Bootstrapping time evolution

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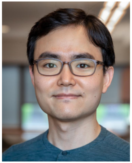
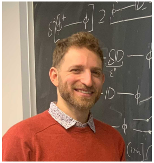
[2511.08560] w/ Minjae Cho, Barak Gabai, Henry Lin, and Zechuan Zheng
[260x.xxxxx] w/ Henry Lin and Zechuan Zheng

Motivation

- We want to compute the time-dependent correlators. For example, the Euclidean two-point function:

$$G(\tau) = \langle \mathcal{O}(\tau)\mathcal{O}(0)\rangle_{\text{g.s.}} = \sum_n |\langle n|X|0\rangle|^2 e^{-(E_n - E_0)\tau}$$

- Common methods: Hamiltonian truncation, tensor networks, Monte Carlo, conformal bootstrap...
- Monte Carlo has sign problems for real-time, which is the relevant quantity for spectral density, transport properties, quasinormal modes...

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- Monte Carlo has sign problems for real-time, which is the relevant quantity for spectral density, transport properties, quasinormal modes...
- In this talk, we will focus on a bootstrap approach directly targeting the time-dependent correlators in a generic quantum mechanical system. We will apply the techniques in large N matrix quantum mechanics.

Quantum mechanics bootstrap

Consider a quantum mechanical system with Hamiltonian H, and we want to know some $\langle \mathcal{O} \rangle$.
Hilbert space positivity $\implies \mathcal{M}_{ij} = \left\langle \overline{\mathcal{O}_i} \mathcal{O}_j \right\rangle \succeq 0$ ("Gram matrix").
Ground state positivity: $\mathcal{N}_{ij} = \left\langle \overline{\mathcal{O}_i} [H, \mathcal{O}_j] \right\rangle \succeq 0$. $\quad (\langle \psi | H | \psi \rangle \geq E_0 \quad \forall \, |\psi\rangle)$

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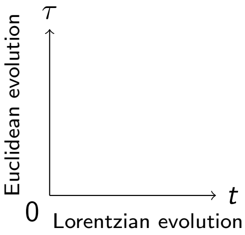
In the SDP form:

$$\begin{aligned} & \min \langle \mathcal{O} \rangle \\ \text{subject to: } & \mathcal{M} \succeq 0 \\ & \mathcal{N} \succeq 0 \\ & \langle [H, \mathcal{O}] \rangle = 0 \end{aligned}$$

From 1pt to 2pt

We will want to consider a Gram matrix with entries separated by time:

$$\mathcal{M}_{ij}(\tau) = \langle \overline{\mathcal{O}_i}(\tau)\mathcal{O}_j(0)\rangle$$



In the SDP form:

$$\begin{aligned} &\min \langle \mathcal{O}_1(\tau)\mathcal{O}_2(0)\rangle \\ \text{subject to } &\mathcal{M}(\tau) \succeq 0 \\ &\text{Heisenberg EOM} \\ &+ \text{relations} \end{aligned}$$

We will obtain two-sided rigorous bounds for time-dependent correlators, with bounds widen at large time.

Positivity

Consider

$$\mathcal{M}_{ij}(\tau) = \langle \overline{\mathcal{O}_i}(\tau)\mathcal{O}_j(0)\rangle = \langle \overline{\mathcal{O}_i}(\frac{\tau}{2})\mathcal{O}_j(-\frac{\tau}{2})\rangle$$

Reflection positivity, i.e. $\overline{\mathcal{O}(\tau)} = \overline{\mathcal{O}}(-\tau)$, implies

$$\boxed{\mathcal{M}(\tau) \succeq 0}$$

Specify we are in ground state or thermal state

- Ground state: Consider

$$\mathcal{N}_{ij}(\tau) = \langle\Omega|\overline{\mathcal{O}}_i(\tau)[H, \mathcal{O}_j(0)]|\Omega\rangle$$

  We have $\mathcal{N}(\tau) \succeq 0$.

