh order poles}} + \cdots$$

AdS Virasoro-Shapiro amplitude

Poles + Single-valuedness + World-sheet intuition/hope

$$\Downarrow$$

Proposal order by order

$$A^{(0)}(s,t) = \int d^2z |z|^{-2\alpha' s-2}|1-z|^{-2\alpha' t-2}$$

$$A^{(1)}(s,t) = \int d^2z |z|^{-2\alpha' s-2}|1-z|^{-2\alpha' t-2} \underbrace{W_3(z,\bar{z})}_{\text{SV polylogs of weight 3}}$$

$$A^{(2)}(s,t) = \int d^2z |z|^{-2\alpha' s-2}|1-z|^{-2\alpha' t-2} \underbrace{W_6(z,\bar{z})}_{\text{SV polylogs of weight 6}}$$

$$\vdots$$

Also consistent with soft graviton theorems.

AdS Virasoro-Shapiro amplitude

A Worldsheet bootstrap

$$A^{AdS}_{VS}(s,t) = \int d^2z \frac{|z|^{-2\alpha' s}|1-z|^{-2\alpha' t}}{|z|^2|1-z|^2}\left(1 + \frac{\alpha'}{R^2}W_3(z,\bar{z}) + \frac{\alpha'^2}{R^4}W_6(z,\bar{z}) + \cdots\right)$$

- Consistency with properties 1,2,3,4.
- Location of poles consistent with the spectrum of $\mathcal{N}=4$ SYM

$$\Delta_{\mathcal{K}} = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \frac{1/2 - 3\zeta(3)}{\lambda^{3/4}} + \cdots$$

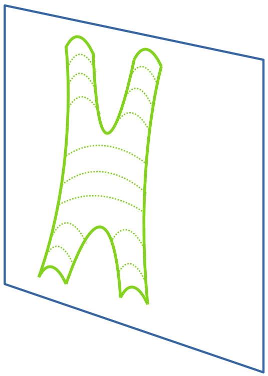
- Consistency with localisation constraints.

We can fix $W_3(z,\bar{z})$, $W_6(z,\bar{z})$ completely! working on $W_9(z,\bar{z})$.

Gluon scattering in AdS

Let's now scatter gluons!

Gluons $\rightarrow$ open strings $\rightarrow$ add branes



- $\mathcal{N} = 4$ SYM $\rightarrow \mathcal{N} = 2$ SYM with flavours

AdS Veneziano amplitude

- Define a Mellin amplitude. Also admits a curvature expansion

$$\mathcal{M}_{open}(s,t) = \underbrace{A^{(0)}_{open}(s,t)}_{\text{Veneziano}} + \underbrace{\frac{\alpha'}{R^2} A^{(1)}_{open}(s,t) + \frac{\alpha'^2}{R^4} A^{(2)}_{open}(s,t) + \cdots}_{\text{curvature corrections}}$$

Problem: now we don't have single valuedness! but

- $A^{(1)}_{open}(s,t), A^{(2)}_{open}(s,t), \cdots$ similar structure of poles as in the closed string case.
- Based on the world-sheet intuition for the closed case, we have a proposal!

Take the integrand for the Veneziano amplitude, and insert harmonic polylogarithms!

AdS Veneziano amplitude

$$A^{AdS}_{open}(s,t) = \int_0^1 dx \frac{x^{-\alpha' s}|1-x|^{-\alpha' t}}{x(1-x)} \left(1 + \frac{\alpha'}{R^2} \underbrace{\mathrm{H}_3(x)}_{\text{HPLs of weight 3}} + \cdots \right)$$

- $\mathrm{H}_3(x), \mathrm{H}_6(x), \cdots$ multi-valued on the complex plane...but well defined on $[0,1]$.
- Structure of poles $\rightarrow$ result to order $1/R^4$ up to one coefficient.
- For $G_F = SO(8)$ we can use localisation results to fix this completely.
- The final expression passes many non-trivial checks!

Monodromy relations on *AdS*

- Is the open string result really a world-sheet correlator?

$$A^{AdS}_{open}(s,t) = \int_0^1 dx \langle V(0)V(z)V(1)V(\infty)\rangle$$

AdS Monodromy relations

- When the vertex operators cross each other we get the usual contribution from $x^{-\alpha' s}(1-x)^{-\alpha' t}$.
- Plus another contribution from the harmonic polylogs! (which have discontinuities around $x=0,1$). We find:

$$e^{\pm i\pi\alpha'(s+K_0)}A_{open}(s,u) + A_{open}(s,t) + e^{\mp i\pi\alpha'(t+K_1)}A_{open}(u,t) = 0$$

- K_0, K_1 implement the disc around 0,1 of polylogs.
- Highly non-trivial! and clear evidence of a world-sheet!

Questions - Discussion

Medium questions

- Is the conjecture correct? do we need to enlarge the family of functions used?
- Can we solve the next order? can we find more patters?
- Double copy relations in AdS?

Big questions

- Can we reproduce this from a world-sheet description for strings on $AdS_5 \times S^5$?
  - String field theory?
  - Pure spinors?
- A quantum curve for the AdS VS amplitude?
- Nature of the $1/R$ expansion? non-perturbative corrections?
- What about genus one amplitudes on AdS?
- AdS amplitudes for massive states?