Carving Out the Space of 3d CFTs with Analytic-Functional Bootstrap

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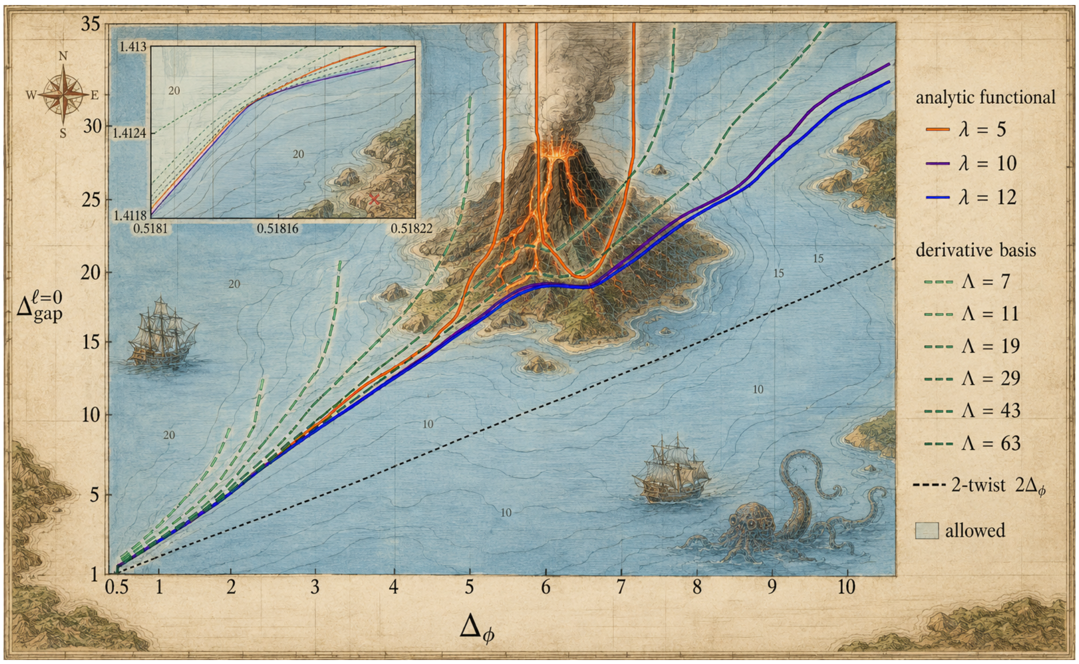
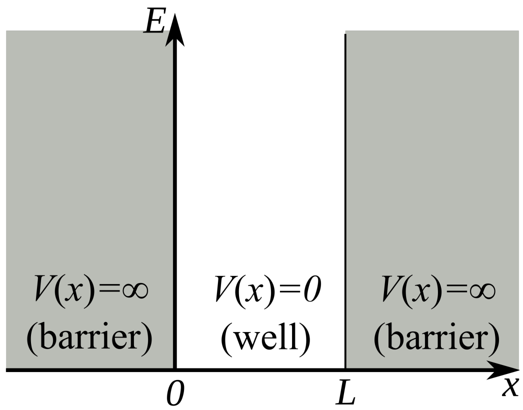


Based on 2608.06497(Today), w/Kausik Ghosh

Particle in a box

We are trying to solve the ground state wave function and energy:



$L = 1$, $H = P^2$,
$\psi\ (x) = \sqrt{2}\ \sin n\pi x$,
$E_n = n^2\ \pi^2$

Power Expansion

```
ψ[x_] := Sum[a[n] (x - 1 / 2) ^ n, {n, 0, Λ}];
Λ = 4;
FindMinimum[{Integrate[ψ[x] (-D[ψ[x], {x, 2}]), {x, 0, 1}],
  Integrate[ψ[x] ^ 2, {x, 0, 1}] == 1 &&
   ψ[0] == 0 &&
   ψ[1] == 0}, a /@ Range[0, Λ]]
```

$\{9.86975, \{a[0] \to 1.4129, a[1] \to 2.18469 \times 10^{-11}, a[2] \to -6.89917, a[3] \to -8.73884 \times 10^{-11}, a[4] \to 4.99035\}\}$

Particle in a box

We are not doing this because we all know the following expansion:

$$\psi\ (x)\ = \sum_{i=0}^{\infty} a_n \,\mathtt{sin}\, n\,\pi\, x$$

- Hamiltonian.
- Boundary condition.
- Orthogonality.
- n=1 gives the optimal ground state.

Conformal Bootstrap: overview

$$\sum_{\Delta,\ell} a_{\Delta,\ell} F_{\Delta,\ell}(z, \bar{z}) = 0 \,, \quad a_{\Delta,\ell} \geq 0$$

- To discretize the crossing equation, we do the power expansion

$$F_{\Delta,\ell}(z, \bar{z}) = \sum_{m,n} \frac{(z - 1/2)^m (\bar{z} - 1/2)^n}{m!\, n!} \partial^m \bar{\partial}^n F_{\Delta,\ell}(z, \bar{z}) \Big|_{z=\bar{z}=1/2}$$

- PMP ⇒SDPB (version 1.0: 2015, 2.0: 2019, 3.0: 2024, ...)
- Patches: Navigator(2021), Skydiving(2023), Optimal Interpolation(2025),...
- Achievement: $(\Delta_\sigma, \Delta_\epsilon) = (0.518148806(24), 1.41262528(29))$ (2411.15300)
  53 crossing equations, derivative order up to Λ=43($m + n \leq \Lambda$)

Other basis?

- Conformal Casimir operator.
- Boundary condition.
- Orthogonality.
- n=0 gives the optimal bound.

1D CFT: Analytic Functional (starting from Mazac 2016) :
$$F_{\Delta}^{\pm}(z) = \sum_n \alpha_n^{\pm}(\Delta) F_{\Delta_n}^{\pm}(z) + \sum_n \beta_n^{\pm}(\Delta) \partial F_{\Delta_n}^{\pm}(z).$$

Main conclusion

- In 3d single-correlator Conformal bootstrap, analytic functional significantly outperform the derivative basis. (≥ 3 orders of magnitude)
- They both converge in a power-law, but analytic functional has a better power.
- Do something...