- Two equivalent ways to specify a thermal state at β:
  1. KMS condition: $\langle\overline{\mathcal{O}_i}(\beta)\mathcal{O}_j(0)\rangle = \langle\mathcal{O}_j(0)\overline{\mathcal{O}}_i(0)\rangle$
  2. Energy-Entropy balance (EEB) inequality [Araki &Sewell, Fawzi, Fawzi, Scalet; Cho, Gabai, Sandor, Yin]:

$$\log\frac{\langle\overline{\mathcal{O}}\mathcal{O}\rangle_\beta}{\langle\mathcal{O}\overline{\mathcal{O}}\rangle_\beta} \leq \beta\frac{\langle\overline{\mathcal{O}}[H, \mathcal{O}]\rangle_\beta}{\langle\overline{\mathcal{O}}\mathcal{O}\rangle_\beta}$$

Primal-dual formalism

Why do we need this? Because the primal problem contains differential equations, which would require time discretization to implement, and one solution is to go to the dual problem.

Primal (variables are x):

Minimize:$c^T x$

subject to:$Ax - b \geq 0$

Dual (variables are λ):

Maximize:$b^T \lambda$

subject to:$A^T \lambda = c, \lambda \geq 0$

They are related by a Lagrangian

$$L(x, \lambda) = c^T x - \lambda^T (Ax - b) = \left(c^T - \lambda^T A\right) x + \lambda^T b$$

Weak duality:

$$p^* \geq d^*$$

Primal-dual formalism: with differential operator

Primal:

$$\begin{aligned} &\text{minimize}\, x(T) \\ &\qquad \text{s.t.}\ \frac{d}{dt}x(t) = 2x(t) + y(t) \\ &\qquad\qquad x(t) \geq 0 \\ &\qquad\qquad y(t) \geq 0 \quad \forall t \in [0, T] \end{aligned}$$

The answer is $x(T) \geq x(0)e^{2T}$.

Primal-dual formalism: with differential operator

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& y(t) \geq 0 \quad \forall t \in [0, T]
\end{aligned}
$$

Dual:

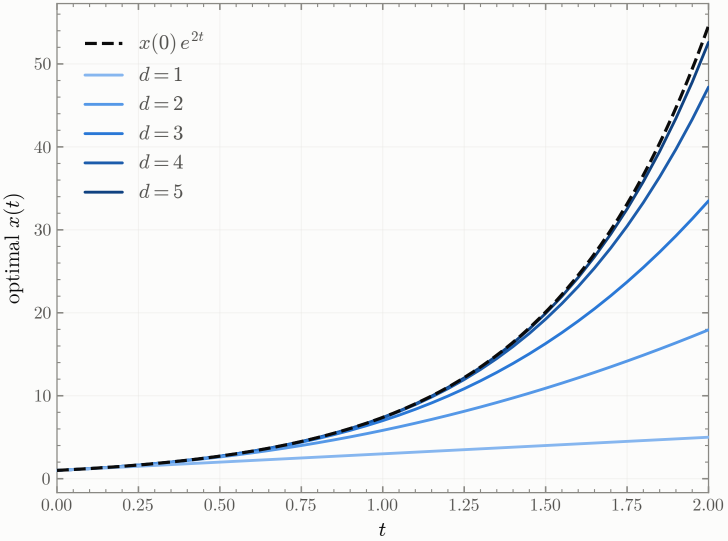
$$
\boxed{
\begin{aligned}
\text{maximize}\, & \lambda_D(0)x(0) \\
\text{s.t.}\, & 1 = \lambda_D(T) \\
& 2\lambda_D(t) + \frac{d}{dt}\lambda_D(t) \succeq 0 \\
& \lambda_D(t) \succeq 0
\end{aligned}
}
$$

The answer is $x(T) \geq x(0)e^{2T}$.

$$
\begin{aligned}
L &= x(T) + \int_0^T dt \left[\lambda_D(t)(2x(t) + y(t) - \frac{d}{dt}x(t)) - \lambda_x(t)x(t) - \lambda_y(t)y(t)\right] \\
&= x(T) + \lambda_D(0)x(0) - \lambda_D(T)x(T) \\
&+ \int_0^T dt \left(2\lambda_D(t) + \frac{d}{dt}\lambda_D(t) - \lambda_x(t)\right) x(t) + \int_0^T dt \left(\lambda_D(t) - \lambda_y(t)\right) y(t)
\end{aligned}
$$

where $\lambda_{x,y}(t)$ is positive to ensure $x(t), y(t) \geq 0$.

