Towards a Quantum Lattice Bootstrap

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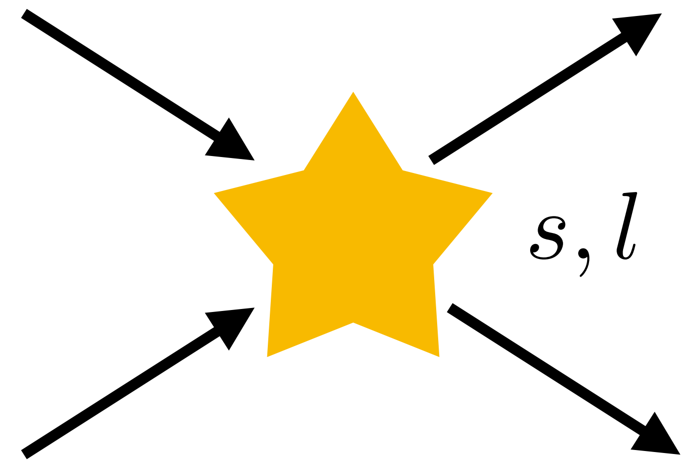
part 0: introduction.

S-matrix inspiration

- The S-matrix
  ⇒ partial wave amplitudes: $T_l(s)$



(1) Unitary ⇒ $0 \leq \mathrm{Im}\, T_l(s) \leq 2$ for s real.

(2) Relativistic causality ⇒ analyticity in s over some domain in $\mathbb{C}$.

- Logic of the S-matrix bootstrap. Suppose you know:
  - the high energy, large s, behaviour
  - the low energy, small s, effective field theory

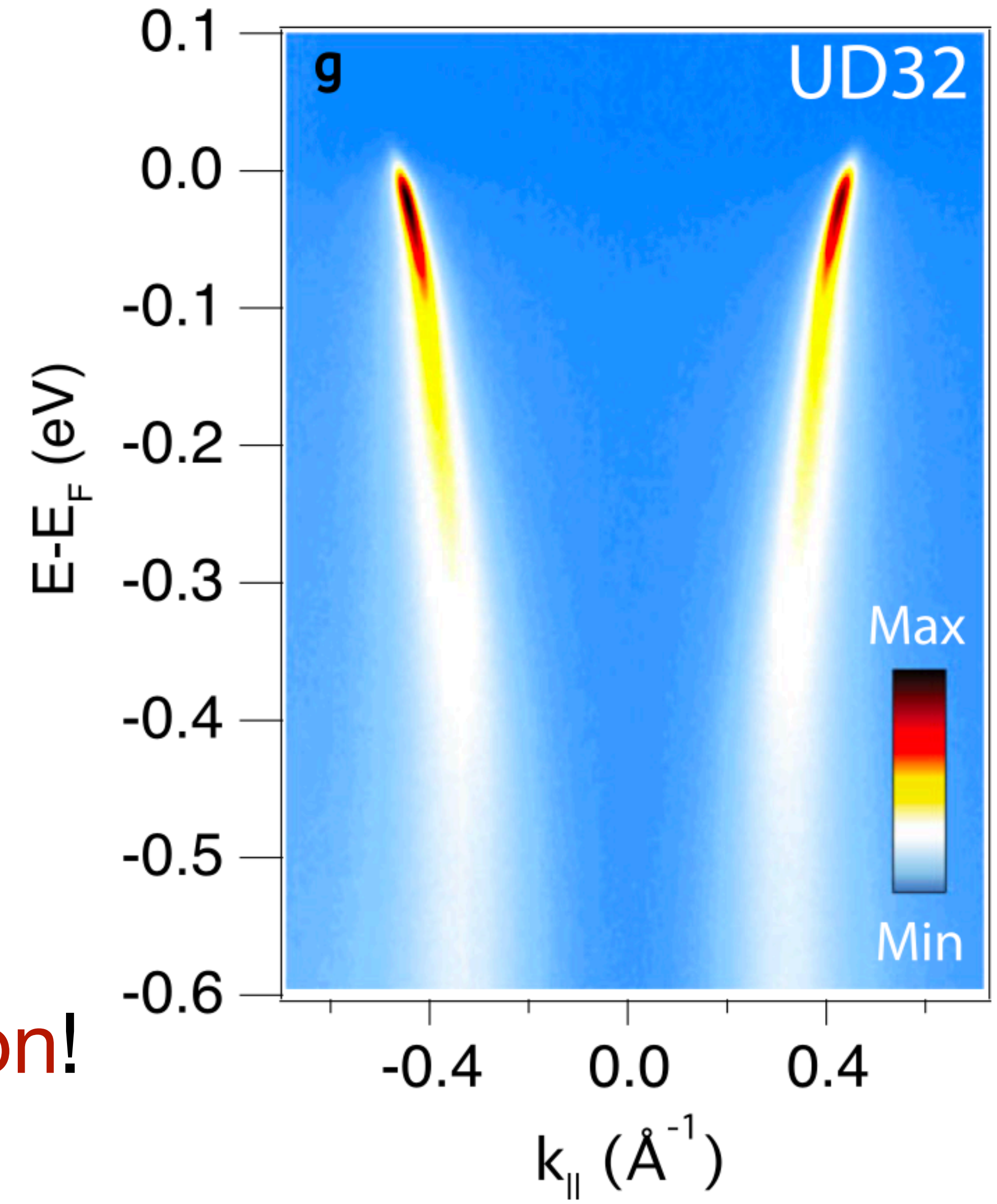
  Then use (1) and (2) to constrain parameters in the effective theory in terms of high energy data.

