Bootstrapping the non-singlet sector

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This talk is based on:

2603.04522 **Klebanov, HL, Meshcheriakov**

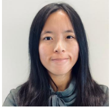
260X.XXXXX **Klebanov, HL, Meshcheriakov, Yeh**

260X.XXXXX **Dommes, HL, Simmons-Duffin, Zheng**

see also:

[Cho, Gabai, Sandor, Yin, 2410.04262]

[Cho, Gabai, HL, Yeh, Zheng, 2511.08560]



Matrix quantum mechanics

Today we will study two models in the planar limit:

- 1-matrix quantum mechanics ($c = 1$)

$$H = \frac{1}{2}\operatorname{Tr} P^2 + \frac{1}{2}\operatorname{Tr} X^2 + \frac{g}{4}\operatorname{Tr} X^4$$

- D0 brane quantum mechanics (BFSS)

$$H = \frac{1}{2}\operatorname{Tr}\left(P_I^2 - \frac{1}{2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right)$$

Both models are 0+1d, with matrix-valued fields that transform in the adjoint of $SU(N)$.

Non-singlet sector

Treat the SU(N) symmetry as a global symmetry $\Rightarrow$ states that transform non-trivially in SU(N).

Equivalently, insert Wilson lines in various representations in the gauged model.

$$U_R(t_2, t_1) = \mathcal{P} \exp \left(\mathrm{i} \int_{t_1}^{t_2} dt \, A_0^a(t) \, T_R^a \right) .$$

Like in AdS/CFT, these non-singlet states should contain open strings, with endpoints in "the UV."

We will study 2 aspects of these models

- Adjoint spectrum
- ungauged thermodynamics.

adjoint spectrum
1-matrix QM
Regge trajectories

adjoint spectrum
BFSS
spinning string

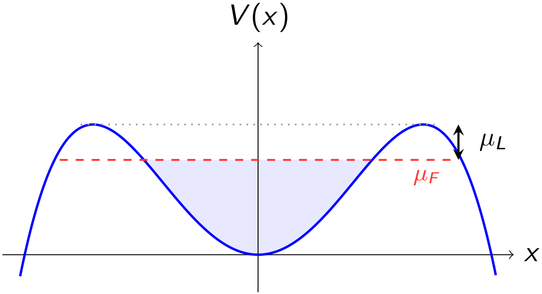
thermodynamics
1-matrix QM
BKT transition?

thermodynamics
BFSS
D0 black hole &
Maldacena-Milekhin

Adjoint spectrum in 1-matrix quantum mechanics

But first, review of singlet sector.

Potential near criticality

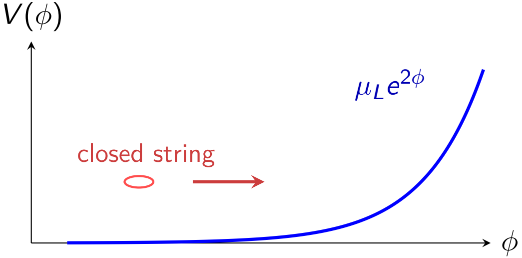


- Singlet sector reduces to N free fermions.
- As $g \to g_c$, $\mu_L = V_{\max} - \mu_F \to 0$.
- g_c is domain of analyticity; typical planar diagrams are huge $\Rightarrow$ closed strings.
- In Hamiltonian language, strings are ripples in the Fermi sea.

2D Liouville String

As $\mu_L \to 0$ singlet excitations are closed strings in a 2D target space:

$$I = \int \mathrm{d}^2\sigma \sqrt{\det g}\left[\frac{1}{4\pi} g^{ab}\left(\partial_a X^0 \partial_b X^0 + \partial_a \phi \partial_b \phi\right) + \frac{1}{4\pi} Q\phi R(g) - \mu_L e^{2\phi}\right]$$



The matrix bootstrap

1. Heisenberg EoM: $\langle [H, O] \rangle = 0$
2. Hilbert space positivity: $\mathcal{M}_{ij} = \langle E | \operatorname{tr} \mathcal{O}_i^\dagger \mathcal{O}_j | E \rangle\,, \quad \mathcal{M} \succeq 0$
3. Ground state positivity:

$$\mathcal{N}_{ij} = \langle \operatorname{tr} O_i^\dagger [H, O_j] \rangle_{\text{gs}}, \quad \mathcal{N} \succeq 0$$

4. Large N factorization $\langle \operatorname{tr} O_i \operatorname{tr} O_j \rangle = \langle \operatorname{tr} O_i \rangle \langle \operatorname{tr} O_j \rangle$.