Primal-dual formalism: with differential operator



Primal-dual formalism for the time-dependent bootstrap

Primal (variables are $\mathcal{M}(\tau)$):

$$\begin{aligned}
\min\ & \operatorname{Tr} \hat{O}\mathcal{M}(T) \\
\text{s.t.}\ & \mathcal{M}(\tau) \succeq 0 \\
& \mathcal{N}(\tau) \succeq 0, \\
& \left(D^\dagger - \partial_\tau\right) \mathcal{M}(\tau) = 0 \quad \text{Heisenberg EOM} \\
& \left.\begin{aligned} &\mathcal{N}(\tau) - \mathcal{M}(\tau)D = 0 \\ &D^\dagger\mathcal{M}(\tau) - \mathcal{M}(\tau)D = 0 \end{aligned}\right\} \text{relations}
\end{aligned}$$

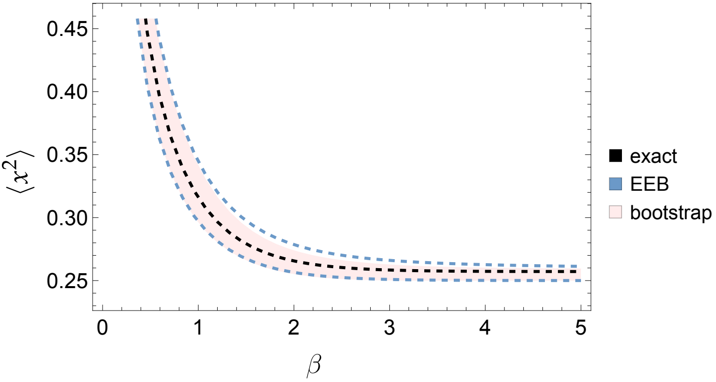
Dual (variables are λ's):

$$\begin{aligned}
\max\ & \operatorname{Tr} \lambda_D(0)\mathcal{M}(0) \\
\text{s.t.}\ & \hat{O} = \lambda_D(T) \\
& \lambda_A D^\dagger - D\lambda_A + \partial_\tau \lambda_D \\
& + \frac{1}{2}[D(\lambda_G+\lambda_D) + (\lambda_G+\lambda_D)D^\dagger] \succeq 0 \\
& - \lambda_G(\tau) \succeq 0
\end{aligned}$$

- This can be a finite-dim problem by picking some bounded-degree ansatz for $\lambda(\tau)$. Any solution that satisfies the dual constraints is a lower bound for the primal problem.

First result: reproduce the thermal bootstrap

Compare this "KMS bootstrap" to the previous thermal bootstrap results using the EEB condition in the anharmonic oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + gx^4$:

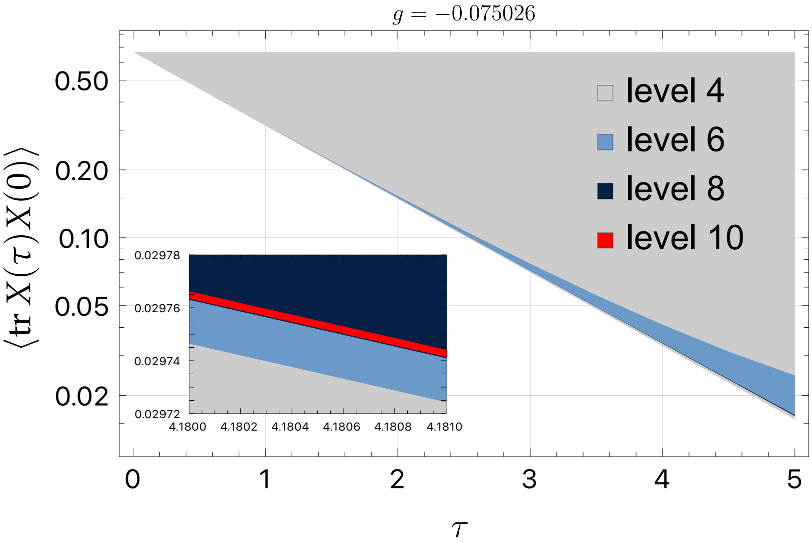


Apply to 1-matrix quantum mechanics (1-MQM)

$$H = \frac{1}{2}\,\mathrm{Tr}\big(P^2 + X^2\big) + \frac{g}{N}\,\mathrm{Tr}\big(X^4\big), \quad [X_{ij}, P_{k\ell}] = \mathrm{i}\delta_{i\ell}\delta_{jk}$$