Green's functions

- The retarded Green's function

$$G^R(\omega, \mathbf{k}) \equiv -i \sum_{\alpha} \int_0^{\infty} dt\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}_\alpha} \langle [\mathcal{O}_\alpha^\dagger(t), \mathcal{O}_0(0)] \rangle$$

  of a local, bounded lattice model.



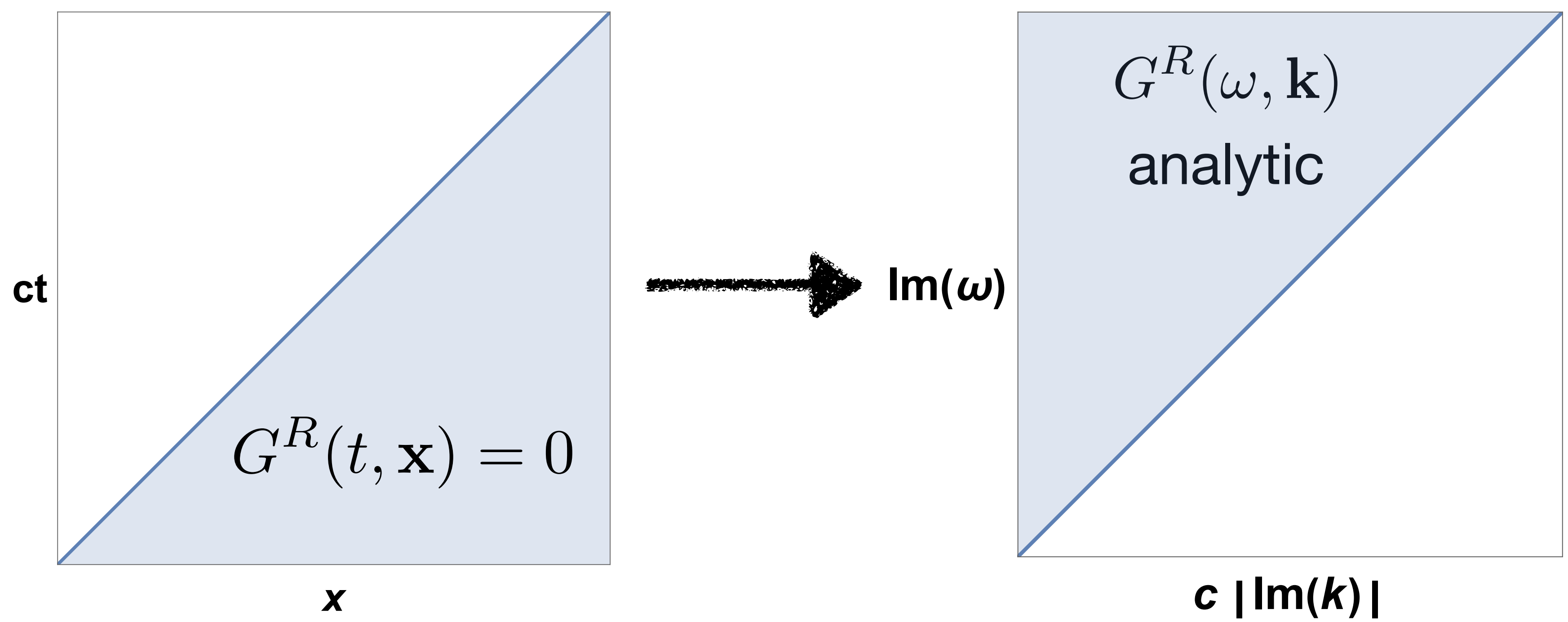
- A fundamental object in quantum many-body physics. Accessible in detail experimentally but often challenging to understand even though UV known.

