Accurate boundary bootstrap for the 3d normal universality class

Wenliang Li

Sun Yat-sen University

Bootstrap 2026, London

Runzhe Hu & WL, 2508.20854
WL & Yuheng Yao, 2608.?????

Outline

- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

Outline

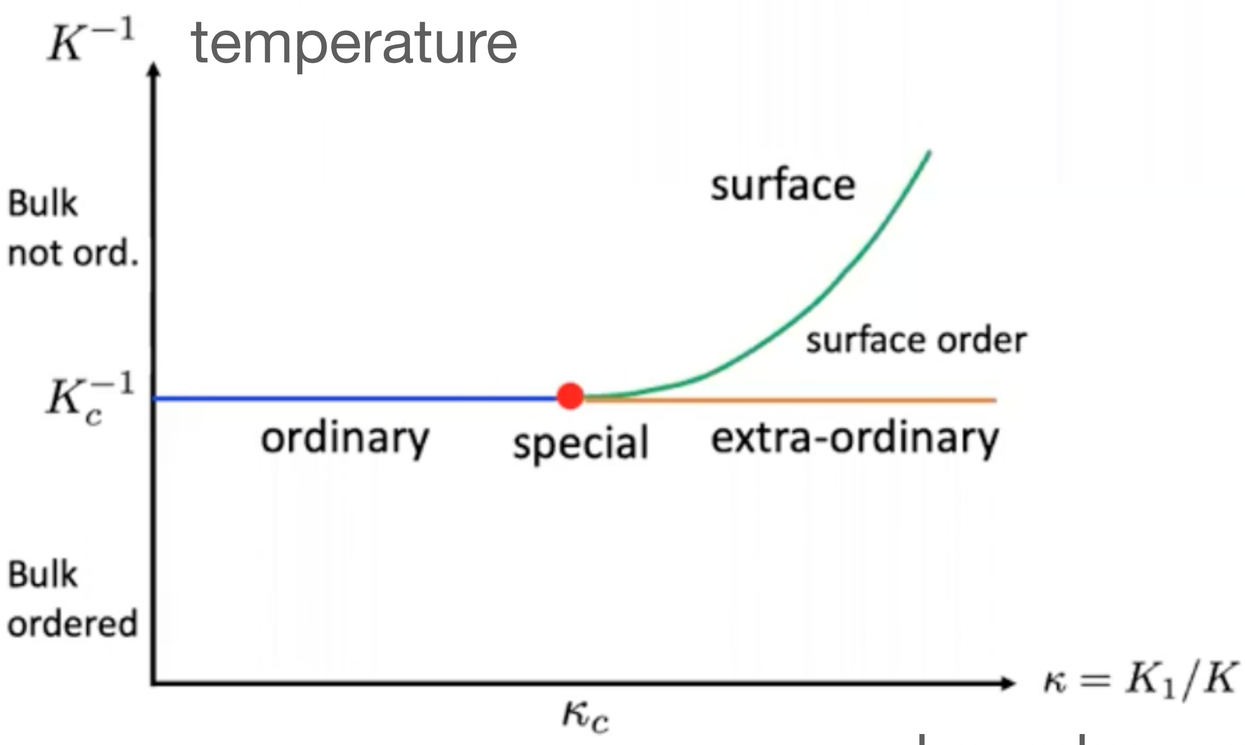
- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

d>3 boundary phase diagram

Classical O(N) model with a boundary

$$\frac{H}{k_B T} = -\sum_{\langle ij \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j$$

$$K_{ij} = \begin{cases} K_1 & i,j \in \text{boundary} \\ K & \text{otherwise} \end{cases}$$



Max A. Metlitski, 2009.05119
Bootstrap Zoom 21

d=3 boundary phase diagram

- N=1: Ising model
  the same phase diagram as d>3

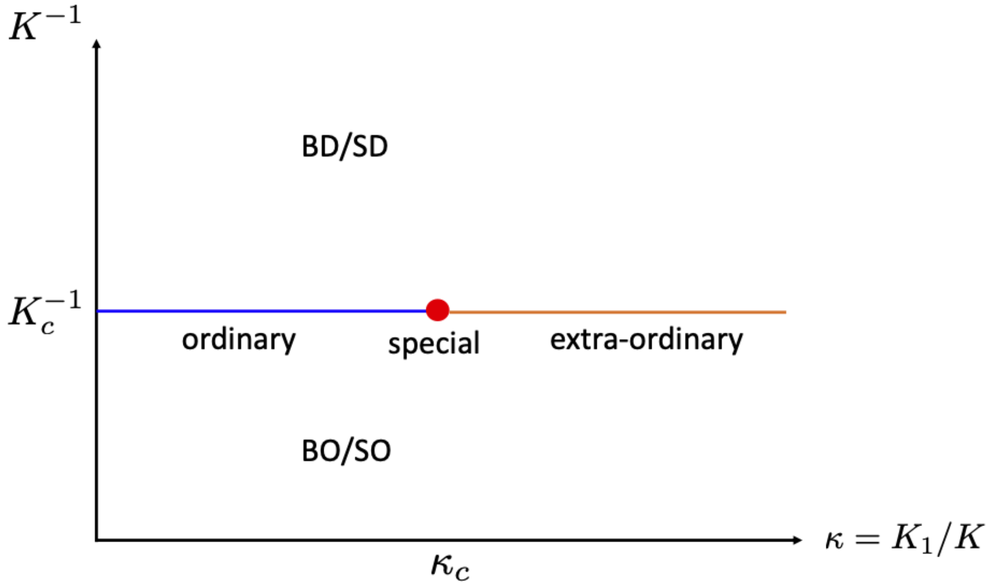
boundary is (3-1)=2d

- N=2: XY model
  the same topology as d>3
  but only **quasi**-long-range boundary order (BKT transition)

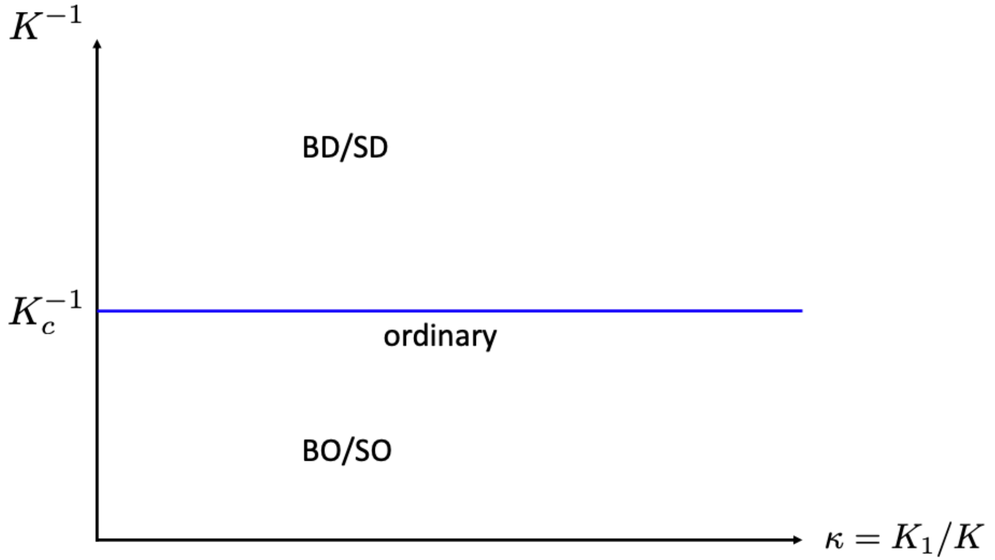
- Large N
  only ordinary transition, no special/extraordinary transition