- The Hamiltonian has the $U(N)$ symmetry $H \to UHU^\dagger$. We will study the ungauged version of the 1-MQM.
- The singlet sector is solvable by mapping to a free fermion system [Brezin, Itzykson, Parisi, Zuber '78].
- Its Feynman diagram expansion coincides with the partition function of 2D quantum gravity coupled to a scalar [Migdal Kazakov '89].
- Bootstrap observables such as $\frac{1}{N^2}\langle \mathrm{Tr}\big(X^2\big)\rangle$ or $\frac{\langle E\rangle}{N^2}$ [Lin '20] [Han, Hartnoll, Kruthoff '20] [Kazakov, Zheng '22].

Euclidean ground state 2pt function at the critical coupling



Adjoint gap in 1-MQM

- The long-time decay behavior is dominated by the first energy gap:

$$\begin{aligned}\langle \mathrm{tr}\, X(\tau)X(0)\rangle &= \sum |\langle 0|X_{ij}|n\rangle|^2 e^{-(E_n-E_0)\tau} \\ &\xrightarrow{\tau\gg 1} |\langle 0|X_{ij}|1\rangle|^2 e^{-\Delta_1\tau}\end{aligned}$$

where $\Delta_1 = E_1 - E_0$ is the *adjoint gap*. Fitting gives $\Delta_1 \approx 0.74436$.

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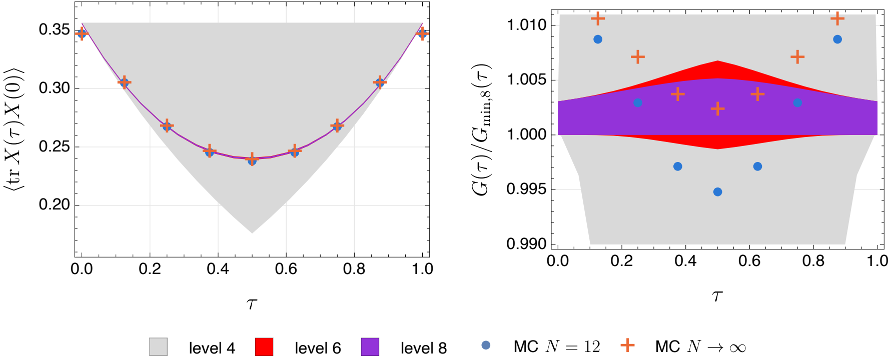
  where $\Delta_1 = E_1 - E_0$ is the *adjoint gap*. Fitting gives $\Delta_1 \approx 0.74436$.
- For 1-MQM, the adjoint sector can be numerically solved via the Marchesini-Onofri (MO) equation:

$$\mathrm{P.V.}\int_{\lambda_-}^{\lambda_+} d\lambda'\,\rho(\lambda')\,\frac{w_n(\lambda)-w_n(\lambda')}{(\lambda-\lambda')^2} \;=\; \Delta_n\, w_n(\lambda)$$

  This gives $\Delta_1 = 0.741573$ at $g = g_c$.

Euclidean thermal state 2pt function - 1-MQM

We demonstrate the bootstrap at $g = 1, \beta = 1$, and compare with the Monto Carlo results.



Lorentzian/complex time: Extended positivity

Naively, we can define $\mathcal{M}_{ij}(t) = \left\langle \mathcal{O}_i^\dagger(t)\mathcal{O}_j(0) \right\rangle$, but then it is not true that $\mathcal{M} \succeq 0$.

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Let $z = \tau + \mathrm{i}t$, consider the Gram matrix formed by $|\mathcal{O}_i(-z/2)\rangle$ and $|\mathcal{O}_i(-\bar{z}/2)\rangle$. Then we have:

$$\widetilde{\mathcal{M}}(z) = \begin{pmatrix} \mathcal{M}(\tau) & \mathcal{M}(z) \\ \mathcal{M}(\bar{z}) & \mathcal{M}(\tau) \end{pmatrix} \succeq 0.$$

The off-diagonal matrix elements contain Lorentzian time separation.

Lorentzian/complex time: Extended positivity

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For anharmonic oscillator, define $\Pi = -ip$. This is time-reversal invariant.

