From Z to a & everything in between

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IHES ✈ TIFR

Bootstrap 2026, London

11 August 2026

based on **2512.00158** and ongoing work

with A. Advant, H. Anand, N. Benjamin, V. Kumar, S. Minwalla, J. Mukherjee, A. Rahaman, P. Ray

How does the operator count grow?

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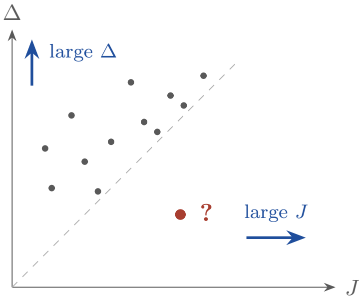
Two directions to go: large Δ, and large spin.



Above two dimensions, the tool is gone

torus T^2

modular invariance

Cardy: the growth is fixed by one number, c

[large spin: . . . , Kusuki; Benjamin, Shao, Ooguri, Wang; SP, Qiao, van Rees]

$S^{d-1} \times S^1$

no modular group

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So what forces universality above two dimensions?

The answer: locality

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Locality — leveraged through the
thermal effective field theory (hydrodynamics)

[**Bhattacharyya, Lahiri, Loganayagam, Minwalla '07; Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma '12; Jensen, Kaminski, Kovtun, Meyer, Ritz, Yarom '12**]

Cardy-like formulas in higher d: [**Benjamin, Lee, Ooguri, Simmons-Duffin '23**; see also Shaghoulian '15; Kang, Lee, Ooguri '22; Benjamin, Lee, SP, Simmons-Duffin, Xu '24; Allameh, Shaghoulian '24; Kusuki, Ooguri, SP '25; Xu, Simmons-Duffin '25; Buric, Mangialardi, Russo, Schomerus, Vichi '25; Banihashemi, Shaghoulian, Shashi '25; David, Kumar '25; Buric, Mangialardi, Russo, Vichi '26; Parmentier, Bobev '26; Mauro, Vichi '26; $\cdots$]

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τ is called the *twist*.

No Casimir energy in D; no scheme dependence.

Nothing should relate these numbers

High temperature is governed by a local derivative expansion.

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High temperature is governed by a local derivative expansion.

Its coefficients are Wilson coefficients — theory-dependent, with no reason to be related to one another.

And a is a Weyl-anomaly coefficient.
Not thermal data in any obvious way.

Spoiler 1: Z knows the a-anomaly

$d = 4$, order β — four parity-even angular structures

$$\log Z\Big|_{O(\beta)} = \frac{\alpha_1 + \alpha_2\left(\omega_1^2 + \omega_2^2\right) + \alpha_3\left(\omega_1^4 + \omega_2^4\right) + \alpha_4\,\omega_1^2\omega_2^2}{\left(1-\omega_1^2\right)\left(1-\omega_2^2\right)}\;\beta$$

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$$\boxed{a = \tfrac{1}{2}\left(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4\right)}$$

One Z, two limits

$$\begin{tikzcd} ? & Z(\beta, \omega_i) \arrow[l, "\beta \to 0"', "\omega_i \text{ fixed}"] \arrow[r, "\nu_i \to 0", "\beta \text{ fixed}"'] & ? \end{tikzcd}$$

$\beta \to 0$ probes large Δ

the Cardy direction

$\omega_i \to 1$ probes large spin

ω is conjugate to J

One Z, two limits

$$? \xleftarrow[\omega_i \text{ fixed}]{\beta \to 0} \boxed{Z(\beta, \omega_i)} \xrightarrow[\beta \text{ fixed}]{\nu_i \to 0} ?$$

$\beta \to 0$ probes large Δ

the Cardy direction

$\omega_i \to 1$ probes large spin

ω is conjugate to J

two expansions of one object — intricately related

Spoiler 2: one pole per rotation plane

$$\boxed{\log Z \xrightarrow[\omega_i \to 1]{} \frac{4\,h(\beta)}{\left(1-\omega_1^2\right)\left(1-\omega_2^2\right)}} \qquad (d=4)$$

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universal

the pole structure

theory-dependent

the residue $h(\beta)$

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Semi-Universality

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman '25]

(see also Komargodski, Popov, Miscioscia '26)

Spoiler 3: subleading semi-universality

$$\log Z = \beta\Big[\, \frac{A(\beta)}{\nu_1\nu_2} + B(\beta)\left(\tfrac{1}{\nu_1}+\tfrac{1}{\nu_2}\right) + C(\beta)\left(\tfrac{\nu_1}{\nu_2}+\tfrac{\nu_2}{\nu_1}\right) + D(\beta) + O(\nu)\Big]$$

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$$\boxed{\tfrac{1}{6}A''(\beta) + B'(\beta) + C(\beta) + D(\beta) \;=\; \frac{2a}{3}}$$

Are they independent?

$$
\begin{tikzcd}
\boxed{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\ +\ \cdots} \arrow[rr, bend right, dashed, leftrightarrow, "?"'] & \boxed{Z(\beta,\omega_i)} \arrow[l, "\beta \to 0"', "\omega_i \text{ fixed}"] \arrow[r, "\nu_i \to 0", "\beta \text{ fixed}"'] & \boxed{\begin{array}{c} A,\ B,\ C,\ D \\ \text{functions of } \beta \end{array}}
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Why high T, and why a derivative expansion?

- The thermal circle is the short direction.
- So $\beta \to 0$ is a genuine KK limit; the reduced theory is gapped, with gap $\sim 1/\beta$.
- Finitely many Wilson coefficients at each order.