- Mermin-Wagner theorem
  no spontaneous breaking of continuous symmetries for short-range interactions in two dimensions

d=3 boundary phase diagram



(a)



(b)

2 < N < Nc

extraordinary-log

Large N

From “normal” to “extraordinary”

- Normal boundary transition
  explicit symmetry breaking field on the boundary
  O(N) -> O(N-1)

$$\beta H = -\sum_{\langle i,j \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j - \sum_{i \in \text{boundary}} \vec{h}_1 \cdot \vec{S}_i,$$

- d>3: extraordinary = normal
  Goldstone modes decouple from the bulk with a normal boundary

Bray, Moore, 1977; Diehl, 1994

- d=3: extraordinary-log

$$S_{IR} = S_{normal} + S_n - s \int d^{d-1}\mathrm{x}\, \pi_i(\mathrm{x}) \hat{\phi}_i(\mathrm{x}) + \delta S,$$

  logarithmic RG flow around the “normal” fixed point
  the coupling s is fixed by restoring O(N) symmetry

$$\phi_i(\mathrm{x}, x_d) \sim \mu_\phi \, (2x_d)^{d-1-\Delta_\phi} \hat{\phi}_i(\mathrm{x}) + \dots$$

$$\vec{n} = (\vec{\pi}, \sqrt{1 - \vec{\pi}^2})$$

Metlitski, 2020; Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021

From “normal” to “extraordinary”

- Logarithmic RG flow around the “normal” fixed point (g=0)
- β function $\beta(g) = \alpha g^2 + O(g^3)$

  nonlinear sigma model $S_n = \int d^{d-1}\mathrm{x} \left(\frac{1}{2g} (\partial_\mu \vec{n})^2 - \vec{h} \cdot \vec{n} \right), \quad \vec{n}^2 = 1 \,.$
- What is **Nc**?
  fixed point stability depends on the sign of $\alpha = \frac{1}{32\pi} \frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$ **Bootstrap target**
- Boundary correlation function

$$\langle \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} \rangle \sim \frac{1}{(\log |\vec{x} - \vec{y}|)^q} \qquad q = \frac{N-1}{2\pi\alpha}$$

Outline

- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

Conformal field theory

- In the RG parameter

$$\alpha = \frac{1}{32\pi}\frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$$

Bootstrap target

$(a_\phi, b_{\phi t})$ are **universal amplitudes** in the normal transition

- In BCFT, $(a_\phi, b_{\phi t})$ are **B**oundary **O**perator **E**xpansion (BOE) coefficients.

$$\mathcal{O}_k(x) = \sum_n b_{kn}\, D_{kn}(x_\perp, \partial_{x_\parallel})\, \hat{\mathcal{O}}_n(x_\parallel) \qquad a_k = b_{kI}$$

- Normal universality class is a natural bootstrap target
  1. leading bulk dimensions known from the bulk bootstrap or Monte Carlo
  2. leading boundary dimensions are protected (tilt, displacement)

Boundary bootstrap

- 2-point function of bulk operators $<\mathcal{O}_1(x)\mathcal{O}_2(y)>$ Liendo, Rastelli, van Rees, 2012

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k \lambda_{12k}\, C_{12k}(x-y,\partial_y)\, \mathcal{O}_k(y)$$

$$\mathcal{O}_k(x) = \sum_n b_{kn}\, D_{kn}(x_\perp, \partial_{x_\parallel})\, \hat{\mathcal{O}}_n(x_\parallel)$$

$$\langle \mathcal{O}_k(x) \rangle = \frac{a_k}{(2x_\perp)^{\Delta_k}}$$

- Bulk channel

$$\phi_a \times \phi_b \sim \sum_S \delta_{ab}\, \mathcal{O} + \sum_T \mathcal{O}_{(ab)} + \sum_A \mathcal{O}_{[ab]}$$

- Boundary channel

$$\phi_N \sim 1 + D + \sum_{\hat{\Delta}>3} \hat{\mathcal{O}}^{(\hat{S})}\,, \quad \phi_i \sim t_i + \sum_{\hat{\Delta}>2} \hat{\mathcal{O}}_i^{(\hat{V})}\,,$$

Boundary bootstrap

- Crossing equation for

$$\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\rangle = \frac{G(\xi)}{(2x_\perp)^{\Delta_1}(2y_\perp)^{\Delta_2}}\,\xi^{-\frac{\Delta_1+\Delta_2}{2}}$$

$$\sum_k \lambda_{12k} a_k\, f^{\Delta_{12}}_{\Delta_k}(\xi) - \xi^{\frac{\Delta_1+\Delta_2}{2}} \sum_n b_{1n} b_{2n} \hat{f}_{\hat{\Delta}_n}(\xi) = 0\,, \qquad \xi = \frac{(x-y)^2}{4x_\perp y_\perp}$$

- Conformal blocks McAvity, Osborn, 1995

bulk channel $$f^{\Delta_{12}}_{\Delta}(\xi) = \xi^{\Delta/2}\,{}_2F_1\Big[\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}; \Delta - \frac{d-2}{2}; -\xi\Big]$$

boundary channel $$\hat{f}_{\hat{\Delta}}(\xi) = \xi^{-\hat{\Delta}}\,{}_2F_1\Big[\hat{\Delta}, \hat{\Delta} - \frac{d}{2} + 1; 2\hat{\Delta} - d + 2; -1/\xi\Big]$$