1d analytic functional(s)

$$G_\Delta(z) = z^\Delta\, {}_2F_1(\Delta, \Delta; 2\Delta; z),$$
$$F^\pm_\Delta(z) = z^{-2\Delta_\phi} G_\Delta(z) \pm (1-z)^{-2\Delta_\phi} G_\Delta(1-z)$$
$$F^\pm_\Delta(z) = \sum_n \alpha^\pm_n(\Delta) F^\pm_{\Delta_n}(z) + \sum_n \beta^\pm_n(\Delta)\, \partial F^\pm_{\Delta_n}(z).$$

- Orthogonality.

$$\Delta^B_n = 2\Delta_\phi + 2n, \quad \Delta^F_n = 2\Delta_\phi + 2n + 1.$$
$$\alpha^{\pm,F}_n\left(\Delta^F_m\right) = \delta_{n,m}, \qquad \partial\alpha^{\pm,F}_n\left(\Delta^F_m\right) = 0,$$
$$\beta^{\pm,F}_n\left(\Delta^F_m\right) = 0, \qquad \partial\beta^{\pm,F}_n\left(\Delta^F_m\right) = \delta_{n,m}$$
$$\alpha^{\pm,B}_n\left(\Delta^B_m\right) = \delta_{n,m}, \qquad \partial\alpha^{\pm,B}_n\left(\Delta^B_m\right) = -c^\pm_n\, \delta_{m,0}$$
$$\beta^{\pm,B}_n\left(\Delta^B_m\right) = 0, \qquad \partial\beta^{\pm,B}_n\left(\Delta^B_m\right) = \delta_{n,m} - d^\pm_n\, \delta_{m,0},$$

1d analytic functional(s)

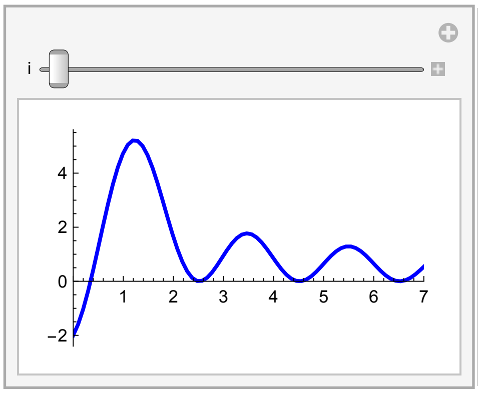
$$\omega_{\pm}\,(\Delta \mid \Delta_{\phi}) = \int_{1}^{\infty} \frac{\mathrm{d}\,z}{\pi}\, h_{\pm}\,(z)\, I_z\, F_{\pm,\Delta}\,(z \mid \Delta_{\phi})\;, \quad I_z\, F\,(z) := \lim_{\epsilon \to 0^+} \frac{F\,(z + i\,\epsilon) - F\,(z - i\,\epsilon)}{2\, i}\,.$$

$$\omega_n\,(\Delta) \equiv \omega_n\,[F_{\Delta}] = 2\,\sin^2\left(\frac{\pi}{2}\left(\Delta - \Delta_n^{B/F}\right)\right) \int_0^1 \mathrm{d}\,z\, g_n\,(z)\, \frac{G_{\Delta}\,(z)}{z^{2\,\Delta_{\phi}}}$$

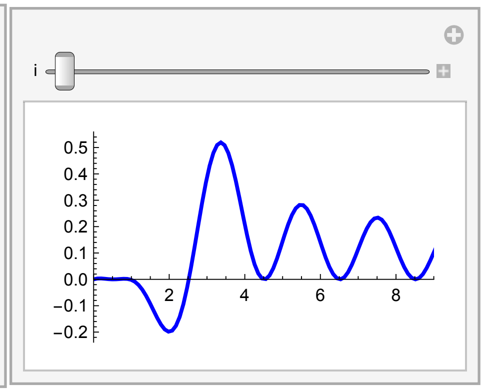
$$f^{F}_{\beta,0}(z) = \frac{-(2\,z-1)}{w^{3/2}} \left({}_3\tilde{F}_2\left(-\frac{1}{2}, \frac{3}{2}, 2\,\Delta_{\phi} + \frac{3}{2}; \Delta_{\phi}+1, \Delta_{\phi}+2; -\frac{1}{4\,w}\right) + \frac{9}{16\,w}\; {}_3\tilde{F}_2\left(\frac{1}{2}, \frac{5}{2}, 2\,\Delta_{\phi} + \frac{5}{2}; \Delta_{\phi}+2, \Delta_{\phi}+3; -\frac{1}{4\,w}\right)\right) \frac{2^{-8\,\Delta\phi - 5}\,\Gamma(4\,\Delta\phi + 4)}{\Gamma(\Delta\phi + 1)^2},$$

$$w = z\,(1 - z), \qquad g\,(z) := -(1 - z)\,{}^\wedge\,(2\,\Delta\phi - 2) * f\left(\frac{1}{1 - z}\right)$$

α functional



β functional



Good basis for numerics?

Given a basis of functional Ω_i:

- Finiteness: $\Omega_i\,(\Delta, l) < \infty$, above unitarity bound
- Swapping: $\Omega_i\left(\sum_{\Delta,l} a_{\Delta,l}\, F_{\Delta,l}\,(z, \bar{z})\right) = \sum_{\Delta,l} a_{\Delta,l}\, \Omega_i\,(F_{\Delta,l}(z, \bar{z}))$
- Completeness: $\sum_{\Delta,l} a_{\Delta,l}\, F_{\Delta,l}\,(z, \bar{z}) = 0\,, \quad \Leftrightarrow \quad \Omega_i\left(\sum_{\Delta,l} a_{\Delta,l}\, F_{\Delta,l}\,(z, \bar{z})\right) = 0, \forall\, i$
- Positivity: $\exists\, b_1, \ldots, b_{N^*}$, such that, $\sum_{j=1}^{N^*} b_j\, \Omega_j\,(\Delta, l) \geq 0$, as long as $\Delta > \Delta^*$ OR $l > l^*$
- Computability

Higher d

$$F_{\Delta,\ell}(z,\bar{z}\mid\Delta_\phi)=\sum_{n,j}\left(\alpha^{-}_{n,j}(\Delta)\,F_{\Delta_n,j}(z,\bar{z}\mid\Delta_\phi)+\beta^{-}_{n,j}(\Delta)\,\partial F_{\Delta_n,j}(z,\bar{z}\mid\Delta_\phi)\right)$$

So far, this doesn’t leads to functionals useful for numerics!