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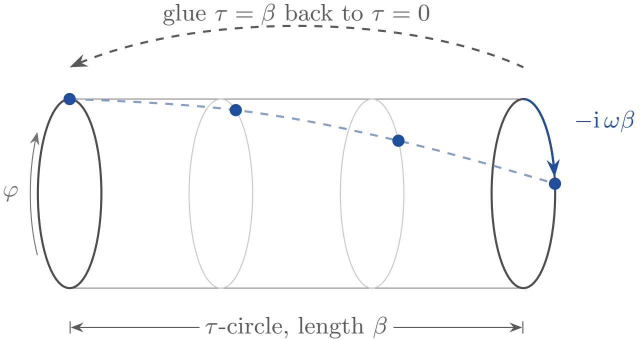
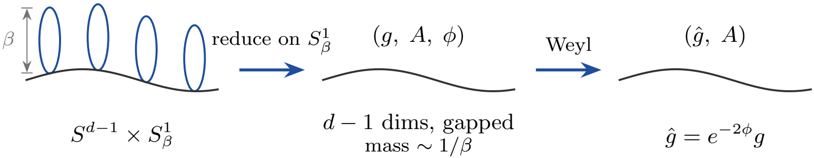
It is the statement that the CFT is local.

An imaginary twist of the thermal circle



$$(\tau, \varphi) \ \sim \ (\tau + \beta,\ \varphi - \mathrm{i}\,\omega\beta)$$

Reduce on the circle, then Weyl



$$S_{\text{th}} = \int d^{d-1}x \sqrt{\hat{g}}\, \beta^{1-d} \Big[-f + c_1 \beta^2 \hat{R} + c_2 \beta^2 F^2 + \cdots \Big]$$

$+\, S_{\text{anomaly}}$ (even d only)

Weyl invariance $\Rightarrow$ only $\hat{g} = e^{-2\phi} g$ appears — up to an anomaly

[Bhattacharyya, Lahiri, Loganayagam, Minwalla; . . . ; Benjamin, Lee, Ooguri, Simmons-Duffin]

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allowed

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produced

separately, there are finitely many local invariants of that derivative order

Does the second list fill the first?

$d = 4$ at order β: four structures

$$\left\{ \; 1, \quad \omega_1^2 + \omega_2^2, \quad \omega_1^4 + \omega_2^4, \quad \omega_1^2 \omega_2^2 \; \right\}$$

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Can a local four-derivative EFT produce all four?

(there are six such invariants)

What we have to compute

	1	$\omega_1^2+\omega_2^2$	$\omega_1^4+\omega_2^4$	$\omega_1^2\omega_2^2$
$\hat{R}^2$	·	·	·	·
$\hat{R}_{ij}\hat{R}^{ij}$	·	·	·	·
$(F^2)^2$	·	·	·	·
$\hat{R}\,F^2$	·	·	·	·
$\hat{R}_{ij}F^{ik}F^{j}{}_{k}$	·	·	·	·
$\nabla_i F_{jk}\nabla^i F^{jk}$	·	·	·	·

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six independent parity-even four-derivative invariants,
four angular structures they could produce.

Six invariants, four structures

	1	$\omega_1^2+\omega_2^2$	$\omega_1^4+\omega_2^4$	$\omega_1^2\omega_2^2$
$\hat{R}^2$	36	-48	$148/3$	$-104/3$
$\hat{R}_{ij}\hat{R}^{ij}$	12	-16	20	-8
$(F^2)^2$	0	0	$64/3$	$64/3$
$\hat{R}\,F^2$	0	-24	$88/3$	$16/3$
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every four-derivative invariant, evaluated on the rotating S^3

Take any row

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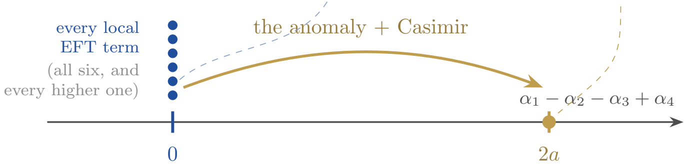
$$0 - 0 - \frac{64}{3} + \frac{64}{3} = 0$$

Why a, not zero

$$\log \mathrm{Tr}\, e^{-\beta(D-\omega_i J_i)} \quad = \quad -S_{\mathrm{th}}[\hat{g}, A] \quad + \quad S_{\mathrm{anomaly}} + \text{Casimir}$$

operators on S^3 — the local EFT — $O(\beta)$ only



$$\boxed{a = \frac{1}{2}\left(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4\right)}$$

the one direction a local EFT cannot reach
is exactly where the anomaly lives

An anomaly coefficient, measured thermally

A Weyl-anomaly coefficient

is measurable from the high-temperature thermodynamics of a rotating S^3.

Example: the Dirac fermion

$$Z = \mathrm{Tr}\, e^{-\beta(D - \omega_i J_i)}$$

no scheme dependence

$$\log Z = \frac{1}{(1-\omega_1^2)(1-\omega_2^2)}\left[\frac{7\pi^4}{90\beta^3} + \frac{\pi^2}{36\beta}\left(-3+\omega_1^2+\omega_2^2\right) + (\cdots)\beta + \text{non-pert}\right]$$

$$(\cdots) = \underbrace{\tfrac{17}{480}}_{\alpha_1}\ \underbrace{-\tfrac{1}{240}}_{\alpha_2}\left(\omega_1^2+\omega_2^2\right)\ \underbrace{-\tfrac{7}{480}}_{\alpha_3}\left(\omega_1^4+\omega_2^4\right)\ +\ \underbrace{\tfrac{1}{144}}_{\alpha_4}\,\omega_1^2\omega_2^2$$

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$$\frac{1}{2}\left(\alpha_1-\alpha_2-\alpha_3+\alpha_4\right) = \frac{1}{2}\left(\frac{17}{480}+\frac{1}{240}+\frac{7}{480}+\frac{1}{144}\right) = \frac{11}{360}$$

which is the a-anomaly of a Dirac fermion. ✓

Four checks, no free parameters

	α_1	α_2	α_3	α_4	$\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)$
free scalar	$\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{1}{144}$	$\frac{1}{360}$ ✓
Dirac fermion	$\frac{17}{480}$	$-\frac{1}{240}$	$-\frac{7}{480}$	$\frac{1}{144}$	$\frac{11}{360}$ ✓
Maxwell	$\frac{11}{120}$	$-\frac{11}{120}$	$-\frac{1}{120}$	$\frac{11}{72}$	$\frac{31}{180}$ ✓
holography $\times \frac{\ell^3}{G_N}$	$\frac{3\pi}{16}$	$-\frac{\pi}{8}$	$\frac{\pi}{16}$	0	$\frac{\pi}{8}$ ✓
$\mathcal{N}=4,\ \lambda=0 \times (N^2-1)$	$\frac{3}{16}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$ ✓