[Anderson, Kruczenski; HL; Han, Hartnoll, Kruthoff; Kazakov, Zheng; ...]

How to estimate the adjoint gap

$$\begin{gathered}\max\ \mu, \quad \mathcal{N} \succeq \mu \mathcal{M} \\ \text{subject to } \mathcal{M} \succeq 0, \quad \mathcal{N} \succeq 0, \quad \text{EoM, etc}\end{gathered}$$

To explain this inequality, recall

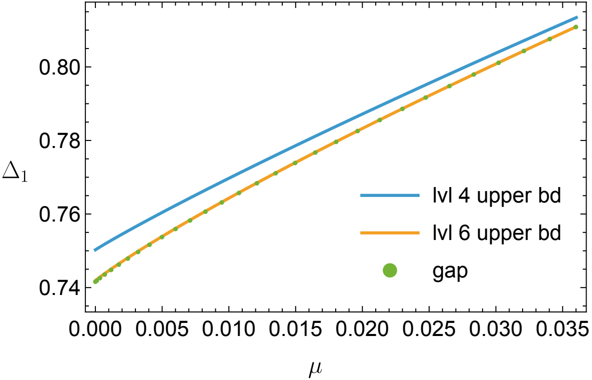
$$\begin{aligned}\mathcal{M}_{ij} &= \langle \mathrm{tr}\, O_i^\dagger O_j \rangle_{\mathrm{gs}} \\ \mathcal{N}_{ij} &= \langle \mathrm{tr}\, O_i^\dagger [H, O_j] \rangle_{\mathrm{gs}},\end{aligned}$$

Consider the diagonal:

$$\frac{\left\langle O^\dagger (H - E_0) O \right\rangle}{\langle O^\dagger O \rangle} \geq \mu$$

We recover the definition of the adjoint gap.

Bound on the adjoint gap at $g = g_c$



We may similarly bound Δ_2, since it is the lowest energy adjoint state in the $\mathbb{Z}_2$ even sector.

A surprise

	1-param fit	2-param fit	MO	lvl 16 bd
Δ_1	0.74436	0.74199	0.74158	0.741573662448591
Δ_2	1.30914	1.30906	1.26130	1.26122
Δ_3	1.71290	1.68859	1.68218	1.68192**

It was suggested [Gross, Klebanov '90] that the adjoint gap *diverges* as $\mu_L \to 0$.

Instead, precision numerics shows the gap **stays finite**: $\Delta_1 \approx 0.7416$.

Question: what is the limiting adjoint spectrum as $g \to g_c$? [Klebanov, HL, Meshcheriakov]

The Marchesini–Onofri equation

The Fermi density is

$$\rho(x) = \frac{1}{\pi}\sqrt{2(\mu_F - V(x))}.$$

The adjoint gaps are the eigenvalues of

$$\Delta_n\, \phi_n(x) = \underbrace{\eta(\mu_L, x)\, \phi_n(x)}_{\text{potential}} - \underbrace{\rho(x) \;\fint_{x_1}^{x_2} \mathrm{d}y\, \frac{\phi_n(y)}{(x-y)^2}}_{\text{“kinetic”}}.$$

where the MO potential is

$$\eta(\mu_L, x) \;\equiv\; \fint_{x_1}^{x_2} \mathrm{d}y\, \frac{\rho(y)}{(x-y)^2}$$

MO equation at criticality

Change to the time-of-flight variable τ:

$$\mathrm{d}\tau = \frac{\mathrm{d}x}{\dot{x}}, \qquad \dot{x} = \pi\rho(x),$$

At $g = g_c$,

$$x(\tau) = \frac{1}{2\sqrt{|g_c|}} \tanh \frac{\tau}{\sqrt{2}},$$
$$\frac{\pi\rho(x)\rho(x')}{(x - x')^2} = \frac{1}{2\pi \sinh^2\left(\frac{\tau-\tau'}{\sqrt{2}}\right)} \equiv K(\tau - \tau').$$

So the MO Hamiltonian *separates*:

$$\mathcal{H}_{\text{quartic}}(p, \tau) = p \coth \frac{\pi p}{\sqrt{2}} + \frac{2}{\pi}\left(\tau \tanh \frac{\tau}{\sqrt{2}} - \sqrt{2}\right)$$

For sufficiently excited states:

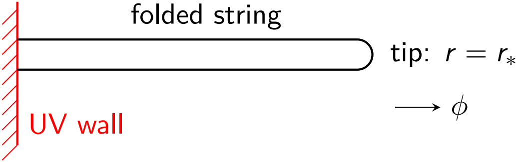
$$\mathcal{H}_{\text{quartic}}(p, \tau) \approx |p| + \frac{2}{\pi}\left(|\tau| - \sqrt{2}\right)$$

massless particle in a **linear potential**.