	Derivative	ω^F, ω^B (1d)	$\nu_{i,j}$ & $\mu_{i,j}$	**Product Functional**
Finiteness	✓	✓	✓	✓
Swapping	✓	✓	✓	✓
Completeness	✓	✓	✓	✓
Positivity	✓	✓	✗	✓
Computability	✓	✓	✓	✓

- Identify the correct basis in higher d
- Numerically implementation

Product functional

$$(\omega^{-}\otimes\omega^{+})\,(F_{\Delta,\ell}(z,\bar{z})) := 2\int_{++}\frac{dz\,d\bar{z}}{\pi^2}\,h_{-}(z)\,h_{+}(\bar{z})\,[I_z I_{\bar{z}} F_{\Delta,\ell}(z,\bar{z}) + I_z I_{\bar{z}} F_{\Delta,\ell}(z,1-\bar{z})]$$

$$\omega_{\pm}(\Delta\mid\Delta_\phi) = \int_1^\infty\frac{dz}{\pi}\,h_{\pm}(z)\,I_z F_{\pm,\Delta}(z\mid\Delta_\phi)\,,\quad I_z F(z) := \lim_{\epsilon\to0^+}\frac{F(z+i\epsilon)-F(z-i\epsilon)}{2i}\,.$$

$$\begin{aligned}\mathcal{F}_{\Delta,l}(z,\bar{z}) = \textstyle\sum_{m,n}\, &[(\alpha_m^-\otimes\alpha_n^+)(\Delta,l)\times\tfrac{1}{2}(F^-{}_{\Delta_m}(z)F^+{}_{\Delta_n}(\bar{z})+F^-{}_{\Delta_m}(\bar{z})F^+{}_{\Delta_n}(z))\\ &+ (\beta_m^-\otimes\alpha_n^+)(\Delta,l)\times\tfrac{1}{2}(\partial F^-{}_{\Delta_m}(z)F^+{}_{\Delta_n}(\bar{z})+\partial F^-{}_{\Delta_m}(\bar{z})F^+{}_{\Delta_n}(z))\\ &+ (\alpha_m^-\otimes\beta_n^+)(\Delta,l)\times\tfrac{1}{2}(F^-{}_{\Delta_m}(z)\partial F^+{}_{\Delta_n}(\bar{z})+F^-{}_{\Delta_m}(\bar{z})\partial F^+{}_{\Delta_n}(z))\\ &+ (\beta_m^-\otimes\beta_n^+)(\Delta,l)\times\tfrac{1}{2}(\partial F^-{}_{\Delta_m}(z)\partial F^+{}_{\Delta_n}(\bar{z})+\partial F^-{}_{\Delta_m}(\bar{z})\partial F^+{}_{\Delta_n}(z))]\end{aligned}$$

Product functional in 2d

$$\mathcal{F}_{\Delta,l}(z,\bar{z}) = \sum_{m,n} [(\alpha_m^- \otimes \alpha_n^+)(\Delta,l) \times \tfrac{1}{2}(F^-{}_{\Delta_m}(z) F^+{}_{\Delta_n}(\bar{z}) + F^-{}_{\Delta_m}(\bar{z}) F^+{}_{\Delta_n}(z))$$

$$+ (\beta_m^- \otimes \alpha_n^+)(\Delta,l) \times \tfrac{1}{2}(\partial F^-{}_{\Delta_m}(z) F^+{}_{\Delta_n}(\bar{z}) + \partial F^-{}_{\Delta_m}(\bar{z}) F^+{}_{\Delta_n}(z))$$

$$+ (\alpha_m^- \otimes \beta_n^+)(\Delta,l) \times \tfrac{1}{2}(F^-{}_{\Delta_m}(z) \partial F^+{}_{\Delta_n}(\bar{z}) + F^-{}_{\Delta_m}(\bar{z}) \partial F^+{}_{\Delta_n}(z))$$

$$+ (\beta_m^- \otimes \beta_n^+)(\Delta,l) \times \tfrac{1}{2}(\partial F^-{}_{\Delta_m}(z) \partial F^+{}_{\Delta_n}(\bar{z}) + \partial F^-{}_{\Delta_m}(\bar{z}) \partial F^+{}_{\Delta_n}(z))]$$

$$G_{\Delta,\ell}^{d=2}(z,\bar{z}) = \frac{1}{2}(k_\tau(z)\, k_\rho(\bar{z}) + k_\tau(\bar{z})\, k_\rho(z)), \quad \tau = \Delta + \ell, \quad \rho = \Delta - \ell$$

$$k_h(z) := z^{\frac{h}{2}}\, {}_2F_1\left(\frac{h}{2}, \frac{h}{2}, h; z\right) = G_{h/2}^{d=1}(z)$$

$$F_{\Delta,\ell}^{d=2}(z, \bar{z} \mid \Delta_\phi) = \frac{1}{4}\left(F_\tau^-(z \mid \Delta_\phi)\, F_\rho^+(\bar{z} \mid \Delta_\phi) + F_\rho^-(z \mid \Delta_\phi)\, F_\tau^+(\bar{z} \mid \Delta_\phi) + (z \leftrightarrow \bar{z})\right)$$

$$1\,D: \; F_\Delta^\pm(z) = \sum_n \alpha_n^\pm(\Delta)\, F_{\Delta_n}^\pm(z) + \sum_n \beta_n^\pm(\Delta)\, \partial F_{\Delta_n}^\pm(z).$$

$$(\omega_1^- \otimes \omega_2^+)\left(F_{\Delta,\ell}^{d=2}(z, \bar{z} \mid \Delta_\phi)\right) := \omega_1^- \otimes \omega_2^+(\Delta, \ell \mid \Delta_\phi) = \frac{1}{2}(\omega_1^-(\tau)\, \omega_2^+(\rho) + \omega_1^-(\rho)\, \omega_2^+(\tau))$$

$$\omega(h) = \omega\left(\frac{h}{2} \,\middle|\, \frac{\Delta_\phi}{2}\right), \quad \tau = \Delta - l, \quad \rho = \Delta + l$$