✓ = agrees with the known a

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$$a(\lambda = 0) = a(\lambda = \infty).$$

The first of infinitely many

order	angular structures that *can* appear	how many the *local EFT* produces	missing = *relations*	
β^{-3}	1	1	0	
β^{-1}	2	2	0	
β^{1}	**4**	**3**	**1**	$= 2a$
β^{3}	6	5	1	$= 0$
β^{5}	9	7	2	$= 0$
β^{7}	12	9	3	$= 0$
$\vdots$			grows like $N^2/12$	

at β^1: the *four columns* of the table, and its *rank three*

only the **first** is fed by the anomaly

Every even d — and only even d

$$\boxed{a_d = \frac{(-1)^{d/2}(d+2)}{2\,\Gamma\left(\frac{d+2}{2}\right)}\ \log Z\Big(\beta,\ \omega_k = e^{2\pi \mathrm{i} k/(d+2)}\Big)\ \Big|_{\beta^1}}$$

$$k = 1, \ldots, d/2$$

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$$k = 1, \ldots, d/2$$

In odd d the EFT produces *every* allowed structure.
No relations at all.

[6d and $d = 8$ checks: backup K3]

The same coefficients control the other corner

$$
\begin{tikzcd}
\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 + \cdots \arrow[rr, <->, dashed, bend right, "?"] & Z(\beta, \omega_i) \arrow[l, "\beta \to 0", "\omega_i \text{ fixed}"'] \arrow[r, "\nu_i \to 0", "\beta \text{ fixed}"'] & \begin{array}{c} A,\ B,\ C,\ D \\ \text{functions of } \beta \end{array}
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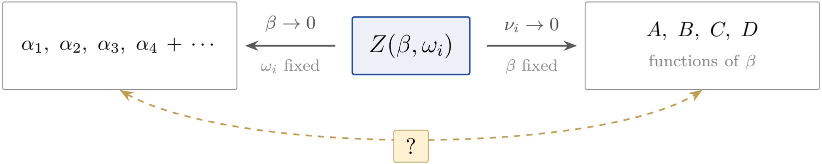
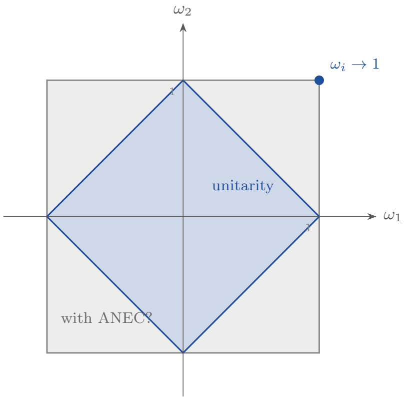


Everything so far was $\beta \to 0$ at fixed ω.
The α's I just constrained are the same data that controls $\omega_i \to 1$.

Semi-Universality

Does the trace even reach the corner?



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$|\omega_1| + |\omega_2| < 1$.
And that is sharp.

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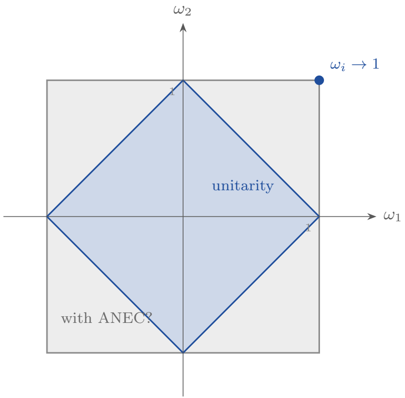

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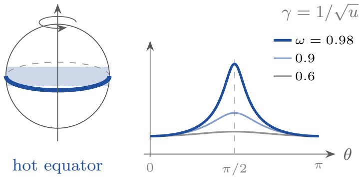
And that is sharp.

The square — hence the corner — needs a bound stronger than unitarity. Conjectural.

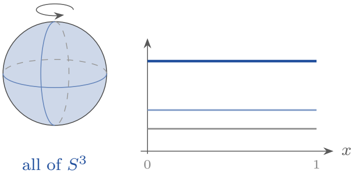
Why a local EFT still controls it

$d = 3$: one rotation plane



a thin belt, width $\sqrt{2\nu/\beta}$

$d = 4$: both planes



no belt — the whole sphere lifts

in the rotating frame the comoving temperature rises and the comoving volume grows

$d = 3$ is the *harder* case: localised heating gives a non-universal density, and only a coarse-grained statement survives.

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

One simple pole per rotation plane

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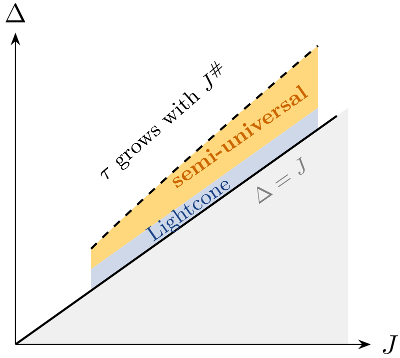
We call this semi-universal.

A physics argument, supported by order-by-order work and by checks.
Not a theorem.