- Today: bootstrap the Green's function!

part I: causality.

Relativistic inspiration

- In relativistic theories real space/time Green's functions vanish outside the lightcone: $G^R(t, \mathbf{x}) = 0 \;\text{ for }\; ct < |\mathbf{x}|$.
- The Fourier transform of this statement is that $G^R(\omega, \mathbf{k})$ is analytic for $\mathrm{Im}\,\omega > c|\mathrm{Im}\,\mathbf{k}|$.



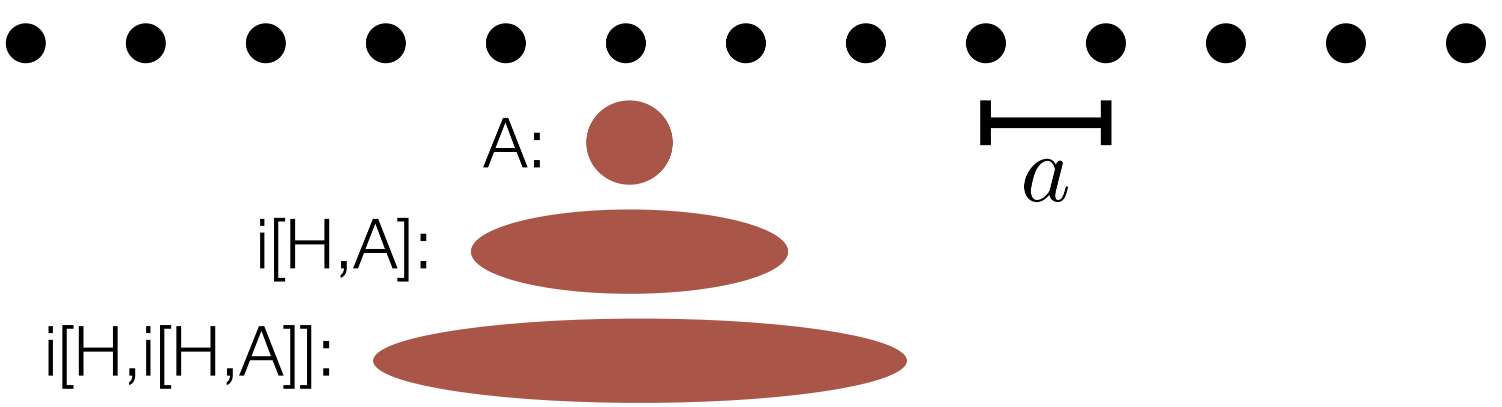
- Used to bound transport in relativistic plasmas!

[Heller et al. PRL 23 & Nat Phys 24]

Lieb-Robinson bound

- The Lieb-Robinson bound follows from the way operators grow under time evolution by a local Hamiltonian.

$$H = \sum_{\alpha} H_{\alpha}$$



- The Lieb-Robinson velocity: $v \propto a||H_{\alpha}||$

Lieb-Robinson bound

- The correlation function is bounded by the operator norm:

$$|\langle[\mathcal{O}_\alpha^\dagger(t), \mathcal{O}_0(0)]\rangle| \leq ||[\mathcal{O}_\alpha^\dagger(t), \mathcal{O}_0(0)]|| \equiv C(t, \mathbf{x}_\alpha)$$

- Which is in turn bounded (e.g. review by Chen-Lucas-Yin)

$$C(t, \mathbf{x}) < ||\mathcal{O}||^2 \begin{cases} K \left(\frac{vt}{x}\right)^{x/a} & (x > vt) \\ 2 & (x < vt) \end{cases}$$