3d : Dimensional reduction

$$G^{d}_{\Delta,\ell}(z,\bar{z}) = \sum_{n=0}^{\infty} \sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\, G^{d-1}_{\Delta+2n,j}(z,\bar{z})$$ (**Hogervorst, 2016**)

$$F^{d}_{\Delta,\ell}(z,\bar{z}) = \sum_{n=0}^{\infty} \sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\, F^{d-1}_{\Delta+2n,j}(z,\bar{z})$$

$$(\omega_1^{-}\otimes\omega_2^{+})\left(F^{d=3}_{\Delta,\ell}(z,\bar{z}\mid\Delta_\phi)\right) = \sum_{n=0}^{\infty} \sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\,(\omega_1^{-}\otimes\omega_2^{+})\left(F^{d=2}_{\Delta,\ell}(z,\bar{z}\mid\Delta_\phi)\right)$$

with $(\omega_1^{-}\otimes\omega_2^{+})\left(F^{d=2}_{\Delta,\ell}(z,\bar{z}\mid\Delta_\phi)\right) := \omega_1^{-}\otimes\omega_2^{+}(\Delta,\ell\mid\Delta_\phi) = \frac{1}{2}\left(\omega_1^{-}(\tau)\,\omega_2^{+}(\rho) + \omega_1^{-}(\rho)\,\omega_2^{+}(\tau)\right)$

Challenge 1: asymptotics

$$\Omega^{\otimes,\mathbb{d}=3}(\Delta,\ell) = \sum_{n=0}^{\infty} \sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\,\Omega^{\otimes,\mathbb{d}=2}(\Delta+2n, j)$$

$$\sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\,\Omega^{\otimes,\mathbb{d}=2}(\Delta+2n, j) \sim \frac{f(\Delta, l, \Delta_\phi)}{(n+\Delta/2)^{6+4\Delta_\phi}}$$

$f(\Delta, l, \Delta_\phi)$ grows in power law w.r.t Δ, l

$$\sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\,\Omega^{\otimes,\mathbb{d}=2}(\Delta+2n, j) = \sum_{i=0}^{\infty} \frac{f_i(\Delta, l, \Delta_\phi)}{(n+\Delta/2)^{6+4\Delta_\phi+i}}$$

$$\Omega^{\otimes,\mathbb{d}=3}(\Delta,\ell) = \sum_{n=0}^{n_{\max}} \sum_{j=\ell,\ell-2,\ldots,\ell \bmod 2} A_{n,j}(\Delta,\ell)\,\Omega^{\otimes,\mathbb{d}=2}(\Delta+2n, j)$$

$$+ \sum_{n=n_{\max}+1}^{\infty} \sum_{i=0}^{i_\Lambda} \frac{f_i(\Delta, l, \Delta_\phi)}{(n+\Delta/2)^{6+4\Delta_\phi+i}} + O\left(\frac{1}{n_{\max}^{6+4\Delta_\phi+i_\Lambda}}\right)$$

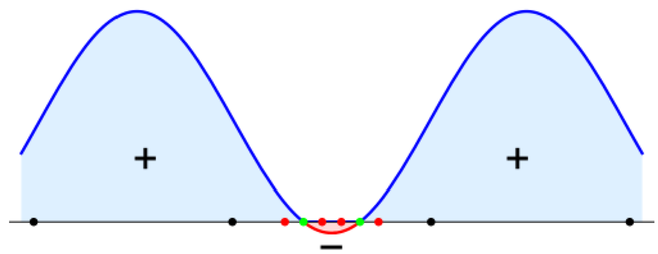
$$\sum_{n=n_{\max}+1}^{\infty} \sum_{i=0}^{i_\Lambda} \frac{f_i(\Delta, l, \Delta_\phi)}{(n+\Delta/2)^{6+4\Delta_\phi+i}} = \sum_{i=0}^{i_\Lambda} \zeta(6+4\Delta_\phi+i,\ n_{\max}+1+\Delta/2)\, f_i(\Delta, l, \Delta_\phi)$$

practically $n_{\max} = 500$, $i_\Lambda = 16$

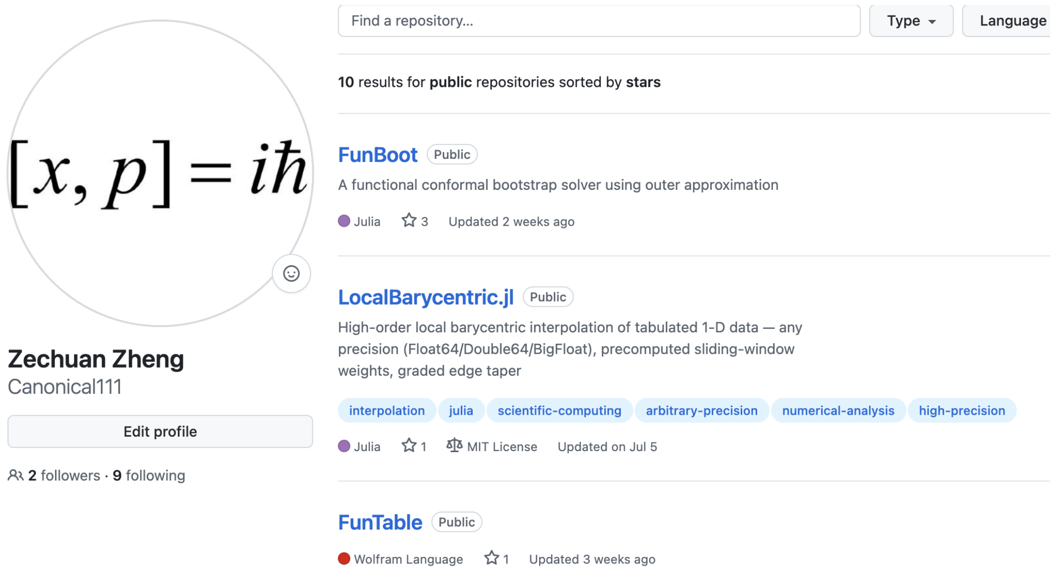
Challenge 2: transcendentality

$$\omega_n(\Delta) \equiv \omega_n[F_\Delta] = 2\sin^2\left(\frac{\pi}{2}\left(\Delta - \Delta_n^{B/F}\right)\right)\int_0^1 \mathrm{d}z\, g_n(z)\,\frac{G_\Delta(z)}{z^{2\Delta_\phi}}$$