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

This is not the lightcone bootstrap



Lightcone: twist held *fixed* as $J \to \infty$.

bounded constituent count — that is its hypothesis

Here: twist grows with $J^{\#}$.

the constituent count grows with the spin — exactly what that hypothesis excludes

[Komargodski, Zhiboedov; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Caron-Huot] recently proven rigorously [SP, Qiao, Rychkov '22; van Rees '24; SP, Qiao, van Rees, ongoing]

Universal powers, theory-dependent coefficients

$$\ln Z \;=\; \sum_k a_k\, f_k(\omega)\, \beta^{-k+r} \;+\; O\Big(e^{-\#/\beta}\Big)$$

- powers β^{-k+r}: **universal** (fixed by d)
- coefficients a_k: **theory-dependent** (Wilson coefficients)
- $f_k(\omega)$: **geometry-dependent** (functions of the rotation)

[Bhattacharyya, Lahiri, Loganayagam, Minwalla ’07; Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma ’12; Jensen, Kaminski, Kovtun, Meyer, Ritz, Yarom ’12]

Cardy-like formula: **[Benjamin, Lee, Ooguri, Simmons-Duffin ’23]**

Take $\omega_i \to 1$ term by term, then resum

EFT term $a_k f_k(\omega)\beta^{-k+r}$ $\rightarrow$ $\omega_i \to 1$ term by term $\rightarrow$ Borel-resum $\Rightarrow$ pole

the pole is a controlled *output* of the EFT, not an assumption

(resummation details: story of another time)

What could break it — and why it doesn't

- assumes terms non-perturbative in β do not conspire
- assumes the order of limits — expand, take $\omega_i \to 1$ at each order, then resum

the free scalar *is* fine-tuned (such terms present) — yet the scaling holds.

Checks against free theory and holography

free scalar, all d ✓

large-N $\mathcal{N} = 4$ SYM ✓

$\hookrightarrow$ rotating black hole · grey galaxy · gas

[Kim, Kundu, Lee, Lee, Minwalla, Patel, 2305.08922; Choi, Jain, Kim, Lee, Minwalla, Patel, Krishna, 2409.18178; Choi, Jain, Kim, Kwon, Lee, Minwalla, Patel, Krishna, 2501.17217; Bajaj, Minwalla, Mukherjee, Rahman, Sharma, 2412.06904]

independent computations agree (explicit forms: backup)

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman '25]

Orders deeper: A, B, C, D

$$\log Z = \beta\Big[\, \frac{A(\beta)}{\nu_1\nu_2} + B(\beta)\Big(\tfrac{1}{\nu_1}+\tfrac{1}{\nu_2}\Big) + C(\beta)\Big(\tfrac{\nu_1}{\nu_2}+\tfrac{\nu_2}{\nu_1}\Big) + D(\beta) + O(\nu)\Big]$$

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$A = \beta h$ — the residue from before.

Are the four independent?

[Advant, Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman, Ray — to appear]

They are not independent

$$\boxed{\tfrac{1}{6}A''(\beta) + B'(\beta) + C(\beta) + D(\beta) = \frac{2a}{3}}$$

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For the local, parity-even part. Theories with a gapless $(d-1)$-dimensional sector — free scalar, Maxwell, $\mathcal{N} = 4$ at $\lambda = 0$ — carry extra β^0 and $\log \beta$ pieces outside the EFT and are excluded.

[Advant, Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman, Ray — to appear]

Same constraint, two expansions

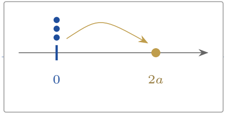
$$\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 2a$$

$+ \cdots$ more equations

$$\xleftarrow[\omega_i \text{ fixed}]{\beta \to 0} \quad \boxed{Z(\beta, \omega_i)} \quad \xrightarrow[\beta \text{ fixed}]{\nu_i \to 0}$$

$$\frac{1}{6}A'' + B' + C + D = \frac{2a}{3}$$

$+ \cdots$ more equations



the same rank deficiency

one rank deficiency, two organizations

Four Weyl invariants generate everything

conformally flat metric, v^μ a conformal Killing vector [Loganayagam '08]

every parity-even local Weyl-invariant scalar is a power series in

$$s := -\tfrac{1}{8}\,\mathrm{Tr}\big(\omega^2\big)\,, \qquad v^2 S, \qquad v^2 B^2, \qquad v^2 \tilde{B}^2$$

$\omega_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ (vorticity) · S the Weyl-covariant Schouten scalar ·
$B_\mu = v^\nu S_{\nu\mu}$ · $\tilde{B}_\mu = \omega_\mu{}^\nu B_\nu$

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and they are graded by how much ν they cost:

$$s = O(1) + O(\nu) \qquad v^2 S = O(\nu) \qquad v^2 B^2 = O(\nu^2) \qquad v^2 \tilde{B}^2 = O(\nu^3)$$

the $\nu \to 0$ counterpart of the derivative expansion we used at $\beta \to 0$

Arbitrary functions in, a constrained answer out

$$\log Z = \beta \int \frac{d\Omega_3}{\left(v^2\right)^2} \sum_{S,\tilde{B}\geq 0} R_{S,\tilde{B}}(s)\ \left(v^2 S\right)^S \left(v^2 \tilde{B}^2\right)^{\tilde{B}}$$

the R's are *arbitrary* — the independent data of the local expansion; terms carrying $v^2 B^2$ are total derivatives away from terms without it

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do every integral:

$$\log Z = \frac{1}{\left(1-\omega_1^2\right)\left(1-\omega_2^2\right)} \sum_{n \geq 0} P_n, \qquad P_n \text{ symmetric, degree } n \text{ in } \left(1-\omega_i^2\right)$$

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the coefficients of the P_n are not all independent.

Identities per order in β

$$\beta^0 \implies \text{the } a\text{-relation}$$

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$$\beta^4 \implies \text{an } N = 4 \text{ relation}$$

$$\vdots$$

One differential identity packaging a whole family of the high-T hierarchy.

What the leading pole says about the spectrum

$$S(\tau, J) = \sqrt{J}\; S_{\text{int}}\Big(\tau/\sqrt{J}\Big)\,\big(1 + o(1)\big) \qquad (d = 3)$$

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

What the leading pole says about the spectrum

$$S(\tau, J) = \sqrt{J}\ S_{\rm int}\Big(\tau/\sqrt{J}\Big)\,\big(1 + o(1)\big) \qquad (d = 3)$$

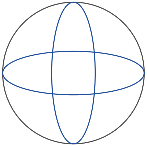
$$\boxed{\text{pole of order } p \implies S \sim J^{\,p/(p+1)}} \qquad \text{here } p = 1$$

regime: τ large, but parametrically below J

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

In $d = 4$ the entropy scales as $(J_1 J_2)^{1/3}$

$$S = (J_1 J_2)^{1/3}\, S_{\text{int}}(\zeta), \qquad \zeta = \frac{\tau}{(J_1 J_2)^{1/3}}$$



$p = 2$

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

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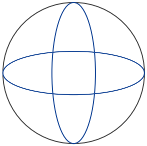


$p = 2$

general d: in the paper

[Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman — 2512.00158]

What the subleading relations do

Leading pole $\longrightarrow$ the leading entropy function.