1 cross ratio, no spin, simpler than 4-pt bulk crossing

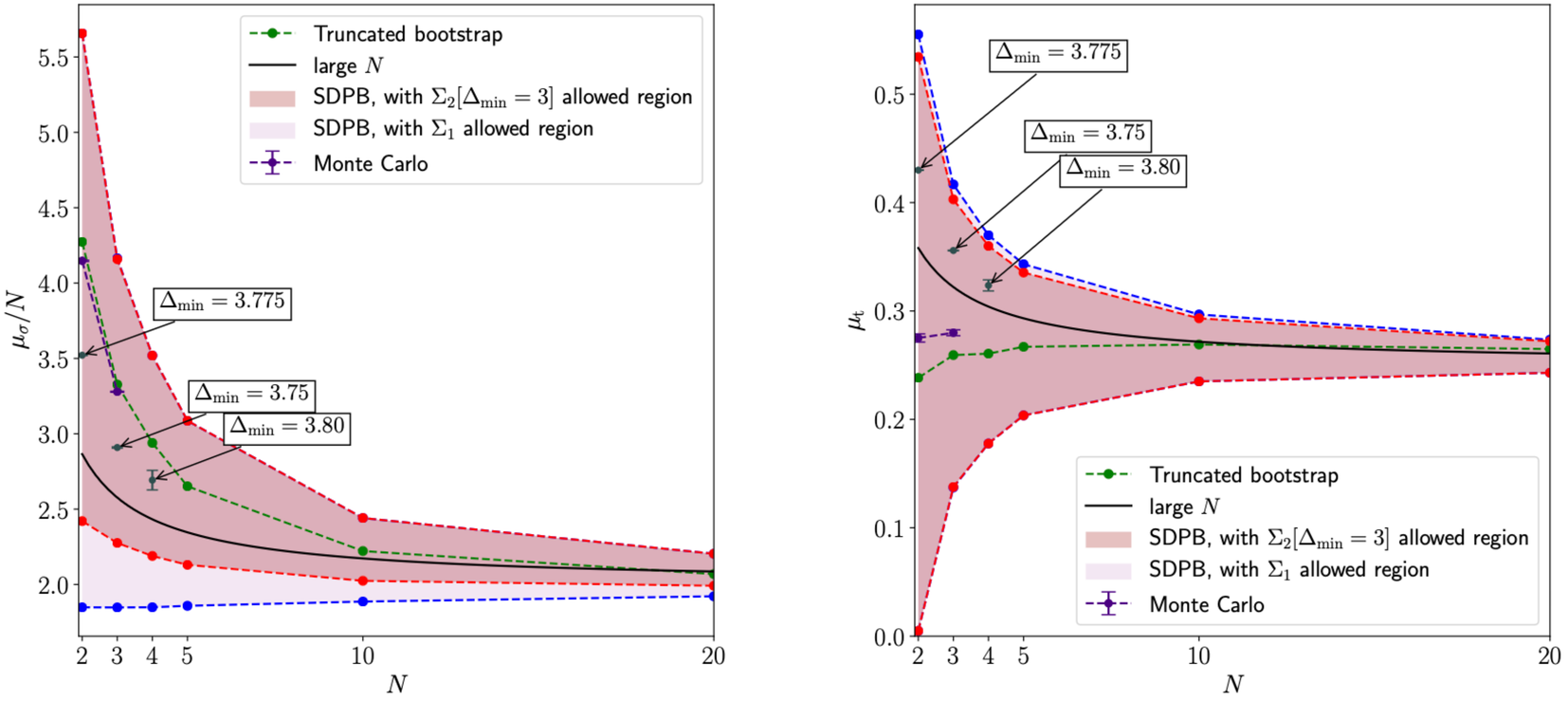
$$\Delta_{12} = \Delta_1 - \Delta_2$$

- Bulk-channel positivity?

Bulk-channel positivity?

- Rigorous bounds if positivity is obeyed

Liendo, Rastelli, van Rees, 2012



Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021

Truncated bootstrap(Gliozzi method)

- Truncate the bootstrap equation to finitely many operators
  determinant: Gliozzi, 2013; Gliozzi, Rago, 2014
  singular value: Esterlis, Fitzpatrick, Ramirez, 2016

- η minimization

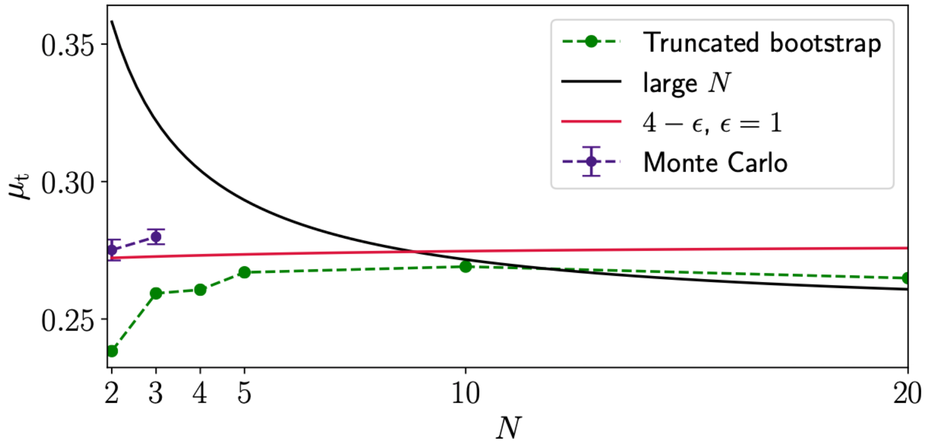
  cost/loss function $$\eta = \sum_j \sum_{m=0}^{M_j} \left| \partial_\xi^m \left(\text{bootstrap equation}_j \right) \right|^2_{\xi=1}$$ Li, 2017

  solve the truncated bootstrap equations by local minimization

- Variants: artificial intelligence, analytic input, random weight, different norm
  Kántor, Niarchos, Papageorgakis, Richmond, Stapleton, Woolley, 2021-2025;
  Li, 2023; Poland, Prilepina, Tadic, 2023-2025; Barrat, Marchetto, Miscioscia, Pomoni, 2024;
  Benjamin, Fitzpatrick, Li, Thaler, 2026

Truncated boundary bootstrap

- Previous truncated boundary bootstrap results are promising
  Gliozzi, Liendo, Meineri, Rago, 2015
  Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021



- **Tension** with Monte Carlo
  truncated bootstrap is not rigorous
  uncontrolled systematic error ? low truncation order ?

- The η minimization allows for significantly higher truncation orders

  We search for the **zeros** of η.
  They are **global** minima.

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
This work	88, 68	76	68	60	88
[6, 38]	9, 8	10	10	9	9

Λ: # bootstrap constraints

How to find a zero?

- Low truncation order: **numerical algebraic geometry**
  rational approximation -> polynomial equations
  Mathematica's NSolve or HomotopyContinuation.jl
  huge number of solutions, few are physical

- High truncation order: **local minimization**
  Mathematica's FindMinimum with LevenbergMarquardt (trust region)
  interpolates between Gauss–Newton and gradient descent

- How to construct a **starting point**?
  low-lying operator dimensions change gently (effective description)
  most operator dimensions are from lower order solution
  scan high-lying operator dimensions (discrete set of values)
  coefficients of conformal blocks =1

Bulk O(N) input

N	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T
2	0.51908(1)[57]	1.51128(5)[57]	3.789(4)[54]	1.23629(**11**)[30]
3	0.518936(**67**)[31]	1.5948(2)[55]	3.759(2)[55]	1.20954(**32**)[31]
4	0.51812(4) [56]	1.66340(35)[56]	3.755(5)[56]	$1.1864^{+0.0024}_{-0.0034}$[23]
5	0.516985(45)[56]	1.7182(10)[56]	3.754(7)[56]	1.1568(10)[23]