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For anharmonic oscillator, define $\Pi = -ip$. This is time-reversal invariant. In fact, let

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathrm{i}\mathbb{I} \\ \mathbb{I} & -\mathrm{i}\,\mathbb{I} \end{pmatrix}$$

Then,

$$W^\dagger \widetilde{M} W = \begin{pmatrix} \mathcal{M}(\tau) + \mathcal{R}(z) & \mathcal{I}(z) \\ \mathcal{I}(z) & \mathcal{M}(\tau) - \mathcal{R}(z) \end{pmatrix} \succeq 0.$$

Now, this is a real-symmetric matrix.

Analytical bound from $\widetilde{\mathcal{M}} \succeq 0$

Use basis $\{x, \Pi\}$. Let $f(t) = \mathrm{Re}\{\langle x(t)x(0)\rangle\}$ and $h(t) = \mathrm{Im}\{\langle x(t)x(0)\rangle\}$[1].

$$\begin{pmatrix} \mathcal{M}(\tau)+\mathcal{R}(z) & \mathcal{I}(z) \\ \mathcal{I}(z) & \mathcal{M}(\tau)-\mathcal{R}(z) \end{pmatrix} = \begin{pmatrix} \langle x^2\rangle + f & \frac{1}{2}-h' & h & f' \\ \frac{1}{2}-h' & \langle \Pi^2\rangle - f'' & f' & -h'' \\ h & f' & \langle x^2\rangle - f & \frac{1}{2}+h' \\ f' & -h'' & \frac{1}{2}+h' & \langle \Pi^2\rangle + f'' \end{pmatrix}$$

This gives differential inequality:

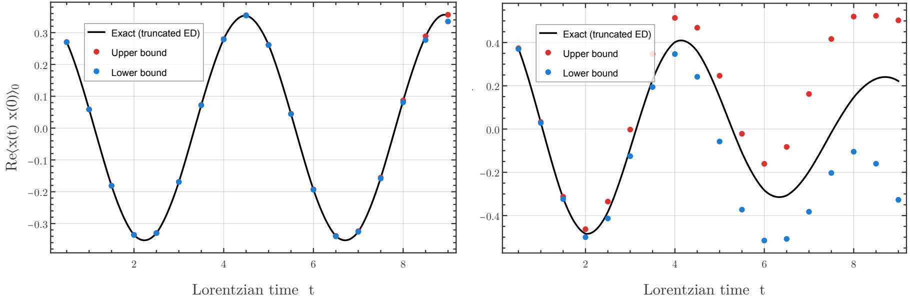
$$f'(t)^2 \le \frac{\langle \Pi^2\rangle}{f(0)}(f(0)^2 - f(t)^2) \implies f(t) \ge f(0)\cos\sqrt{\frac{\langle \Pi^2\rangle}{f(0)}}\,t$$

The SHM saturates this bound.

[1] Then $f'(t) = \mathrm{Im}\{\langle \Pi(t)x(0)\rangle\}$ and $f''(t) = \mathrm{Re}\{\langle \Pi(t)\Pi(0)\rangle\}$

Lorentzian ground/thermal 2pt - anharmonic oscillator

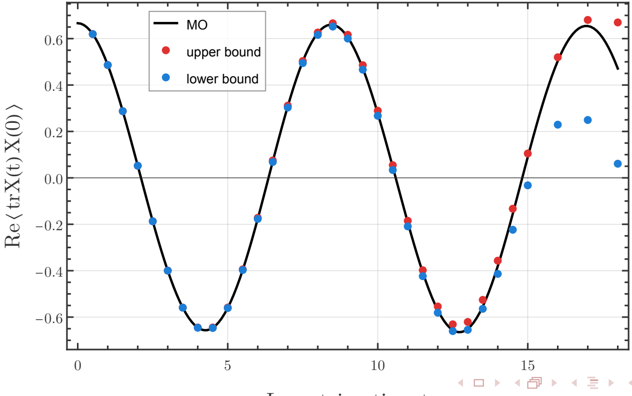
Benchmark: Lorentzian correlators of the anharmonic oscillator, compared with Hamiltonian truncation method.