Subleading functions constrained, not arbitrary

$\longrightarrow$ so is the subleading large-spin data.

[Advant, Anand, Benjamin, Kumar, Minwalla, Mukherjee, SP, Rahaman, Ray — to appear]

Back to general d: unitarity alone is not enough

In $d > 2$, unitarity is weak about how low the twist can go.

Causality (ANEC) hints at more. [Córdova–Diab '17; Manenti, Stergiou, Vichi '19]

What does Z itself force?

Bound A: twist can't be too negative

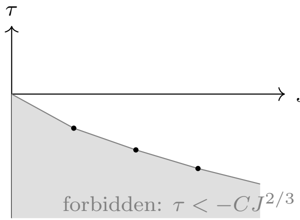
Bound A [SP, Benjamin, . . .]

finiteness of $\log Z$ + the thermal-EFT growth law (no analyticity):

$$\tau \ \gtrsim\ -C\,J^{2/3} \quad (d=4)$$

sharp — saturated by the tower $(\tau, J) = (-n^2, n^3)$.

causality forbids the twist from running off too fast.



[discussion with Shiraz Minwalla, Jeremy Mann]

Bound B: twist positivity $\Delta \geq |J|$

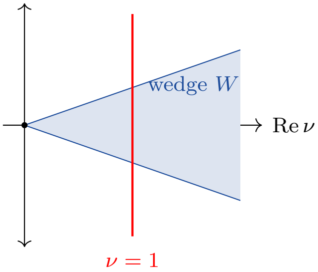
Bound B [SP, …]

wedge zero-freeness of $\log Z$ + $J \to -J$:

$$\Delta \;\geq\; |J|$$

$F_\beta = Z^{1/\beta}$: $F := \lim_{\beta\to\infty} F_\beta = 1$ on the line, then Vitali $\Rightarrow F \equiv 1$ on the wedge $\Rightarrow \inf_n(\tau_n + \nu J_n) = 0$.

strictly stronger than unitarity; matches Córdova–Diab.



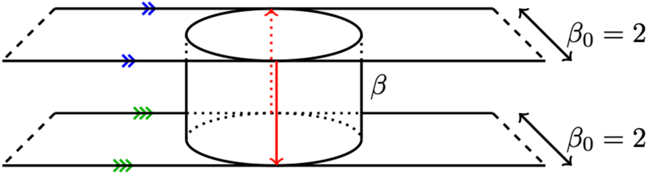
$\nu = 1 - \omega$

Ongoing: averaged OPE data at large spin

averaged thermal one-point coefficient × density of states:

$$\log\left[\,\rho(\tau, J)\,\overline{b^2}\,\right] \;\sim\; J^{1/3}\, s_b\!\left(\tau/J^{1/3}\right),$$

and the *same* $J^{1/3}$ scaling holds for the three-point coefficient of highly-spinning symmetric-traceless operators.



glue two thermal copies along a cylinder (β); complexify the gluing → large spin

[geometry: Benjamin, Lee, Ooguri, Simmons-Duffin]

How much universality is possible?

How much universality is possible?

In 2d there is essentially one Weyl invariant, so the theory-dependent function collapses to a number.

[. . . , Kusuki; Benjamin, Shao, Ooguri, Wang; SP, Qiao, van Rees]

the all-orders statement is about $h(\beta)$; microcanonical density is true non-perturbatively, and needs a twist gap [SP, Qiao, van Rees — 2505.02897]

Three takeaways

2512.00158

1. $d > 2$: semi-universal — a universal shape, a theory-dependent function. (physics argument; checked in gravity & free theory)

2. 2d: super-universal — one number c, to all orders. (theorem)

3. consequences: spectral bounds *beyond* unitarity; averaged OPE data; dense large-spin spectra.

Three takeaways

ongoing

1. $Z \to a$: a local thermal EFT does not generate everything — and the first combination it misses fixes the anomaly.
   (every even d; nothing in odd d)

2. Semi-universality at leading *and* subleading order — the subleading functions are not independent.

3. consequences: an infinite hierarchy of relations; a_d in every even d; subleading large-spin data constrained.

Thank you

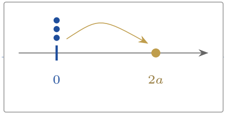
$$\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 2a$$

$+ \cdots$ more equations

$$\xleftarrow[\omega_i \text{ fixed}]{\beta \to 0} \boxed{Z(\beta, \omega_i)} \xrightarrow[\beta \text{ fixed}]{\nu_i \to 0}$$

$$\frac{1}{6}A'' + B' + C + D = \frac{2a}{3}$$

$+ \cdots$ more equations



the same rank deficiency

one rank deficiency, two organizations

with A. Advant, H. Anand, N. Benjamin, V. Kumar,
S. Minwalla, J. Mukherjee, A. Rahaman, P. Ray

2512.00158 · to appear

Backup slides

K1 — where does Z converge?