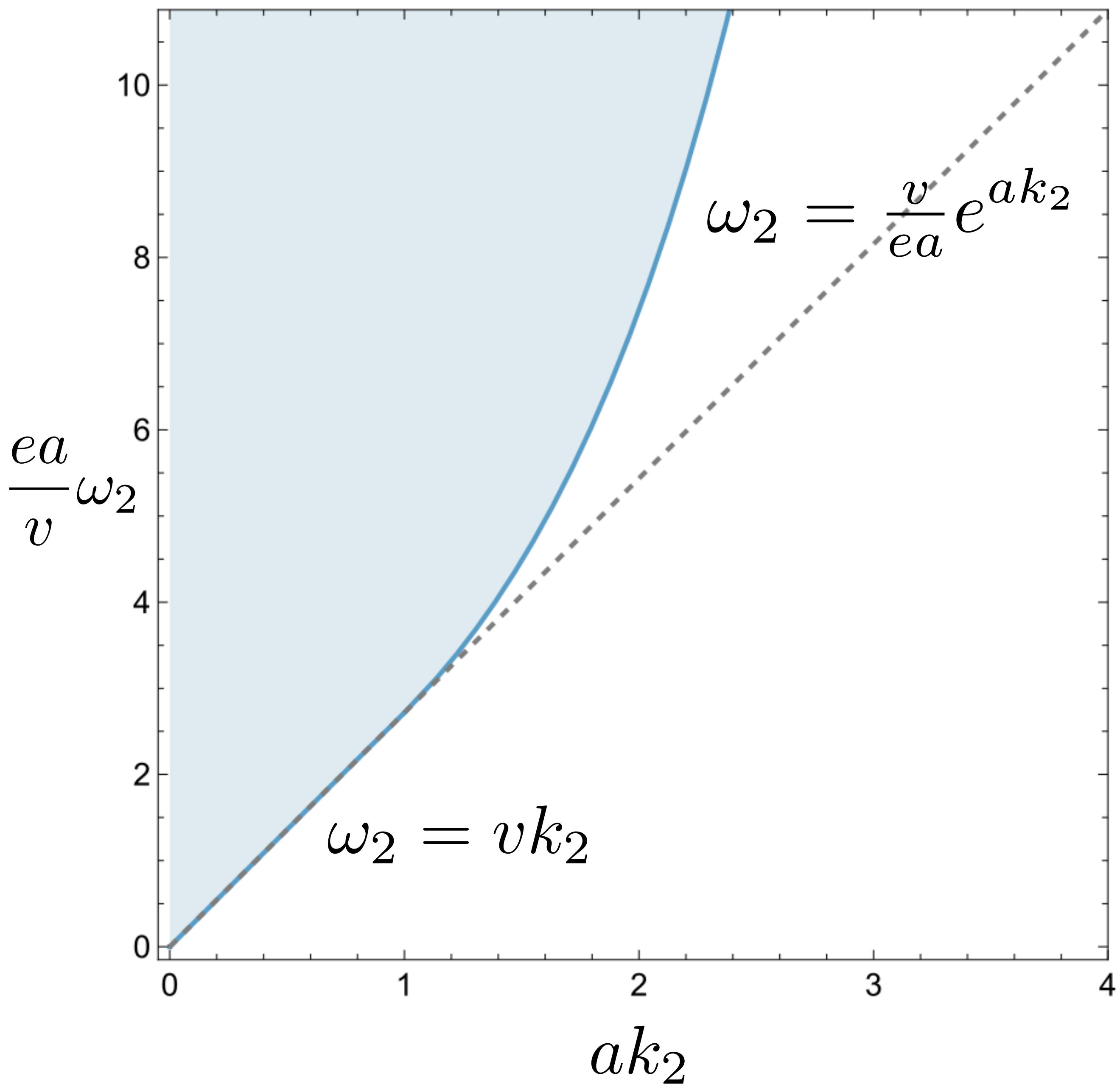
- That is, correlations decay exponentially outside the Lieb-Robinson light cone.

- a and v are the microscopic data.

Analyticity

- Consider $\omega = \omega_1 + i\omega_2\,, \quad \mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$
- The imaginary parts lead to exponential growth/decay in the integrand for G^R.
- The integral will converge when any exponential growth is overcome by exponential Lieb-Robinson decay.
- Standard arguments show that for regions of ω,k where the integral converges, it is in fact analytic.
- It is easy to determine the domain of analyticity.