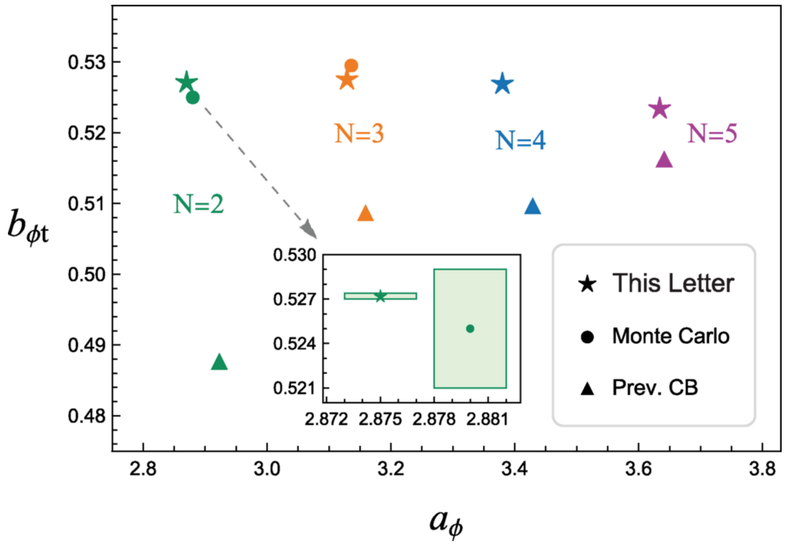
- Bulk bootstrap

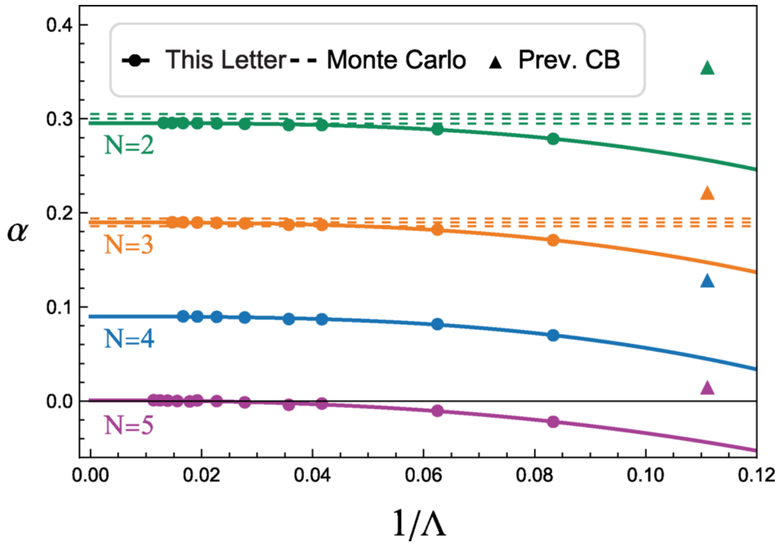
  [23] Kos, Poland, Simmons-Duffin, 2013
  [30, 31] Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019 + 2020

- Monte Carlo

  [54-57] Hasenbusch, 2019 + 2020 + 2021 + 2025

Truncated bootstrap results





$N = 2$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	2.875(2)	0.5272(2)	0.2440(4)	0.2957(6)
MC [7]	2.880(2)	0.525(4)		0.300(5)
CB [6]	2.923	0.4882	0.2701	0.3567

$N = 3$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.129(3)	0.5278(2)	0.2406(6)	0.1903(7)
MC [7]	3.136(2)	0.529(3)		0.190(4)
CB [6]	3.159	0.5092	0.2690	0.2236

$N = 4$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.380(7)	0.5272(13)	0.2369(24)	0.0906(35)
CB [6]	3.429	0.5105	0.2758	0.1304

$N = 5$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.634(5)	0.5235(5)	0.2390(9)	0.002(2)
CB [6]	3.641	0.5166	0.2653	0.0166

$$\alpha = \frac{1}{32\pi}\frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$$

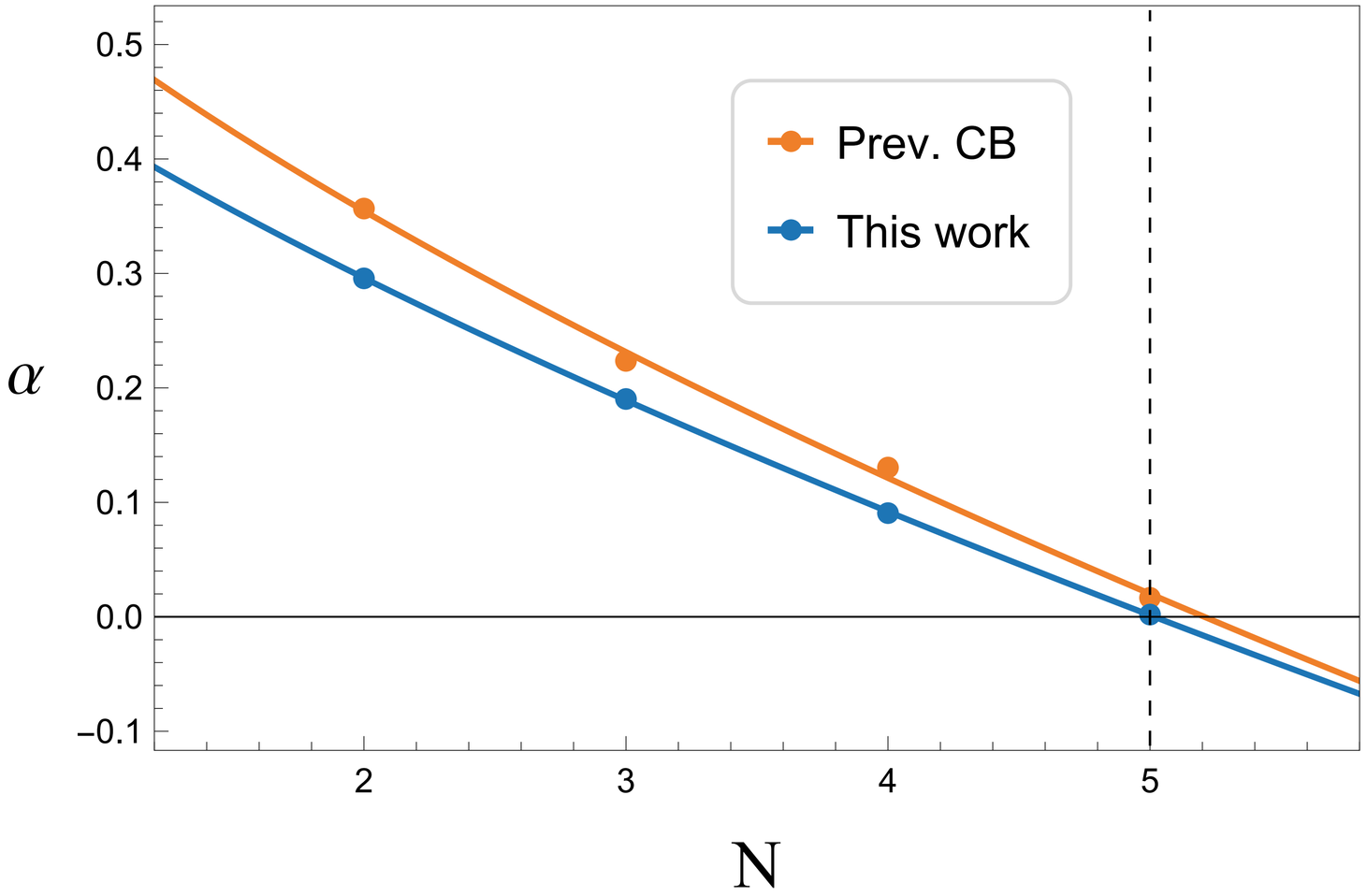
MC: Monte Carlo
Toldin, Metlitski, 2021

Toldin et al., unpublished
N=4: α = 0.097(3)

3.386(2) 0.524(2)

N=6: α = -0.091(9)

Where is Nc?