Unitarity alone gives the diamond $|\omega_1| + |\omega_2| < 1$. For (j_L, j_R) with $j_L j_R \neq 0$, $\Delta \geq 2 + j_L + j_R$ gives

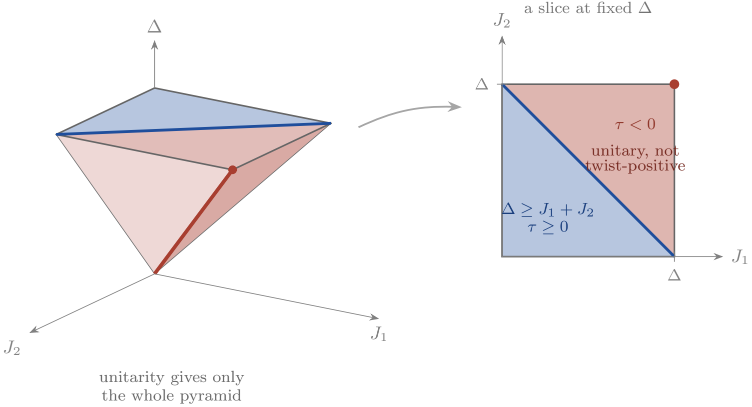
$$\Delta - \omega\cdot J \;\geq\; 2 + j_L\,(1 - |\omega_1 + \omega_2|) + j_R\,(1 - |\omega_1 - \omega_2|)\,,$$

and $\max(|\omega_1 + \omega_2|, |\omega_1 - \omega_2|) = |\omega_1| + |\omega_2|$.

Sharp: the chiral tower $(j_L, j_R) = (J, 0)$, $\Delta = 1 + J$, $(J_1, J_2) = (J, J)$ saturates the unitarity bound. Note $\tau = 1 - J < 0$ — these are exactly the negative-twist states.

The square $|\omega_i| < 1$, and hence the corner, needs an ANEC-type bound or something like that beyond unitarity. This is assumed, not proved. Free-field constructions violating it exist, and there $h(\beta)$ diverges.

K2 — where states can sit in (Δ, J_1, J_2)



Unitarity confines states to the pyramid $\Delta > \max(J_1, J_2)$. Twist positivity $\Delta \geq J_1 + J_2$ is strictly stronger.

K3 — general even d, and the checks

$$a_d = \frac{(-1)^{d/2}(d+2)}{2\,\Gamma\left(\frac{d+2}{2}\right)} \log Z\Big(\beta,\ \omega_k = e^{2\pi\mathrm{i}k/(d+2)}\Big)\Big|_{\beta^1}, \qquad k = 1, \ldots, \tfrac{d}{2}$$

$d = 2$	prefactor -2	$a_2 = c/6$
$d = 4$	prefactor $3/2$	gives $\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)$
$d = 6$	prefactor $-2/3$	scalar $\frac{1}{9072}$, Dirac $\frac{191}{45360}$, hol. $\frac{\pi^2}{48}\,\ell^5/G_N$
$d = 8$	prefactor $5/24$	checked in gravity

In $d = 4$ the four structures evaluate to $(1, -1, -1, 1)$ at the cube roots and the prefactor $(1-\omega_1^2)(1-\omega_2^2)$ to 3 — which is why $3/2$ gives $\frac{1}{2}$.

K4 — how many relations, order by order

At order β^{2N-3}:

$$\dim \mathrm{Rel}_N = \sum_{k=1}^{N} \left(\lfloor \tfrac{k}{2} \rfloor - \lfloor \tfrac{k}{3} \rfloor \right) = \left\lfloor \frac{N^2}{12} + \frac{N}{6} + \frac{1}{3} \right\rfloor$$

N	0	1	2	3	4	5	6	7	8	9	10	11
structures	1	2	4	6	9	12	16	20	25	30	36	42
rank	1	2	3	5	7	9	12	15	18	22	26	30
relations	0	0	**1**	1	2	3	4	5	7	8	10	12

Only $N = 2$ has an inhomogeneous source. Every relation at $N \neq 2$ has right-hand side zero.

K5 — the next order, $O(\beta^3)$

Seventeen six-derivative invariants, evaluated on the same background, span a *five*-dimensional subspace of the six angular structures

$$\left\{1,\ \sum\omega^2,\ \sum\omega^4,\ \omega_1^2\omega_2^2,\ \sum\omega^6,\ \omega_1^2\omega_2^4+\omega_1^4\omega_2^2\right\}.$$

Writing a general term as $a_1 + a_2\sum\omega^2 + \cdots + a_6(\dots)$, every one obeys

$$a_1 - a_3 - a_5 + a_6 = 0.$$

Right-hand side *zero* — no anomaly source at this order. Verified for the holographic $\log Z$, where each coefficient separately has α' and G_N corrections but the combination vanishes exactly.

K6 — the relation in fluid variables

With $s_1 = 1 - \omega_1^2$, $s_2 = 1 - \omega_2^2$, write the local contribution through angular degree two as

$$\log Z_{\text{loc}} = \frac{A_0(\beta)}{s_1 s_2} + A_1(\beta)\left(\tfrac{1}{s_1} + \tfrac{1}{s_2}\right) + A_{20}(\beta)\left(\tfrac{s_1}{s_2} + \tfrac{s_2}{s_1}\right) + A_{11}(\beta) + O(s_i).$$

Four coefficient functions, generated by three underlying fluid functions. Eliminating them:

$$L(L-1)A_0 + 6(L-1)A_1 + 6\,(A_{20} + A_{11}) = 0, \qquad L = \tfrac{1}{2}\,(\beta\partial_\beta + 3)\,.$$

L acts on β^{2N-3} by multiplication by N. In the operator-counting convention the right-hand side is $4a\beta$, not 0, with a the anomaly.

Changing variables to $\nu_i = \beta(1 - \omega_i)$ gives $\frac{1}{6}A'' + B' + C + D = \frac{2a}{3}$.

K7 — why roots of unity, and what is being evaluated

At $(\omega_1^2, \omega_2^2) = (\zeta, \bar{\zeta})$ with $\zeta = e^{2\pi \mathrm{i}/3}$:

$$\sum \omega^2 = -1, \quad \sum \omega^4 = -1, \quad \omega_1^2 \omega_2^2 = 1, \quad (1-\omega_1^2)(1-\omega_2^2) = 3,$$

so the β^1 Laurent coefficient is $\frac{1}{3}(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)$ and $\frac{3}{2}\times$ that is a.

These points have $|\omega| = 1$, outside every domain in which the trace converges. What is being evaluated is the Laurent coefficient — a rational function of ω — not Z itself.