Folded strings in 2D string theory

2D string theory: target space has time X^0 and Liouville direction ϕ.

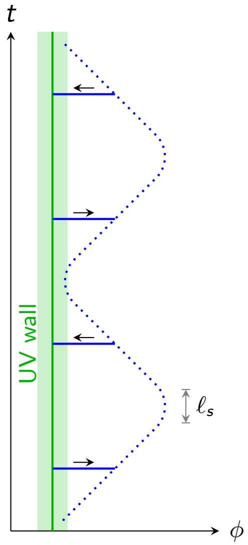
A folded open string extends from a "UV wall" at $\phi = 0$ into the bulk. The fold acts as a massless particle. [Bardeen, Bars, Hanson, Peccei '76; Maldacena '05]



String tension $= \frac{1}{2\pi\alpha'}$. Two segments of length $|\phi|$ give:

$$E = |p_\phi| + \frac{1}{\pi\alpha'}|\phi| + \text{const}$$

Oscillating folded string

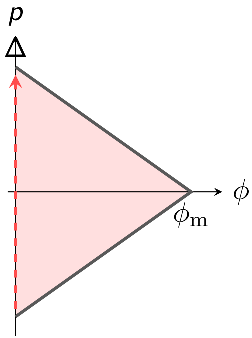


Bohr–Sommerfeld quantization

$$\mathcal{H} \approx |p| + \frac{|\phi|}{\pi\alpha'}$$

Phase space orbit:

- turning point at $\phi_{\rm m} = \pi\alpha'\Delta$
- maximum momentum $p_{\rm max} = \Delta$



Phase space integral:

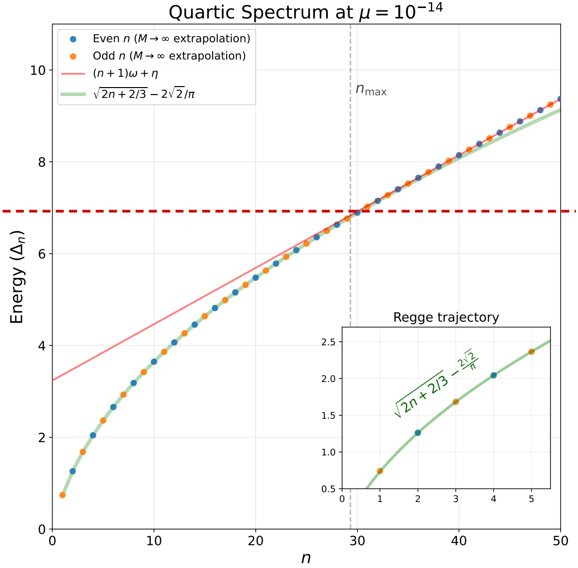
$$\underbrace{\phi_{\rm m} \cdot \Delta}_{\oint p d\phi} = \pi\alpha'\Delta^2 = 2\pi n \qquad \Longrightarrow \qquad \boxed{\Delta^2 = \frac{2n}{\alpha'}}$$

Regge trajectories

A slightly more careful treatment gives

$$\boxed{\Delta_n^{\mathrm{Regge}} \approx \sqrt{2n + \frac{2}{3}} - \frac{2\sqrt{2}}{\pi}}$$

In principle $n \gg 1$, but numerics shows that this formula works well for small n.



Smooth interpolation between Regge ($n \lesssim n_{\max}$) and WKB ($n \gtrsim n_{\max}$) regimes.