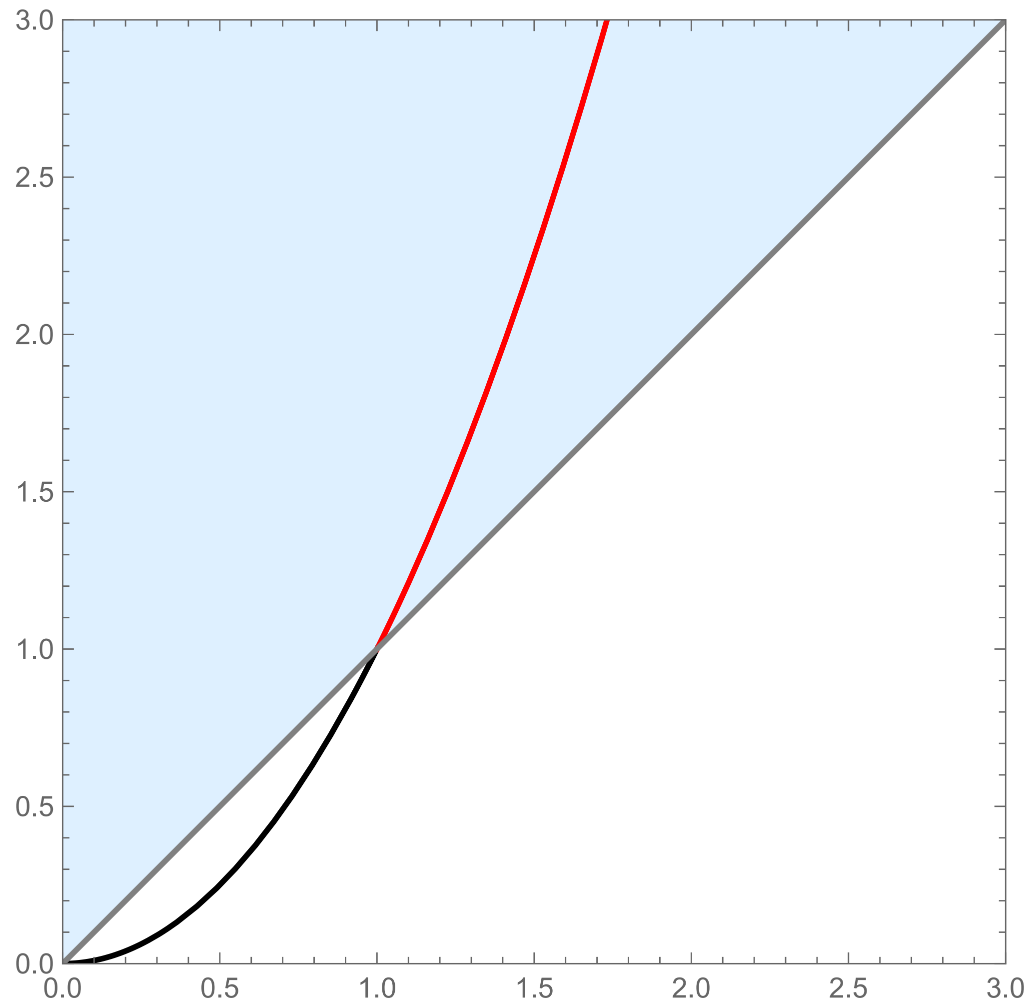
Analyticity



- Lieb-Robinson is fuzzy in real space but sharp in momentum space!
- (i.e. Fourier transform of exponentially decaying function has strong analyticity properties).
- No Brillouin zone for k_2.
- For low energy EFTs: as good as a relativistic lightcone!

Case I: Diffusion bound

- If the mode is diffusive at long wavelengths, the retarded Green's function has a pole at: $\omega = -iDk^2$
- For purely imaginary momenta and frequencies: $\omega_2 = Dk_2^2$



- The pole is not allowed to enter the shaded region.
- This region must be beyond the diffusion timescale τ. Requires:

$$D \leq v\tau^2$$

(Sharp version of SAH-Hartman-Mahajan '17)

Case II: non-Fermi liquids

- Suppose the low energy description of a system with a Fermi surface is a Hertz liquid:

$$G^R(\omega, k) = \frac{\mathcal{A}}{\omega - v_F^\star k_\perp - \Sigma(\omega)} \qquad \Sigma(\omega) = -i\lambda \frac{(-i\omega)^\alpha}{\Lambda^{\alpha-1}}$$