The evaluation is a universal relation only when $N \equiv 2 \pmod 3$; at other orders an allowed moment survives at the same point, so it gives nothing.

K8 — the same argument in 2d

$$\log \mathrm{Tr}\ e^{-\beta(H-\omega J)} \xrightarrow[\omega\to 1]{} \frac{L}{\beta\,(1-\omega^2)}\,\frac{2\pi c}{12}$$

Path integral on the twisted $S^1_\beta \times S^1_L$:

$$ds^2 = d\tau^2 + dx^2, \qquad (\tau, x) \sim (\tau+\beta,\ x - \mathrm{i}\omega\beta).$$

Untwist with $x' = x + \mathrm{i}\omega\tau$:

$$ds^2 = \left(1-\omega^2\right)\left(d\tau - \tfrac{\mathrm{i}\omega}{1-\omega^2}dx'\right)^2 + \frac{dx'^2}{1-\omega^2}, \qquad (\tau, x') \sim (\tau+\beta, x').$$

So $\beta \to \left(1-\omega^2\right)^{1/2}\beta$ and $L \to L\left(1-\omega^2\right)^{-1/2}$, hence $\frac{L}{\beta} \to \frac{L}{\beta(1-\omega^2)}$.

The twist is a rescaling of the torus: the comoving temperature rises (β shrinks) and the comoving volume grows (L stretches). Cardy on the rescaled torus produces the pole, with residue $2\pi c/12$.

K9 — hot-equator geometry ($d = 3$)

On S^2, $ds^2 = d\tau^2 + d\theta^2 + \sin^2\theta\, d\phi^2$ with $(\tau, \phi) \sim (\tau + \beta, \phi - \mathrm{i}\omega\beta)$. Undo the twist with $\phi' = \phi + \mathrm{i}\omega\tau$ and Weyl-rescale by the resulting $g_{\tau\tau} = (1 - \omega^2 \sin^2\theta)$:

$$ds^2_{\text{weyl}} = \left(d\tau - \tfrac{\mathrm{i}\omega \sin^2\theta}{1-\omega^2\sin^2\theta} d\phi'\right)^2 + \tfrac{d\theta^2}{1-\omega^2\sin^2\theta} + \tfrac{\sin^2\theta\ d\phi'^2}{(1-\omega^2\sin^2\theta)^2}.$$

Singular at the equator. Setting $\theta = \pi/2$:

$$ds^2_{\text{eff}} = \left(d\tau - \tfrac{\mathrm{i}\omega}{1-\omega^2} d\phi'\right)^2 + \tfrac{d\phi'^2}{(1-\omega^2)^2}.$$

The comoving temperature diverges and the angular proper volume diverges as $\omega \to 1$ — both, which is why the high-T EFT controls the limit. But in $d = 3$ the local density is not universal — it is $a_0\gamma^3 + a_1(\gamma')^2/\gamma + \ldots$, and only the coarse-grained δ-function coefficient is fixed.

K10 — why $d = 4$ has no equator

Write S^3 as $d\Omega_3^2 = \frac{dx^2}{4x(1-x)} + x\,d\phi_1^2 + (1-x)\,d\phi_2^2$. With $v = \omega_1 x\,d\phi_1 + \omega_2(1-x)\,d\phi_2$,

$$u \equiv 1 - |v|^2 = 1 - \omega_1^2 x - \omega_2^2(1-x) = s_2 + (s_1 - s_2)x, \qquad s_i = 1 - \omega_i^2.$$

For $\omega_1 = \omega_2$ this is constant over the whole sphere. For unequal velocities it interpolates between s_2 and s_1 — both going to zero.

So there is no localisation in $d = 4$ with both planes scaled: $\gamma = 1/\sqrt{u}$ blows up everywhere at once — the comoving temperature rises over the *whole* sphere, and the comoving volume with it — and $L \propto \gamma^4$ is the correct density. Integrating,

$$\int_0^1 dx\ \gamma^4 = \frac{1}{s_1 s_2} = \frac{\beta^2}{4\nu_1\nu_2}\left(1 + O(\nu)\right),$$

which against a finite volume gives the double pole directly.

$L \propto \gamma^d$ holds precisely when $d = 2n$ and all n potentials are scaled.

K11 — the residue for a free scalar

$$h_{d=4}(\beta) = \sum_{q\geq 1} \frac{\operatorname{csch}(q\beta)}{2q^3\beta^2}$$

$$= \frac{\pi^4}{180\beta^3} - \frac{\pi^2}{72\beta} + \frac{3\zeta(3)}{8\pi^2} - \frac{7\beta}{1440} + \sum_{m\geq 1}\sum_{\ell | m} \frac{(-1)^{\ell+1}}{\pi^2\ell^3}\, e^{-2\pi^2 m/\beta}$$

$$h_{d=6}(\beta) = \sum_{q\geq 1} \frac{\operatorname{csch}^2(q\beta)}{4q^4\beta^3}, \qquad h_{d=3}(\beta) = \sum_{q\geq 1} \frac{\operatorname{csch}(q\beta/2)}{2q^2\beta}$$

The $d = 3$ series is asymptotic and carries a $\log\beta$ from the gapless mode. The β^0 constant in $d = 4$ is the gapless-sector contribution — the same piece that puts the free scalar outside the differential relation.

K12 — the residue for a rotating black hole

$$h_{\mathrm{BH}}(\beta) = \frac{N^2}{64}\left[\frac{2\pi}{\beta^3}\left(2\beta^2+\pi^2\right)^{3/2} + \frac{2\pi^4}{\beta^3} - \frac{10\pi^2}{\beta} - \beta\right]$$

(the $2\pi/\beta^3$ multiplies only the $(2\beta^2+\pi^2)^{3/2}$ term)

Small β: $h_{\mathrm{BH}} \to N^2\pi^4/16\beta^3$, convergent, with no non-perturbative corrections — in contrast to the free scalar.