The Liouville wall and n_{max}

The Liouville potential becomes appreciable $\mu_L\, e^{2\phi/\sqrt{\alpha'}} \sim 1$, or

$$\phi_{\mathsf{max}} \;\sim\; \frac{\sqrt{\alpha'}}{2} \log \frac{1}{\mu_L}$$

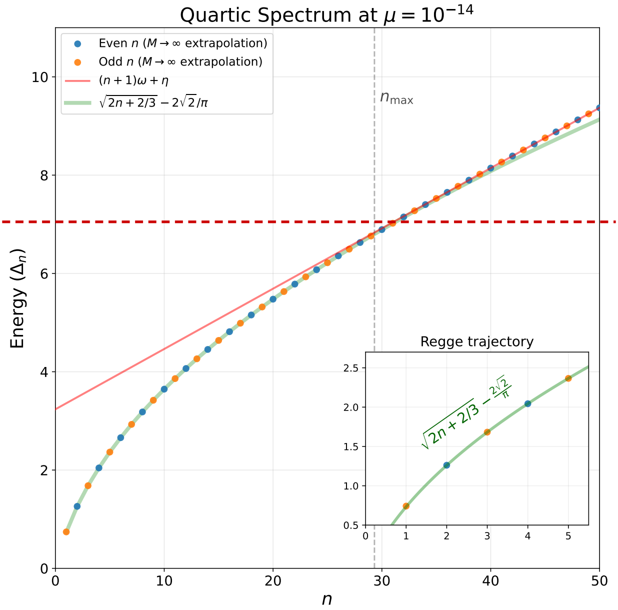
A fold with energy Δ_n reaches

$$\phi_{\rm tip}(n) = \pi\alpha'\,\Delta_n \sim \pi\sqrt{2n\alpha'}$$

Crossover: $\phi_{\rm tip}(n_{\mathsf{max}}) = \phi_{\mathsf{max}}$,

$$n_{\mathsf{max}} \sim \frac{\log^2(1/\mu_L)}{8\pi^2}$$

$n \gg n_{\mathsf{max}}$ *long strings* [Maldacena; Balthazar, Rodriguez, Yin $\cdots$]



Regge ($n \lesssim n_{\max}$) to long strings ($n \gtrsim n_{\max}$) regimes.

A factor of two

D.J. Gross, I.R. Klebanov / $c = 1$ matrix model 473

find that the vortex–antivortex correction is $\sim \mu^{2R}$. Thus, for complete agreement with the continuum methods, our estimate of the adjoint energy gap, eq. (3.18), has to be multiplied by 2. Apart from this minor discrepancy in the transition radius, our matrix model results are in accord with the continuum methods.

Matrix bootstrap helped us find the factor of 2 from 36 years ago!

Adjoint spectrum in BFSS

For SU(N) singlets, the BFSS model is famously believed to be gapless.

What about in the adjoint sector? [Maldacena & Milekhin] conjectured that there is an $\sim O(1)$ gap in 't Hooft units.

Some evidence from Monte Carlo [Berkowitz, Hanada, Rinaldi, Vranas '18; Pateloudis et al '22].

Supercharge bootstrap

For BFSS, one can bootstrap the singlet ground state using the supercharge equations of motion $\langle \{Q_\alpha, O_\alpha\} \rangle = 0$. [HL & Zheng].

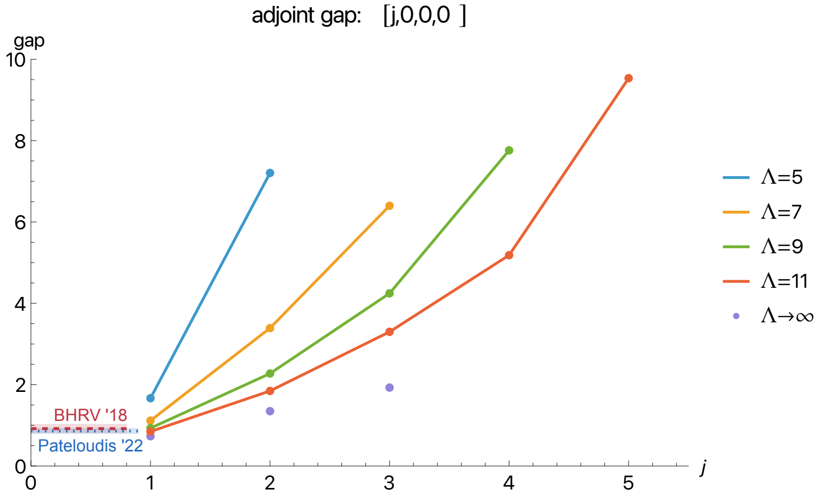
Lots of improvements to the BFSS bootstrap [wip w/ Dommes, HL, DSD, Zheng].

We compute bounds on the adjoint spectrum in various SO(9) sectors.

$$\mathcal{N} - \mu \mathcal{M} \succeq 0$$

This involves scanning over $\langle \mathrm{tr}\, X^2 \rangle$ and μ and then using non-linear relaxation over other vars [Kazakov & Zheng, HL & Zheng].