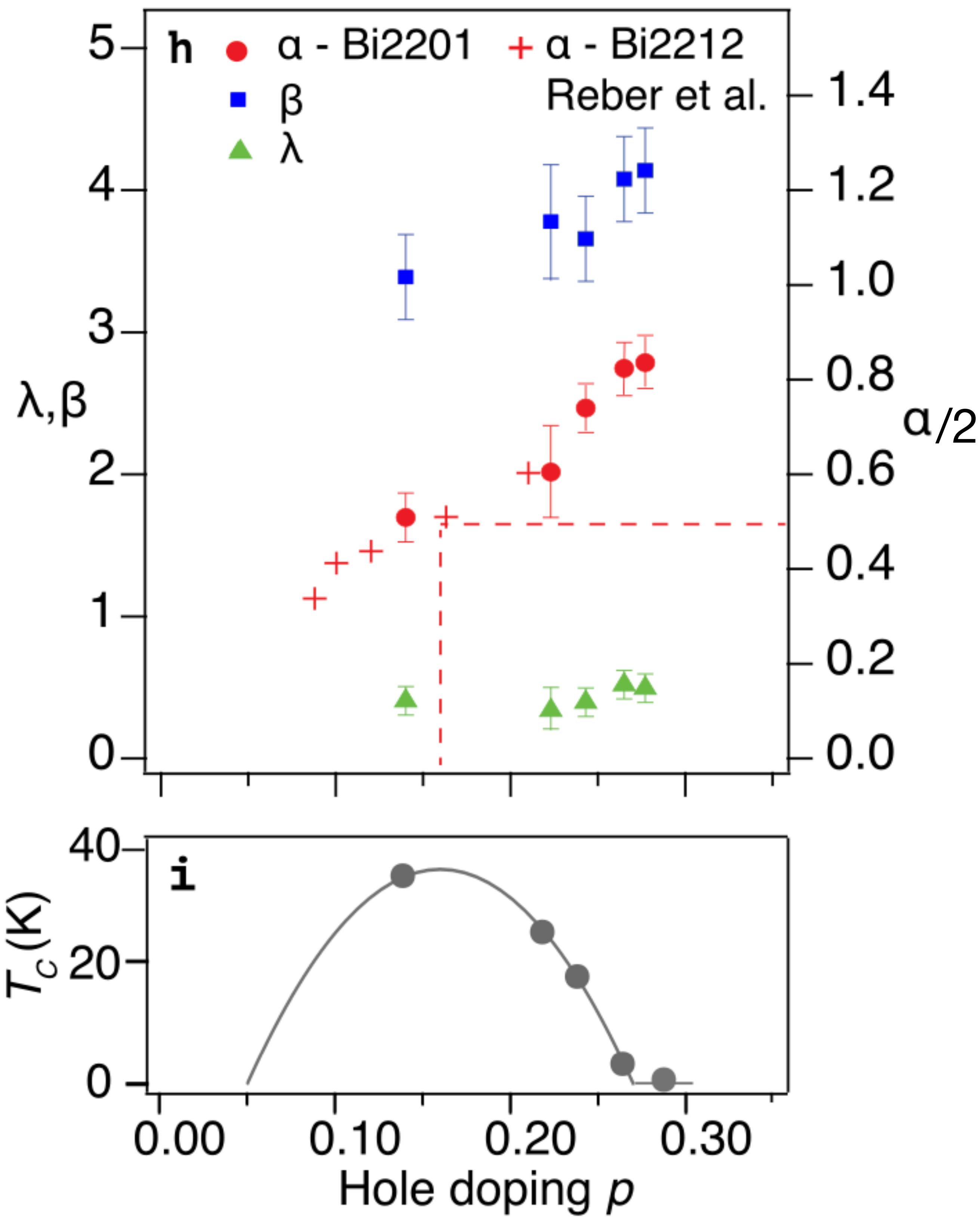
- Here λ is a dimensionless coupling.

- $\alpha < 1$: quasiparticle is totally destroyed. All dispersion is very slow and causality doesn't seem to constrain much.

- For $1 < \alpha < 2$ the quasiparticle lifetime is shorter than in a Fermi liquid. Follow the pole at imaginary momenta …