Parametrically, with r_+ the horizon radius,

$$\frac{h_{\mathrm{BH}}(r_+)}{N^2} = \frac{\pi\left(r_+^2-1\right)^3}{8\,r_+\left(2r_+^2-1\right)}.$$

K13 — the three holographic phases

As a function of $\tau_s = \tau/(N^2 J_1 J_2)^{1/3}$, at large N and rapid rotation:

- **Black hole:** $T \geq 1/2\pi$, i.e. $\tau_s \geq (1/2)^{1/3}$.
- **Grey galaxy:** between the two. Here $S_{\text{int}} = 2\pi\tau$ exactly — linear.
- **Gas:** 10d supergravity on $\text{AdS}_5 \times S^5$; the free-gas approximation needs $\sqrt{J_1 J_2} \ll N^6$.

K14 — the entropy law in general d

Scaling p of the $\lfloor d/2 \rfloor$ angular momenta:

$$\frac{S}{(J_1 \cdots J_p)^{1/(p+1)}} = S_{\text{int}}\left(\frac{\tau}{(J_1 \cdots J_p)^{1/(p+1)}},\ \ldots\right)$$

with $\lfloor d/2 \rfloor + 1 - p$ scaled charges in all.

$d = 3$, $p = 1$: $\sqrt{J}$. $\qquad$ $d = 4$, $p = 2$: $(J_1 J_2)^{1/3}$, one scaled variable.

Asymmetric case: $d = 4$ with only one potential scaled has *two* scaled variables, $S/\sqrt{J_1} = S_{\text{int}}\big(\tau/\sqrt{J_1},\ J_2/\sqrt{J_1}\big)$.

All of these are the exponent law $J^{p/(p+1)}$ at different p — as is the negative-twist bound, one-sided.

K15 — averaged OPE data: the square root

Complexify the gluing, $\theta = \mathrm{i}\eta$ with $\eta = \beta\omega$. The physical data is the conformal monodromy around the hot-spot cycle, and the map to it is nonlinear. At the endpoint

$$\beta_{\mathrm{rel}} = B_\star(\beta) + 2\sqrt{\nu} + O(\nu), \qquad \nu_{\mathrm{rel}} = 4\sqrt{\nu} + O(\nu),$$

$$B_\star(\beta) = \operatorname{arccosh}\!\left(2e^{2\beta} - 1\right).$$

So a simple pole in the *relative* variable becomes a square-root singularity in ν: $\log Z_{\mathrm{1pt}} \sim C(\beta)/\sqrt{\beta(1-\omega)}$.

Inverting, $p = \frac{1}{2}$ and $p/(p+1) = \frac{1}{3}$:

$$\log\!\left[\rho(\tau, J)\,\overline{b^2}\right] \sim J^{1/3} s_b\!\left(\tau/J^{1/3}\right).$$

The exponent is structural and d-independent; the residues are theory-dependent.

K16 — three heavy operators, and a spin triangle

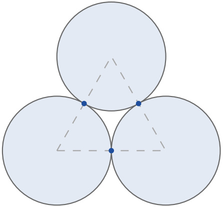
Three mutually tangent hot spots. All three reach their null simultaneously: $\nu_{\rm rel}^{(ij)} = 2\sqrt{2(\nu_i + \nu_j)}$, so

$$\log Z_c \sim \sum_{(ij)} \frac{D_{ij}}{\sqrt{\nu_i + \nu_j}}.$$

Inverting,

$$\log A \sim \tfrac{3}{2} \sum_{(ij)} D_{ij}^{2/3} \left(J_i + J_j - J_k\right)^{1/3} + \sum_i \beta_i^* \tau_i.$$

The saddle exists only if $J_i + J_j > J_k$.



K17 — the 2d statement, precisely

Theorem [SP, Qiao, van Rees — 2505.02897]

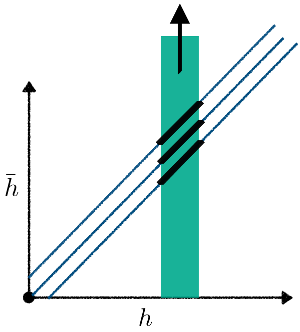
Unitary, $c > 1$, twist gap $\Rightarrow$ the smoothed spin-J density is fixed by c, to all orders in $1/J$.

$$\lim_{J\to\infty} \frac{\int_\Omega d\tau\ \varphi(\tau)\,\rho_J(\tau)}{\frac{1}{\sqrt{2J}} e^{4\pi\sqrt{\frac{c-1}{24}J}}} = \int_\Omega d\tau\ \varphi(\tau)\,\rho_0(\tau)$$

The limit *exists* — no $\limsup / \liminf$, no averaging over spins. Twist accumulates in every window above $\frac{c-1}{12}$, and the maximal gap between spin-J operators falls at least as fast as $J^{-1/4+\epsilon}$.

Distinct from the all-orders statement about $h(\beta)$, which follows because in 2d there is essentially one Weyl invariant.

K18 — the 2d universal profile



With $A = \frac{c-1}{24}$,

$$\rho_0(h) = \sqrt{\frac{2}{h-A}}\left[\cosh\left(4\pi\sqrt{A(h-A)}\right) - \cosh\left(4\pi\sqrt{(A-1)(h-A)}\right)\right]\Theta(h-A)$$

Fixed by the single number A. Above threshold the windowed density is nonzero; below it, growth is strictly slower.

The mechanism: modular covariance of the vacuum-reduced partition function, analysed as $\beta_R \to 0$, then projected onto fixed spin. The twist gap is what makes the vacuum module dominate the dual channel.

K19 — spin refinement

The statement can be refined by inserting a spin fugacity at a root of unity:

$$\log \operatorname{Tr}\, e^{-\beta(D-\omega_i J_i)+2\pi \mathrm{i}(p_i/q_i)J_i} \;\sim\; \frac{1}{q}\,\frac{h(q\beta)}{\prod_i (1-\omega_i^2)}.$$

This constrains the spectrum at fixed residue class of J, not merely on average.

Proved in 2d by modular transformations; argued in $d > 2$ by a folding trick. Caveat: if the fugacity has a fixed point on S^{d-1}, defect contributions arise, and one must scale all $\omega_i \to 1$ together.