Adjoint gap in BFSS (preliminary)



Spinning string

$$\mathrm{d}s^2 = -\frac{r^{7/2}}{\sqrt{\lambda d_0}}\mathrm{d}t^2 + \frac{\sqrt{\lambda d_0}}{r^{7/2}}\mathrm{d}r^2 + \frac{\sqrt{\lambda d_0}}{r^{3/2}}\mathrm{d}\Omega_8^2.$$

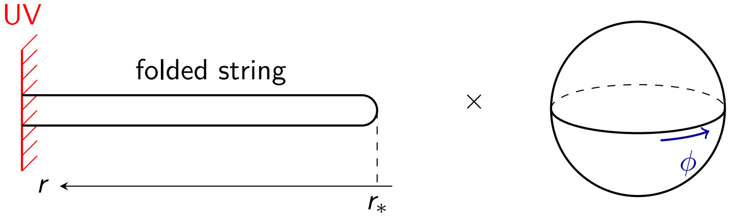
A string propagating on this background is governed by the Nambu–Goto action

$$S_{\mathrm{NG}} = -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{-\det h_{ab}}.$$

A solution is

$$t = \tau, \qquad r = \sigma, \qquad \theta = \frac{\pi}{2}, \qquad \varphi = \Phi(\tau, r).$$

Folded string solution



The tip $r_* \propto \lambda^{1/3} J^{-2/3}$. The solution is reliable in the **large charge limit** when $J \gg 1$.

The string extends deep into the bulk when $J \gg 1$. Prediction:
$E - E_\infty = 2.55388 \lambda^{1/3} J^{-2/3}$

Angular momentum barrier?

We can understand this intuitively by modeling the tip of the string as a massless particle in a confining potential:

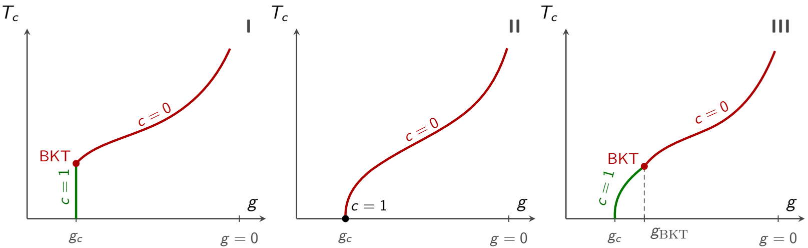
$$E = \sqrt{\dot{r}^2 + \underbrace{\frac{J^2 r^5}{\lambda d_0}}_{\text{angular momentum barrier}}} - \underbrace{\frac{r}{\pi}}_{\text{string tension}}$$

The "angular momentum barrier" is attractive in this background, because the S_8 shrinks.

Minimizing this effective potential gives $E \propto r_* \propto J^{-2/3}$.

Thermodynamics

What is the phase diagram of the ungauged MQM in (g, T) in the infinite N 't Hooft limit?


T_c
I
BKT
$c = 0$
$c = 1$
g
g_c
$g = 0$
T_c
II
$c = 0$
$c = 1$
g
g_c
$g = 0$
T_c
III
BKT
$c = 0$
$c = 1$
g
g_c
g_{BKT}
$g = 0$

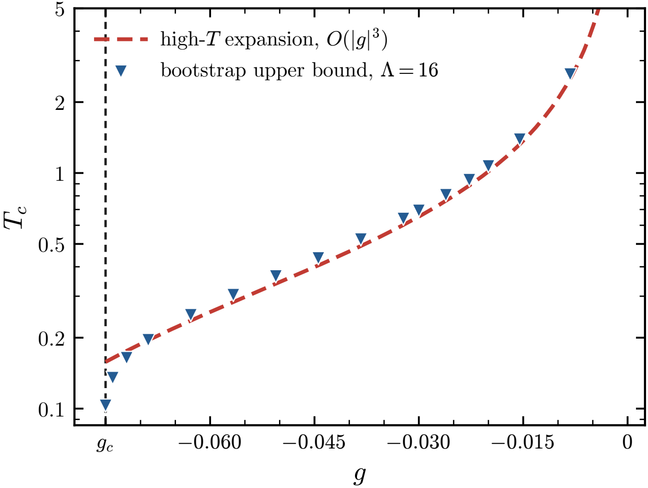
High temperature expansion

Small $\beta \Rightarrow$ Matsubara non-zero models are massive; integrate them out $\Rightarrow$ 0d matrix integral. From [BIPZ, Klebanov & Hashimoto], solve this matrix integral.