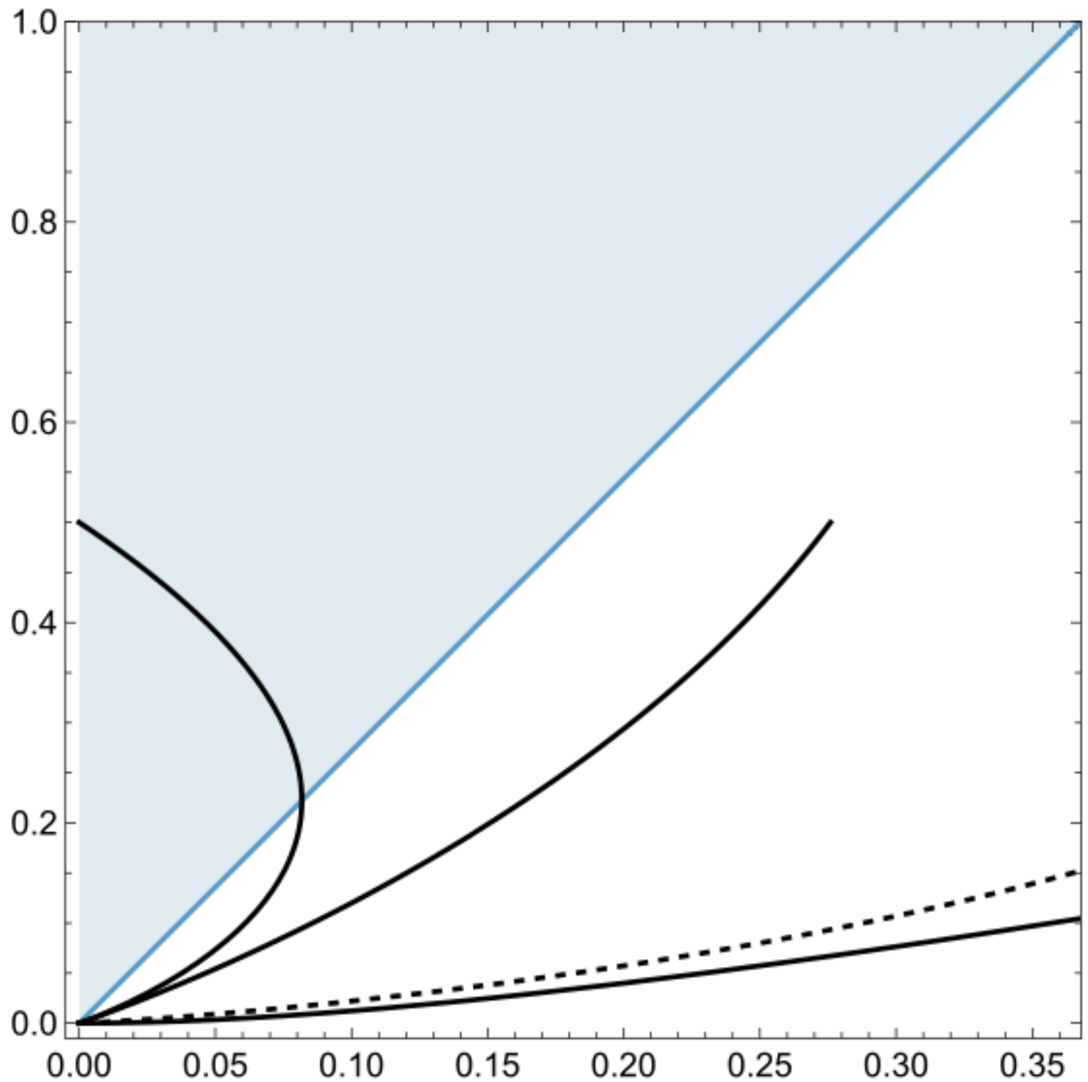
$$\omega_2 - v_F^\star k_2 - \lambda \frac{\omega_2^\alpha}{\Lambda^{\alpha-1}} = 0$$

Hertz liquid fits



- Nodal ARPES data in cuprates fit to this form in [Reber et al. Nat Comm 19, Smit et al. Nat Comm 24].
- $1 < \alpha < 2$ in overdoped region.
- Coupling λ remains bounded as MFL is approached at optimal doping.
- (Differs from our coupling by a factor of Σ' so can't compare quantitatively).

Case II: non-Fermi liquids



- Clearly one must have $v_F^\star < v$ for the pole to start in the allowed region.
- For 1 < α < 2, for the pole remains in the allowed region for all ω₂ < Λ if:

$$|\lambda| < 1 - \frac{v_F^\star}{v} \approx 1$$

- I.e. Lieb-Robinson causality sharply bounds the dimensionless coupling of a Hertz liquid.

part II: spectral weight.

Spectral weight upper bound

- Using similar steps to [Abanin et al. PRL '15], a local, bounded Hamiltonian implies