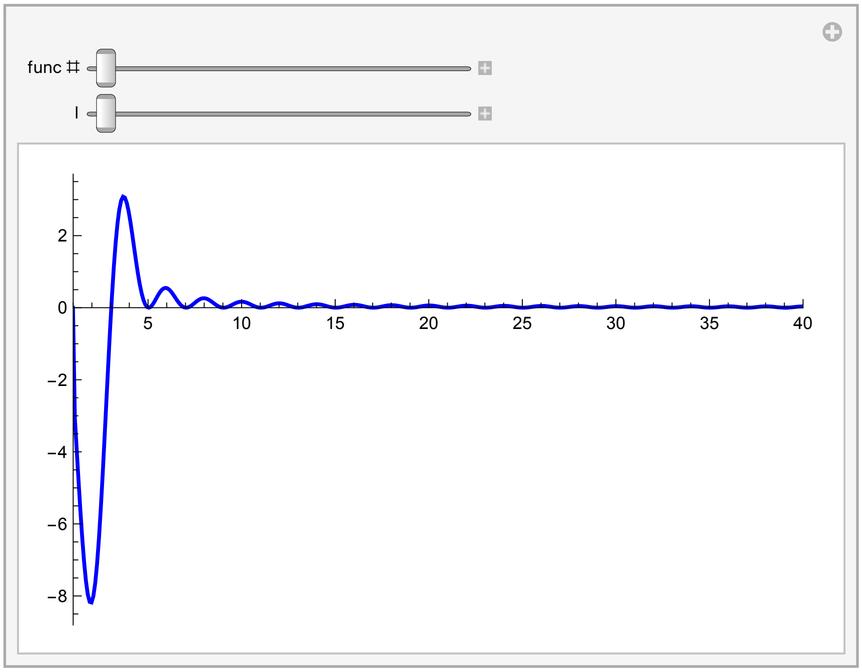
- SDPB not directly applicable
- Outer-approximation: arbitrary order interpolation
- Dynamical precision: adjust the precision by the condition number



Open-Source Software



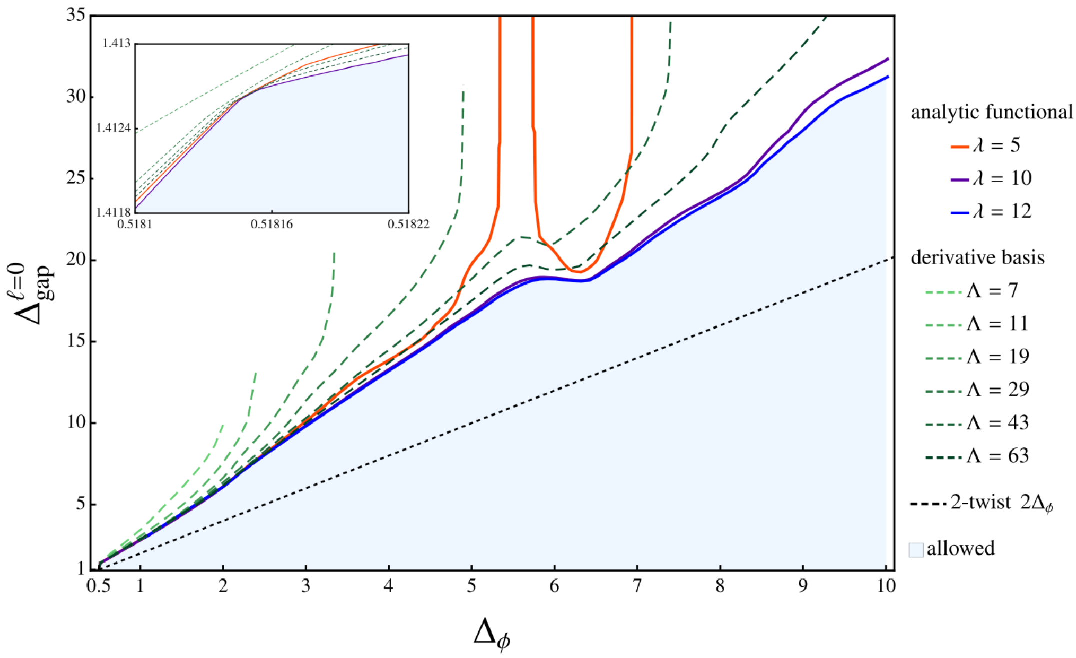
Functional Profile (3d)




```
printFunc[7]
```

Map of 3d CFT: bootstrap perspective

$$N_\lambda = 2\ (\lambda+1)^2\ ,\quad N_\Lambda = \frac{(\Lambda+1)\ (\Lambda+3)}{8}\ ,\quad N_{\lambda=12} = 338\ ,\quad N_{\Lambda=63} = 528$$