This predicts $T_c = \frac{1}{48g} + \cdots$. Reliable prediction at $g \ll 1$.

The continuum theory on this curve is the $c = 0$ minimal string.

Bootstrapper vs perturbationist



Consistent with thermal bootstrap [Fawzi, Fawzi, Scalet; CGSY '24].

But what about strong coupling $g \approx g_c$??

Low temperature expansion

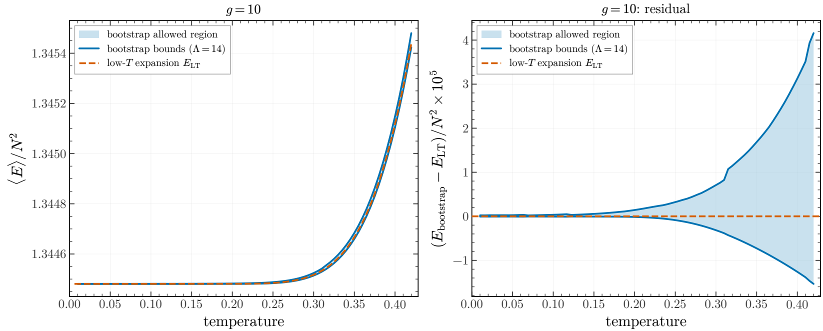
E vs T. Uses the adjoint spectrum $\{\Delta_n\}$ (and at higher orders, eigenfunctions) as input.

$$E(\beta)/N^2 = E_0/N^2 + \Delta_1 e^{-\beta\Delta_1} + \Delta_2 e^{-\beta\Delta_2} + \cdots$$

$$\langle C_2\rangle(\beta)/N^3 = e^{-\beta\Delta_1} + e^{-\beta\Delta_2} + \cdots$$

[CGSY '25 developed the low T expansion; we improved it.]

Low-temperature expansion at strong coupling



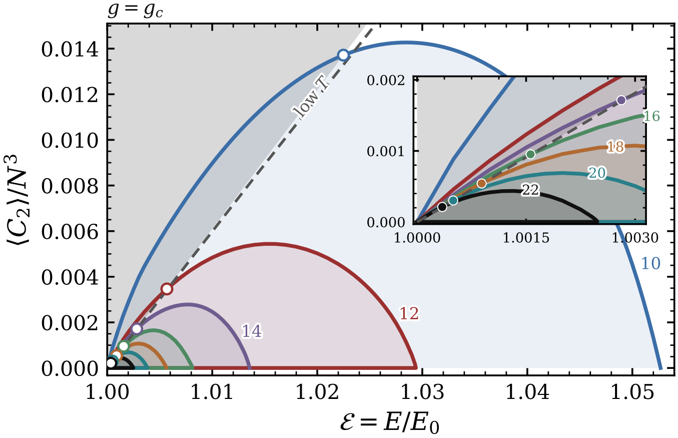
Used solution to MO equation Δ_1, Δ_2 as input.

Low temperature expansion

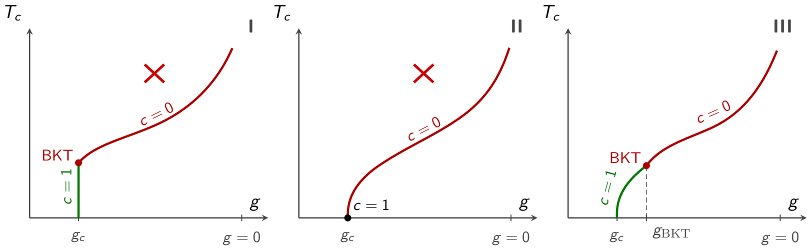
Another approach to compute T_c near $g = g_c$: assume we are in the low T regime and compute $E(T)$ or $\langle C_2(T) \rangle$.

microcanonical bootstrap to find the allowed region $(E, \langle C_2 \rangle)$.

Microcanonical bootstrap



Lack of an appreciable "overhang" at high levels
$\Rightarrow$ tension with (I).



Current (preliminary) evidence disfavors I and II.

(II) unlikely because the Villain model on a fixed lattice has a BKT transition at finite $1/R > 0$.

Discussion and future directions

- Despite being ~ 50 years old, precision numerics from the bootstrap led to new results about the $c = 1$ matrix model.
- Relation to the [Kazakov Kostov Kutasov] Euclidean black hole?
- Thermodynamics of BFSS & the D0 black hole?
- Other matrix models
- Real time physics? [Jessica's talk]

Thanks!