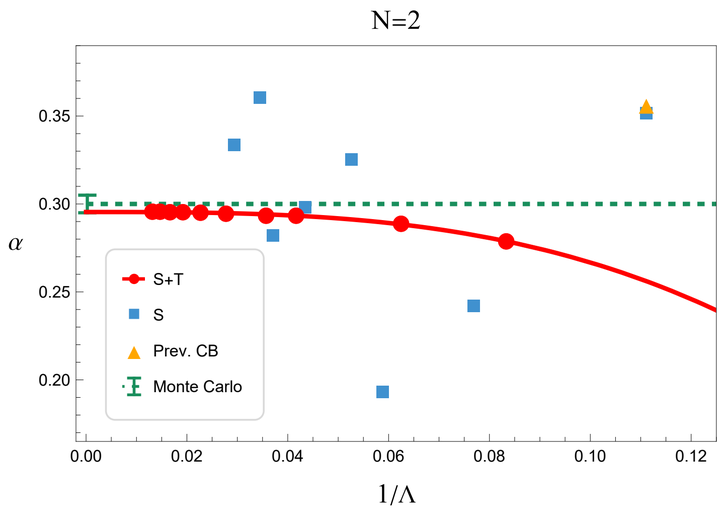
Previous: Nc ≈ 5.2

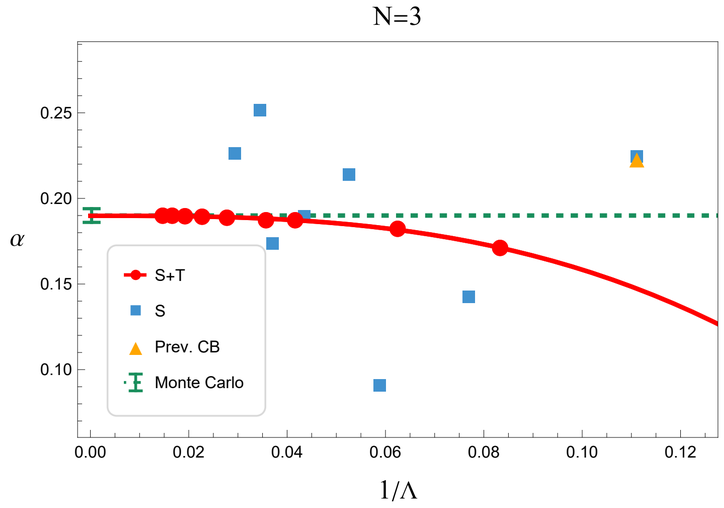
This work: Nc = 5.016(28)

Exactly at 5? Why?

Importance of bulk T

- $\langle \phi_a(x)\, \phi_b(y) \rangle$ involves two O(N-1) singlets a,b = (i,N) and i = 1,2,...,N–1
- Two crossing equations $\langle \phi_N(x)\, \phi_N(y) \rangle$, $\sum_i \langle \phi_i(x)\, \phi_i(y) \rangle$
- Previously, the bulk T contributions were projected out, i.e. only 1 crossing eq
  Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021





Other bootstrap results

- Subleading O(N) traceless-symmetric dimensions for N=2,3,4,5,6

$$\Delta_{T'} = 3.6484(24)\,,\ 3.559(4)\,,\ 3.49(3)\,,\ 3.354(17)\,,\ 3.31^{+0.07}_{-0.11}$$

  consistent with the O(2) bulk bootstrap estimate 3.650(2)

- New bulk 1-point coefficients

$$\begin{aligned} N=2: &\quad a_S = 5.571(12)\,, &\quad a_T = 3.897(6)\,, \\ N=3: &\quad a_S = 5.369(12)\,, &\quad a_T = 8.406(12)\,. \end{aligned}$$

  used the bulk OPE coefficients ($\lambda_{\phi\phi S}$, $\lambda_{\phi\phi T}$) from the O(2), O(3) bulk bootstrap.

Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019 + 2020

Higher accuracy?

- We can systematically increase the truncation order.
- The accuracy is mainly limited by bulk input uncertainties.
- More precise bulk input, more accurate bootstrap results.
- Let’s demonstrate this by the Ising boundary bootstrap!
- Ising normal = Ising extraordinary

Outline

- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

Bulk Ising input

- We consider <σ(x) σ(y)> and <σ(x) ϵ(y)>.
- The mixed correlator is nonzero due to symmetry breaking.
- High precision input from the bulk bootstrap

$$\Delta_\sigma^{\text{input}} = 0.518148806(\mathbf{24})\,,\ \Delta_\epsilon^{\text{input}} = 1.41262528(\mathbf{29})\,,$$

$$\lambda_{\sigma\sigma\epsilon}^{\text{input}} = 1.05185373(11)$$

Chang, Dommes, Erramilli, Homeric, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin, 2024

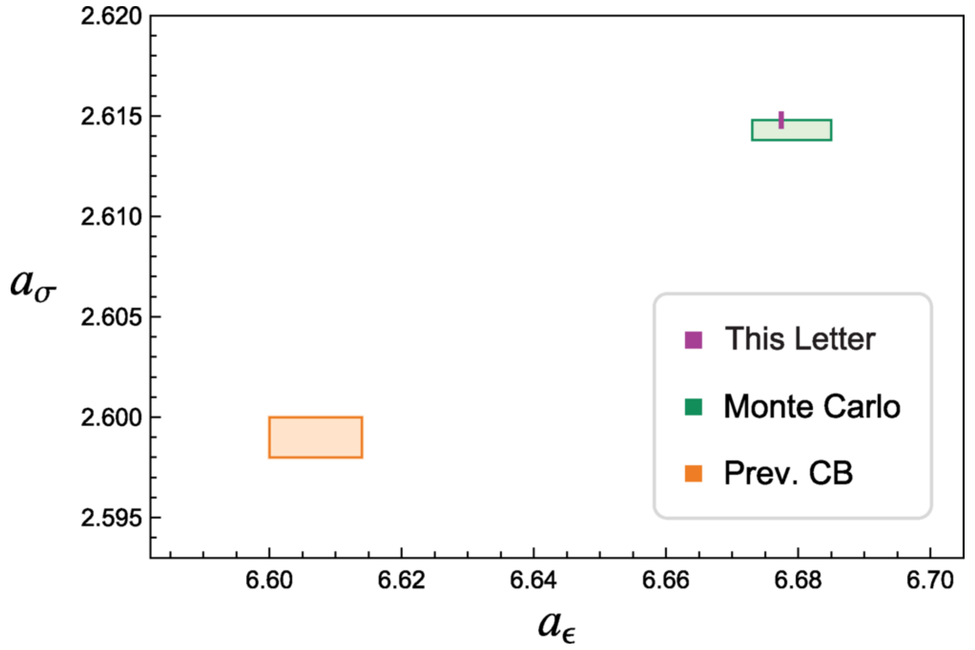
- Main source of error $\Delta_{\epsilon'}^{\text{input}} = 3.82951(\mathbf{61})$ Reehorst, 2021

Ising boundary bootstrap

- Fusion rules

$$\sigma \times \epsilon \sim \sigma + \sigma' + \sigma'' + \dots$$
$$\sigma \times \sigma \sim 1 + \epsilon + \epsilon' + \epsilon'' + \dots$$
$$\sigma, \epsilon \sim 1 + D + \hat{N} + \hat{N}' + \dots$$

Monte Carlo: Przetakiewicz, Wessel, Toldin, 2025
Previous: Gliozzi, Liendo, Meineri, Rago, 2015



- In the bulk channel, the Z2 odd spectrum has a **large gap**.
  $\Delta\sigma \approx 0.52$, $\Delta\sigma' \approx 5.3$ (multiplet recombination)

  Highly accurate results from the $\langle\sigma(x)\,\epsilon(y)\rangle$ crossing.