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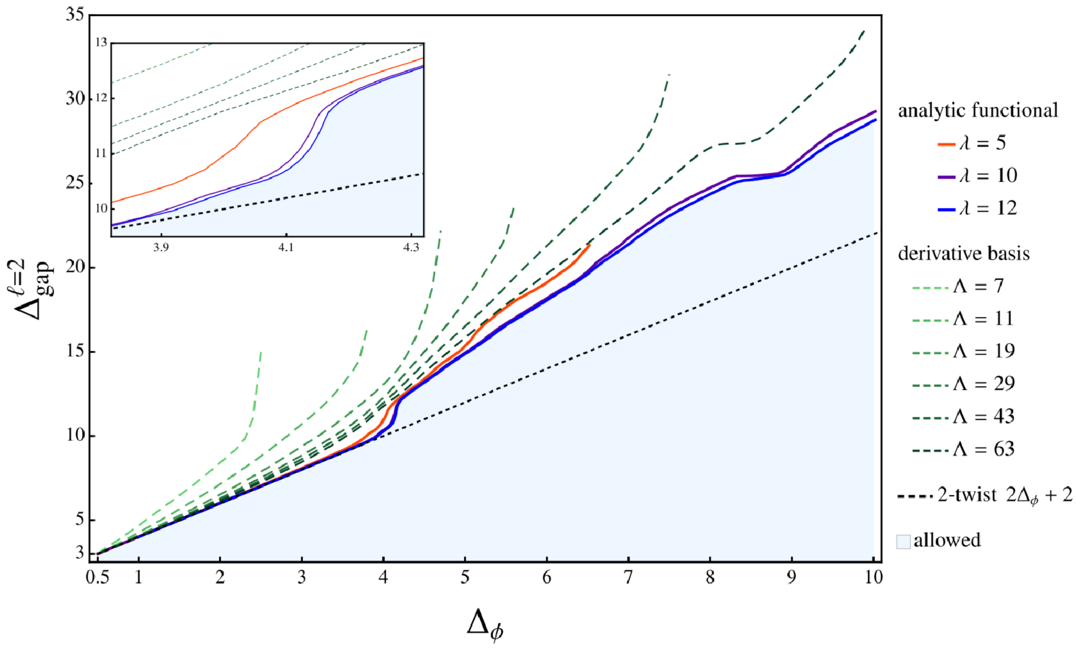
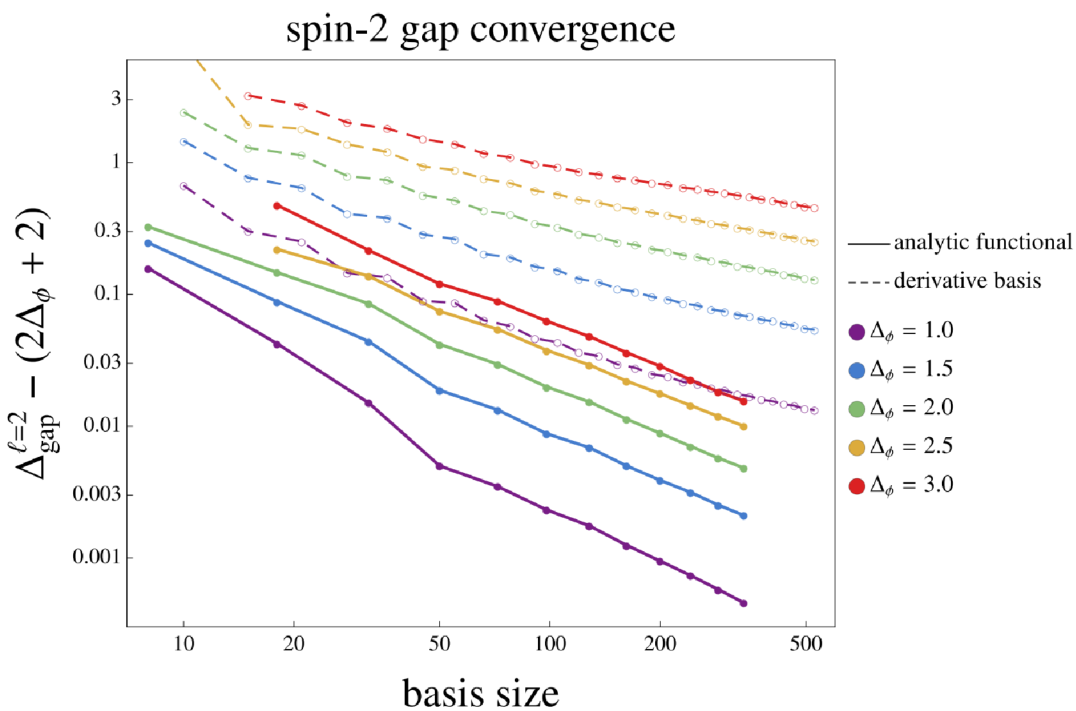
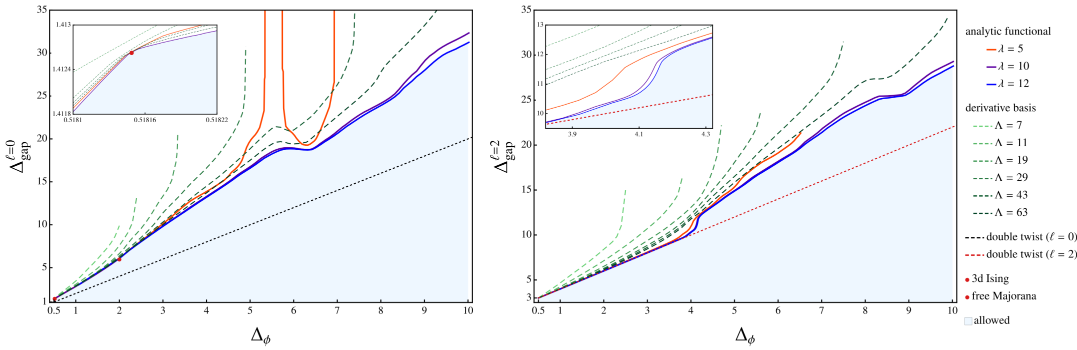


Convergence

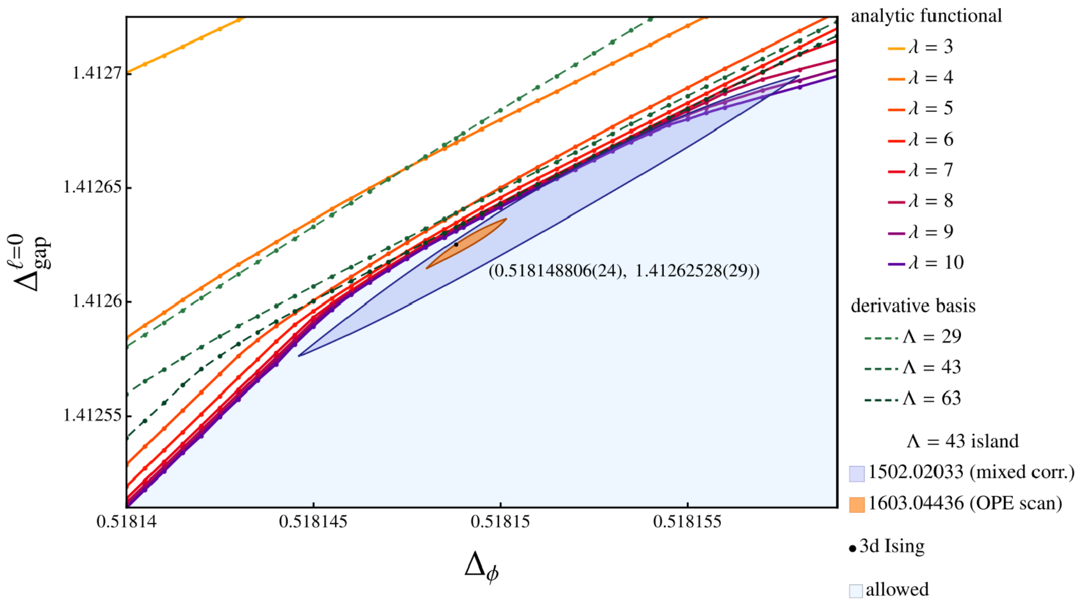


Do something?