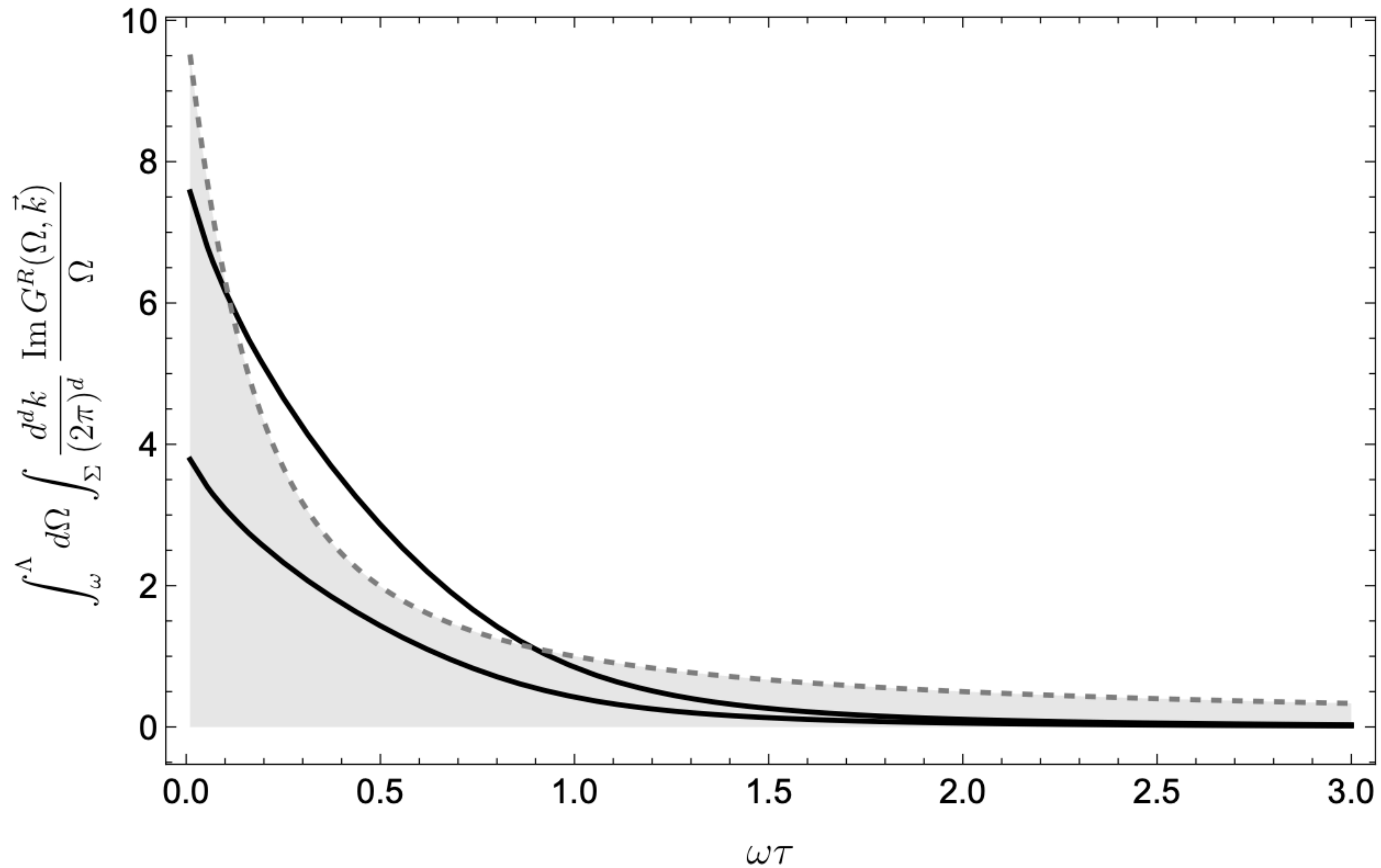
$$\int_{\omega}^{\Lambda} \frac{d\Omega}{\pi} \int_{\Sigma} \frac{d^d k}{(2\pi)^d} \frac{\operatorname{Im} G^R(\Omega, \vec{k})}{\Omega} \leq \frac{1}{\omega n_B(\omega)} \frac{||n_0||^2}{\nu^2} \times \begin{cases} 1 & \text{for } \frac{\omega}{\varepsilon} \leq 1 \\ \frac{\omega}{\varepsilon} e^{2(1-\omega/\varepsilon)} & \text{for } \frac{\omega}{\varepsilon} \geq 1 \end{cases}$$

- For example, suppose that scales below Λ are described by the Drude-Kadanoff-Martin dispersion:

$$\frac{\operatorname{Im} G^R(\Omega, \vec{k})}{\Omega} = \frac{\chi D k^2}{\Omega^2 + (Dk^2 - \tau\Omega^2)^2}$$

(It is immediate from the exponential bound that this power law behaviour cannot persist to microscopic scales)

The Mott-Ioffe-Regal bound



- We showed that the model is only consistent with the upper bound when the collective mean free path $\ell \equiv \sqrt{D\tau}$ obeys: $\boxed{a \lesssim \ell}$ (A precise inequality, but depends on parameter regime)

Conclusions

- Lieb-Robinson causality implies sharply defined domains of analyticity on quantum lattice models.
- Analyticity can constrain low energy dynamics.
- Lattice Green's functions are subject to upper bounds that also constrain low energy dynamics.
- A full "quantum lattice bootstrap" would tie these features together systematically, as is done in the S-matrix bootstrap. Potential for many applications.