Ising bootstrap results

- Selected results

MC: Przetakiewicz, Wessel, Toldin, 2025
Fuzzy Sphere: Zhou, Zou, 2024
Previous: Gliozzi, Liendo, Meineri, Rago, 2015
Bulk Bootstrap: Simmons-Duffin, 2016

Method	a_ϵ	a_σ	$b_{\epsilon D}$	$b_{\sigma D}$
This work	6.677424(16)	2.6148(3)	1.7234(5)	0.24757(4)
MC [40]	6.679(6)	2.6143(5)	1.69(1)	0.242(2)
FS [61]	6.4(9)	2.58(16)	1.74(22)	0.254(17)
CB [38]	6.607(7)	2.599(1)	1.742(6)	0.25064(6)

- Boundary scaling dimensions BOE coefficients

$$\hat{\Delta}_{\hat{N}} = 5.8792(12)\,, \quad \hat{\Delta}_{\hat{N}'} = 8.086(24)\,,$$
$$b_{\epsilon\hat{N}} = 0.2147(23)\,, \quad b_{\epsilon\hat{N}'} = 0.0464(42)\,,$$
$$b_{\sigma\hat{N}} = 0.00946(11)\,, \quad b_{\sigma\hat{N}'} = 0.00130(12)\,.$$

Fuzzy Sphere: 5.858 Dedushenko, 2024

Method	$\Delta_{\sigma'}$	$\Delta_{\epsilon''}$	$\Delta_{\sigma''}$
This work	5.28901(3)	6.873(7)	8.42915(36)
CB [29]	5.2906(11)	6.8956(43)	
CB [38]	5.49(1)	7.27(5)	10.6(3)

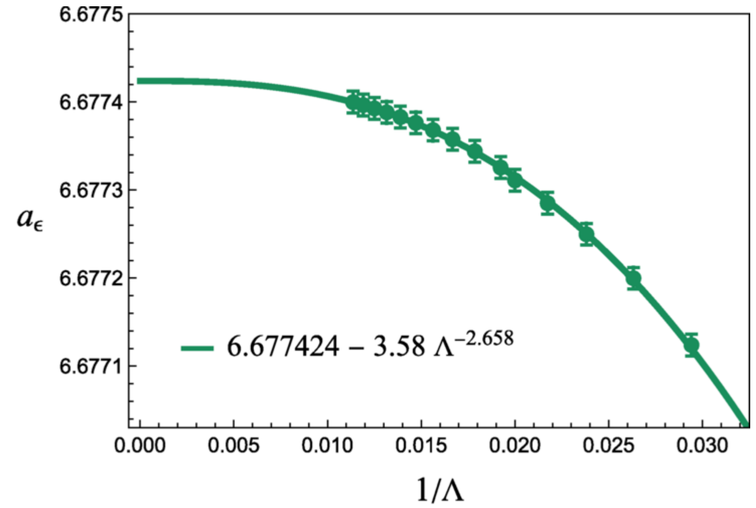
- New bulk 1-point coefficients

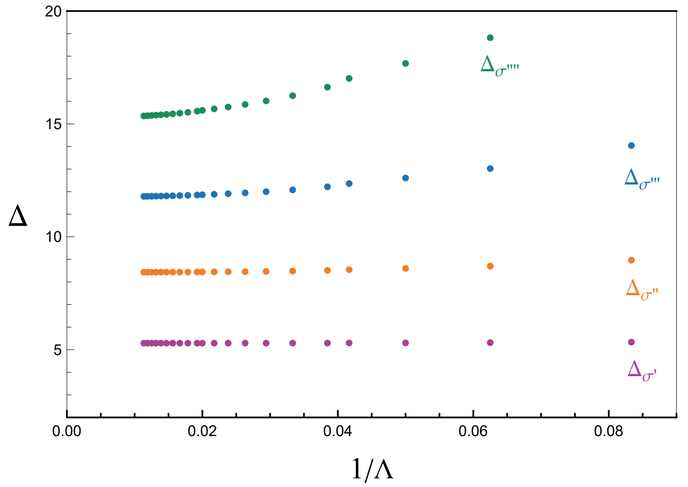
$$a_{\epsilon'} = 42.46(14)\,,\ a_{\sigma'} = 110(4)\,,\ a_{\epsilon''} = 267.8(15)$$

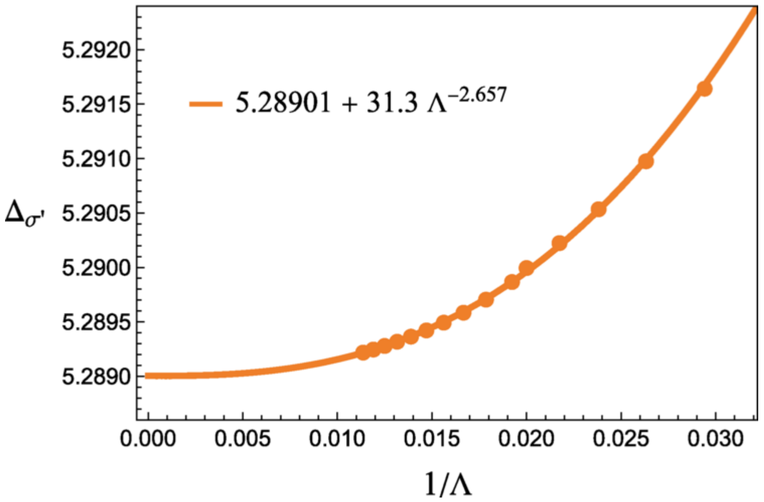
Use bulk OPE coefficients from
Simmons-Duffin, 2016
Reehorst, 2021