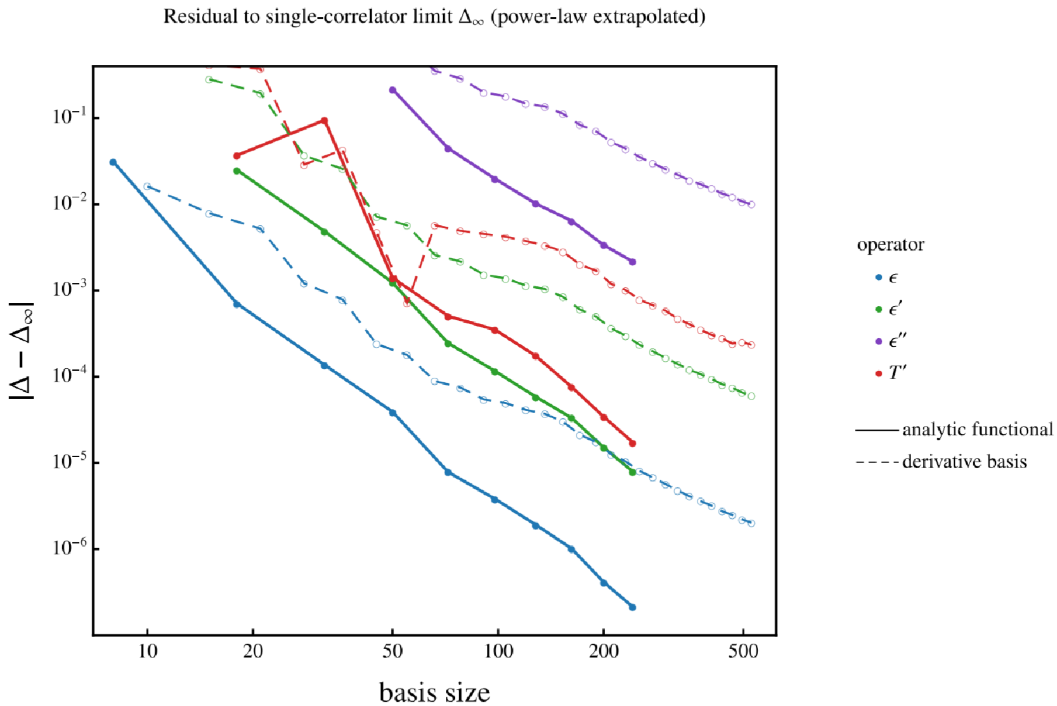


- Extremal solution vs physical theory
- 3d Ising
- free-Majorana-fermion CFT
- GFF

Further Zoom in

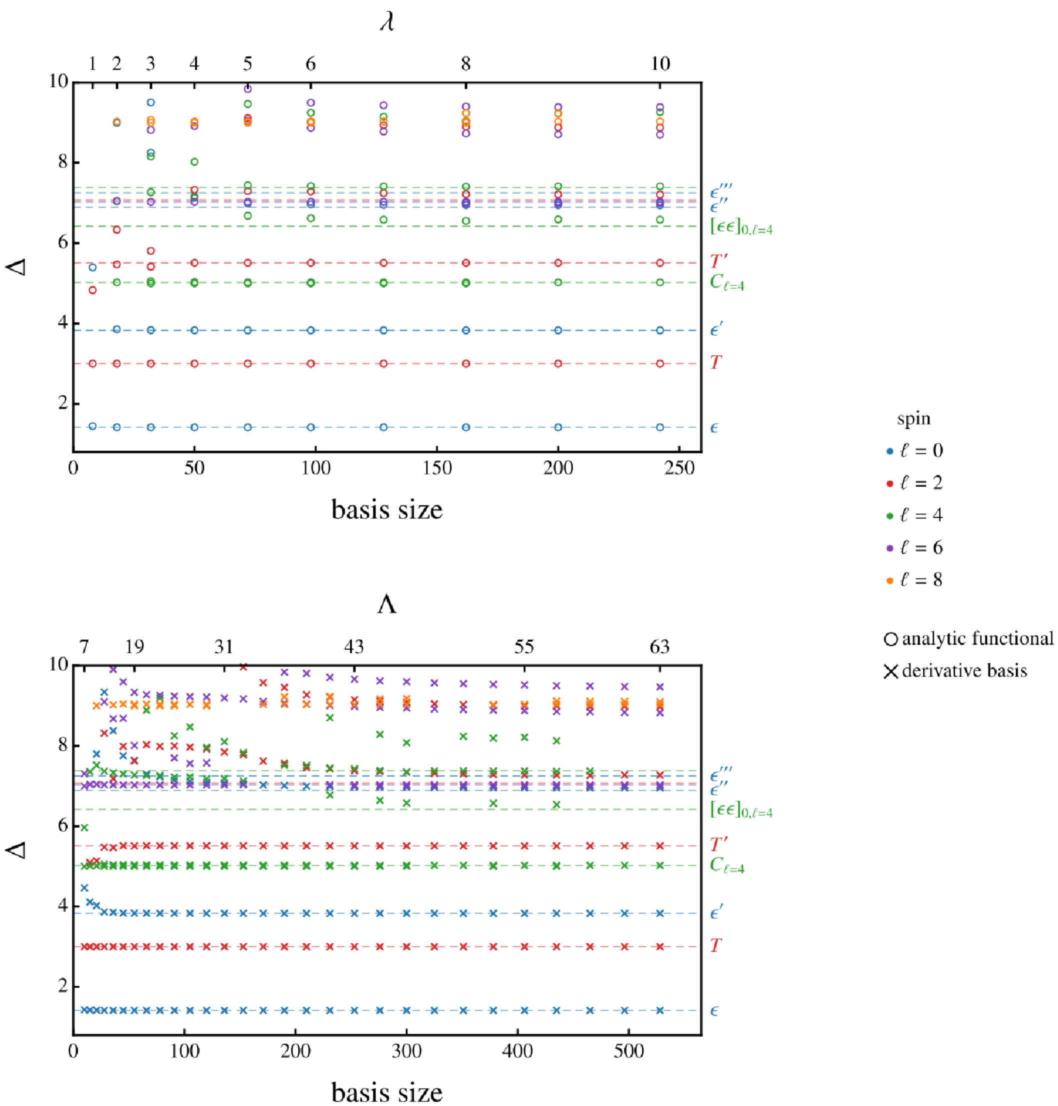


Extrapolation?