Lorentzian ground state 2pt function - 1-MQM

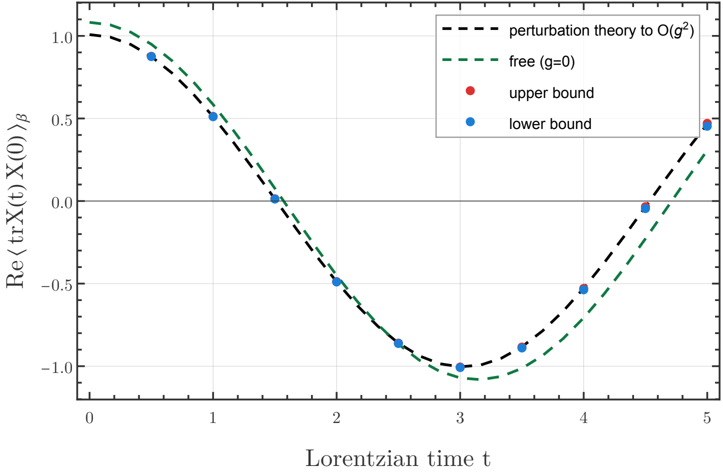
In 1-MQM, the ground state 2pt can be computed numerically from the adjoint spectrum by numerically solving the Marchesini-Onofri equation.

$$\langle \mathrm{tr}\, X(t)X(0)\rangle_{gs} = \sum_n |\langle 0|X_{ij}|n\rangle|^2 e^{-i\Delta_n t}$$



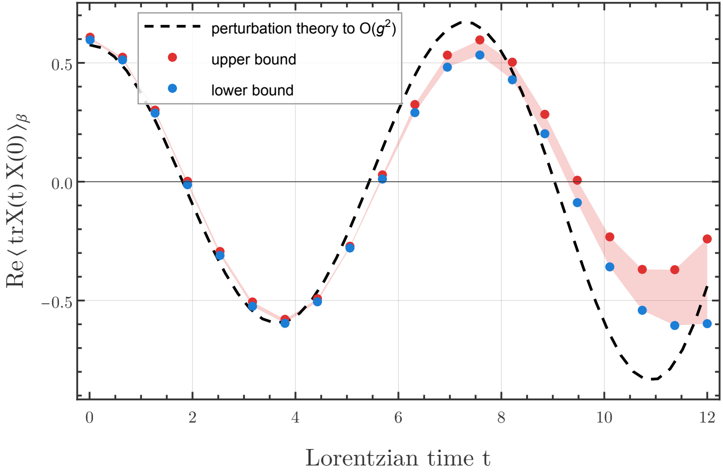
Lorentzian thermal 2pt - 1-MQM

Compare with perturbation theory at $g = 0.01$:



Lorentzian thermal 2pt - 1-MQM

Another example at $g = -0.06$ (recall that $g_c \approx -0.075026$) and $\beta = 20$:



Non-rigorous alternatives:

- Discretizing the primal problem
- Analytical continuation. The naive version of reconstructing Lorentzian correlators from sampled Euclidean data is ill-conditioned.

$$G(\tau) = \sum_n |\langle n|X|0\rangle|^2 e^{-(E_n - E_0)\tau} \to \sum_n |\langle n|X|0\rangle|^2 e^{-i(E_n - E_0)t}$$

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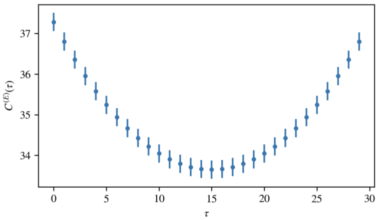
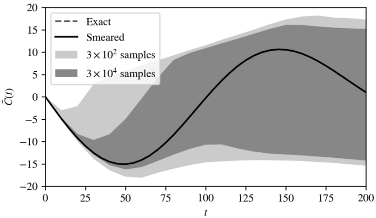
Rigorous alternatives:

- Methods based on the positivity of the spectral function $\rho(\omega)$ (Fourier transform of the real-time correlator) [Lawrence 2408.11766].



Sampled Euclidean data



Reconstructed (smeared) Lorentzian

Summary and Future directions

Summary:

- We propose a positivity framework for two-point function bootstrap in Euclidean and Lorentzian signatures.
- The method is applicable in generic quantum systems, such as large N matrix quantum mechanics, spin chains,...

Future directions:

- Multi-point correlators, OTOCs
- Multi-matrix quantum mechanics models
- Towards spectral function and QNMs