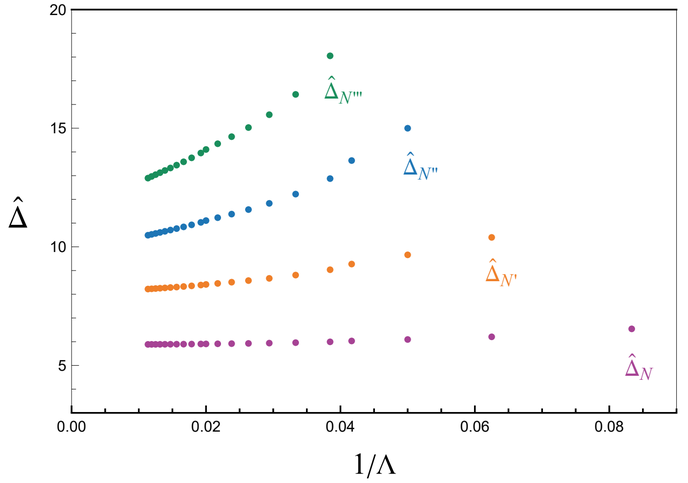
Convergence and extrapolation

selected results from the <σ(x) ϵ(y)> crossing

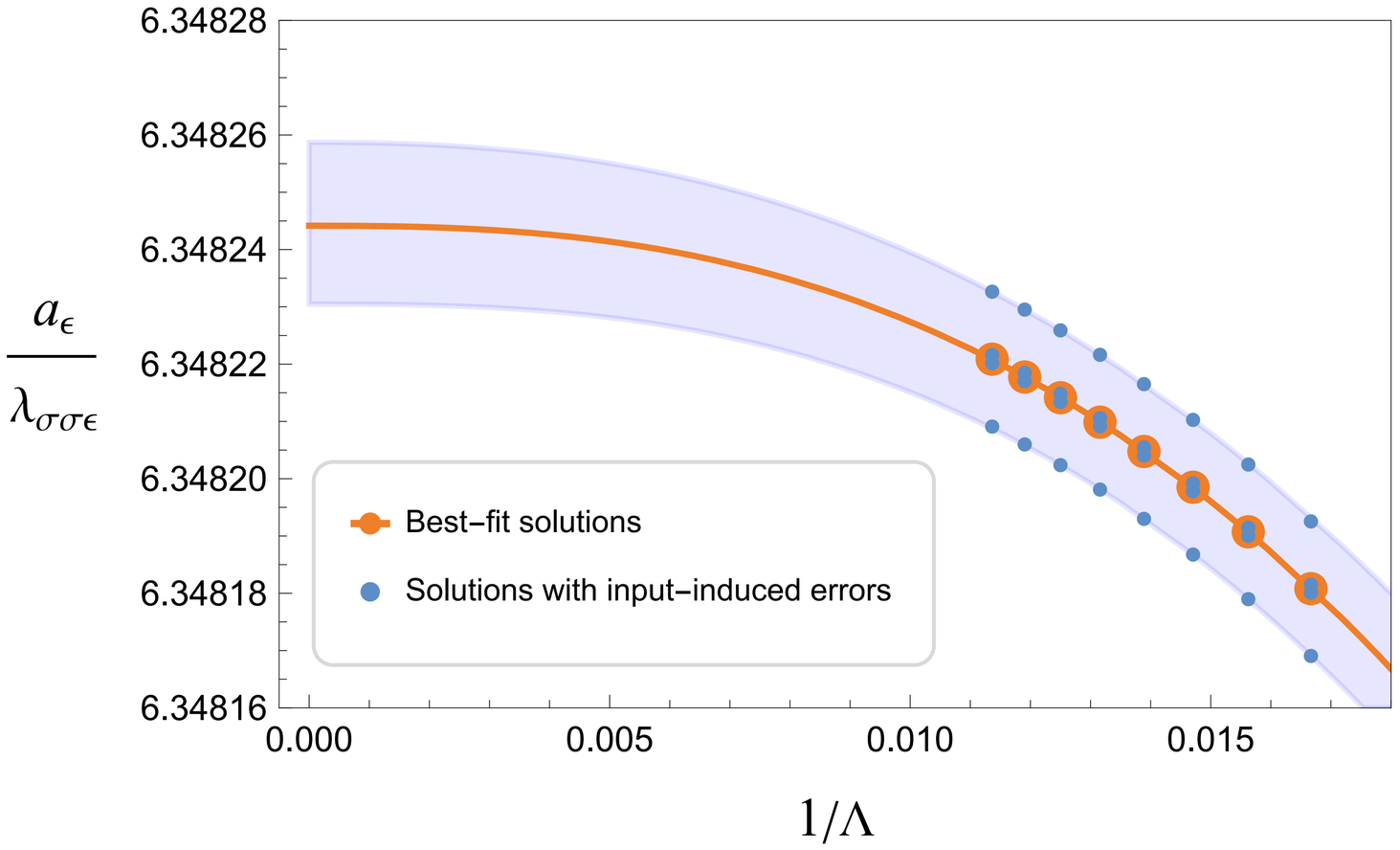








Error analysis



input induced error + extrapolation uncertainty

Are errors reliable?

- Bulk OPE coefficient from our boundary bootstrap results

$$\lambda_{\sigma\sigma\epsilon}^{\text{this work}} = \sqrt{\left(\frac{a_\epsilon}{\lambda_{\sigma\sigma\epsilon}}\right)^{-1}(\lambda_{\sigma\sigma\epsilon}\, a_\epsilon)} = 1.05184(13)$$

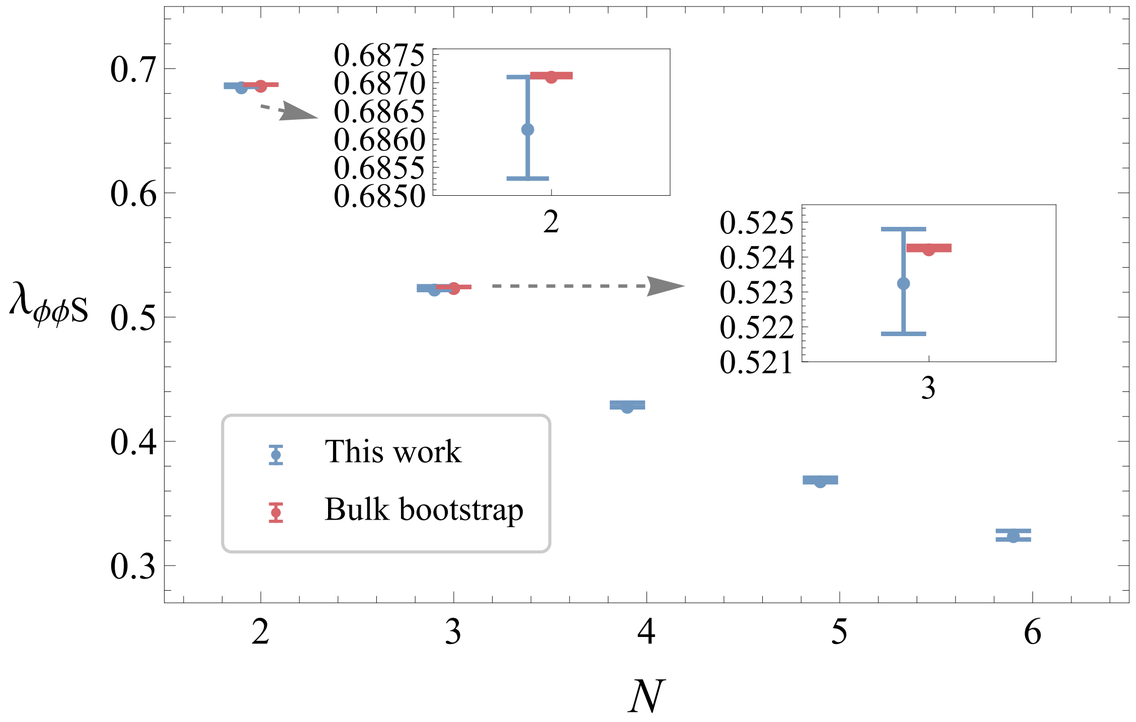
bulk bootstrap determination $\lambda_{\sigma\sigma\epsilon}^{\text{input}} = 1.05185373(11)$

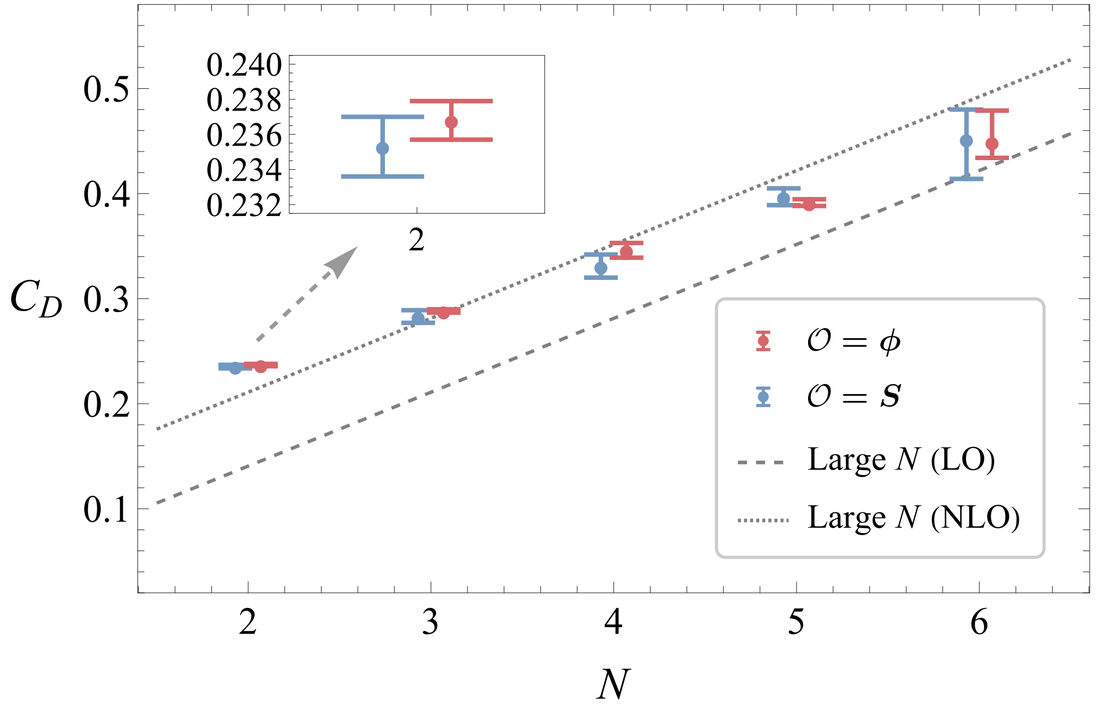
- Zamolodchikov norm of the displacement operator

$$C_D = \left(\frac{\Delta_{\mathcal{O}}\, a_{\mathcal{O}}}{4\pi\, b_{\mathcal{O}D}}\right)^2 = \begin{cases} 0.18966(10) & \mathcal{O} = \sigma \\ 0.18970(10) & \mathcal{O} = \epsilon\,. \end{cases}$$

Ward identity

More check for N>1





from <ϕϕ> and <ϕS>

N>1 bulk OPE coefficients

- Boundary bootstrap estimates for N = 2, 3, 4, 5, 6

$$\lambda_{\phi\phi S}^{\text{this work}} = \sqrt{(\lambda_{\phi\phi S} a_S)\left(\frac{a_S}{\lambda_{\phi S\phi}}\right)^{-1}}$$
$$= 0.6862(9), 0.5233(15), 0.4290^{+0.0021}_{-0.0016}, 0.3689(18), 0.3247^{+0.0032}_{-0.0037}$$

- Previous bulk bootstrap determinations for N = 2, 3

$$\lambda_{\phi\phi S}^{\text{prev}} = 0.687126(27)\,,\ 0.524261(59)$$

Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019 + 2020

Outline

- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

Summary

- Truncated boundary bootstrap becomes more systematic
  higher truncation orders, better error estimates, new bulk+boundary results

- Accurate bootstrap results from precise bulk input
  resolve Bootstrap/Monte Carlo discrepancies, 2 orders more accurate than the latest MC, Nc = 5 (?)

- Larger bootstrap system?
  correlators of higher points, boundary operators, other bulk operators (spinning?), integrated constraints(?)

- Other defects? other boundary, interface (folding trick), line, ...

Other bootstrap targets?

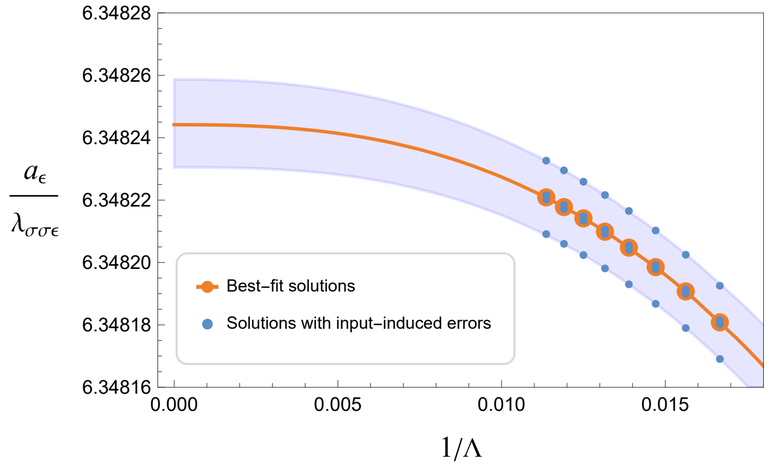
- Nontrivial manifolds? real projective space, …. ; anomaly?

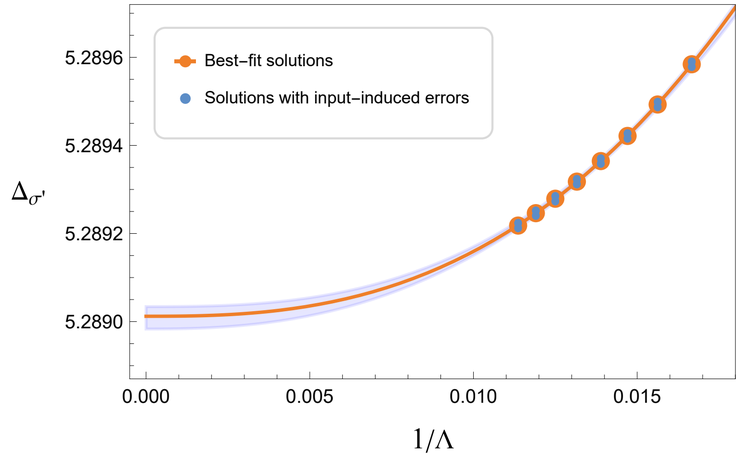
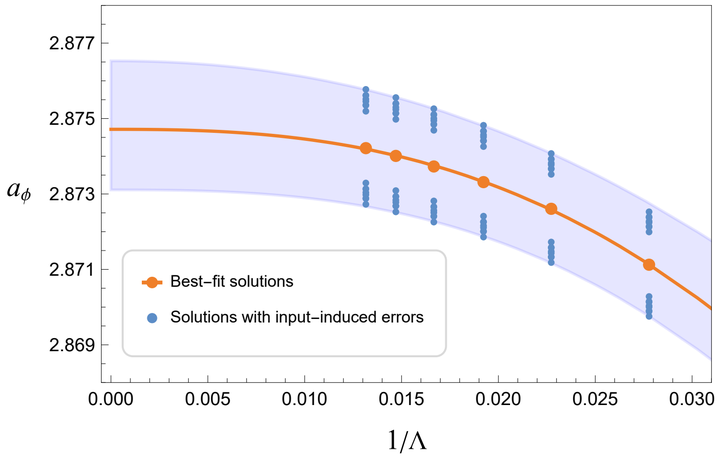
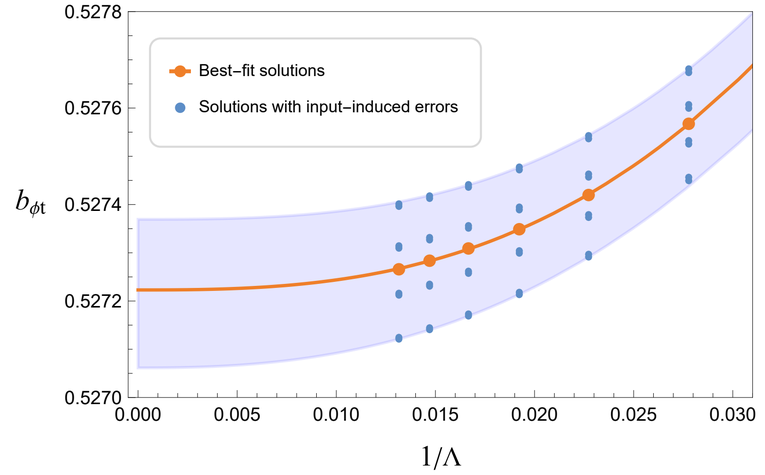
- Systematic non-unitary bulk bootstrap?
  Yang-Lee edge singularity, percolation, polymer, disorder, turbulence(?),
  complex CFT (conformality loss? conformal window?), …

Thank you!

More error analysis

Ising





O(2)