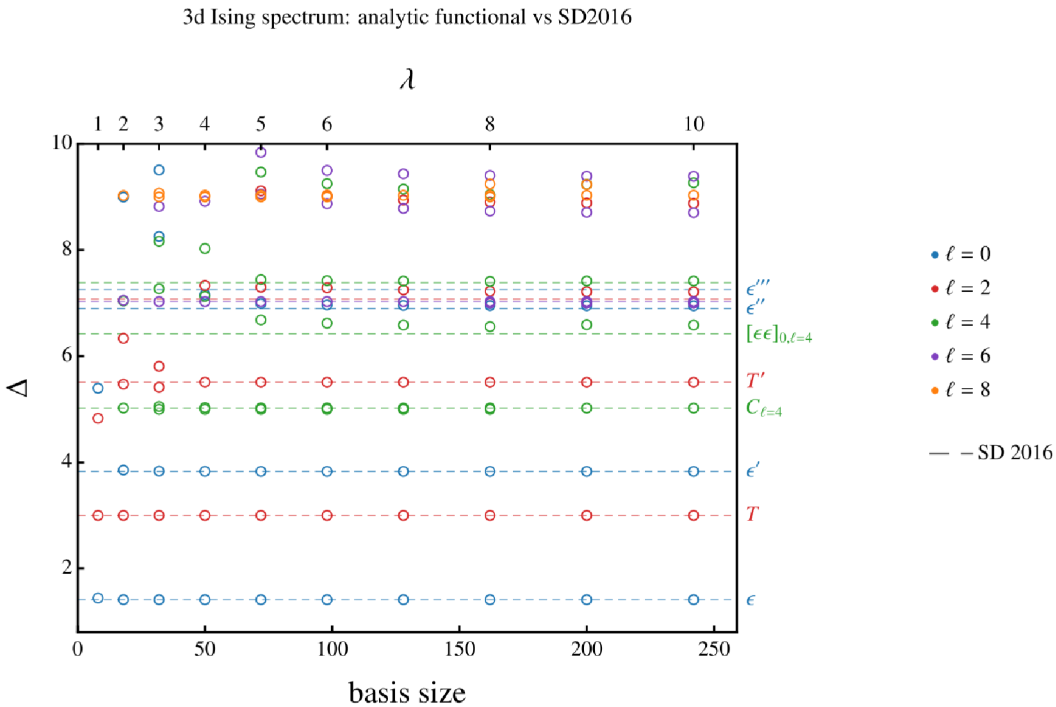


operator	ℓ	Δ_∞ (single corr.)	Δ (3 d Ising)	$\Delta_\infty - \Delta_{\text{Ising}}$
ϵ	0	1.4126311	1.412625	$+6\times10^{-6}$
ϵ'	0	3.82979	3.82968	$+1.1\times10^{-4}$
ϵ''	0	6.9453	6.8956	$+5.0\times10^{-2}$
T'	2	5.50993	5.50915	$+7.8\times10^{-4}$

Spectrum


λ
Δ
basis size
ϵ'''
ϵ''
$[\epsilon\epsilon]_{0,\ell=4}$
T'
$C_{\ell=4}$
ϵ'
T
ϵ
Λ
spin
$\ell = 0$
$\ell = 2$
$\ell = 4$
$\ell = 6$
$\ell = 8$
analytic functional
derivative basis

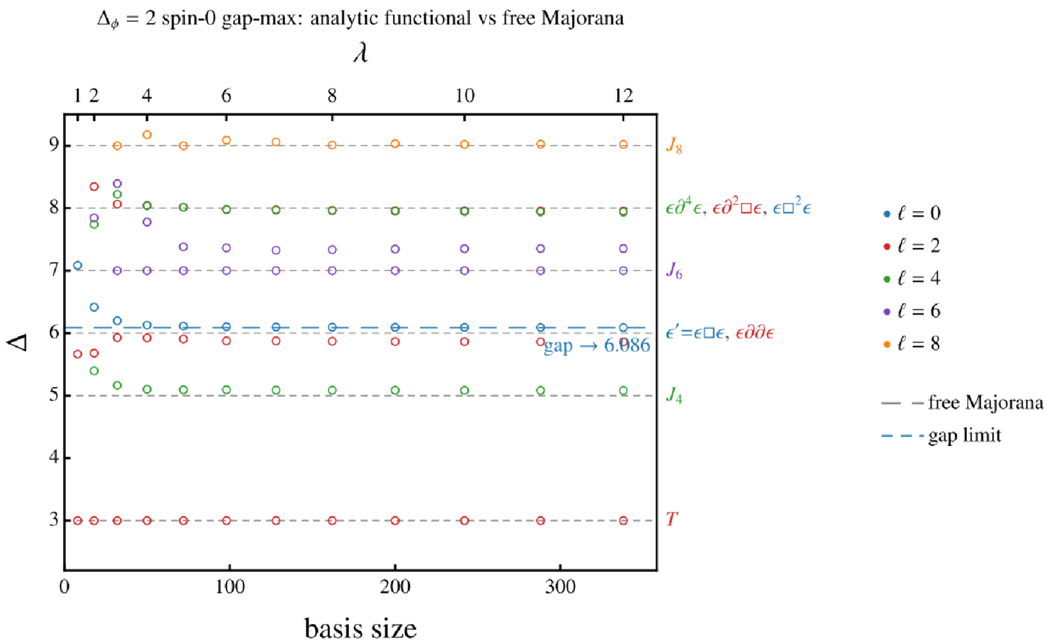
Missing ϵ'''



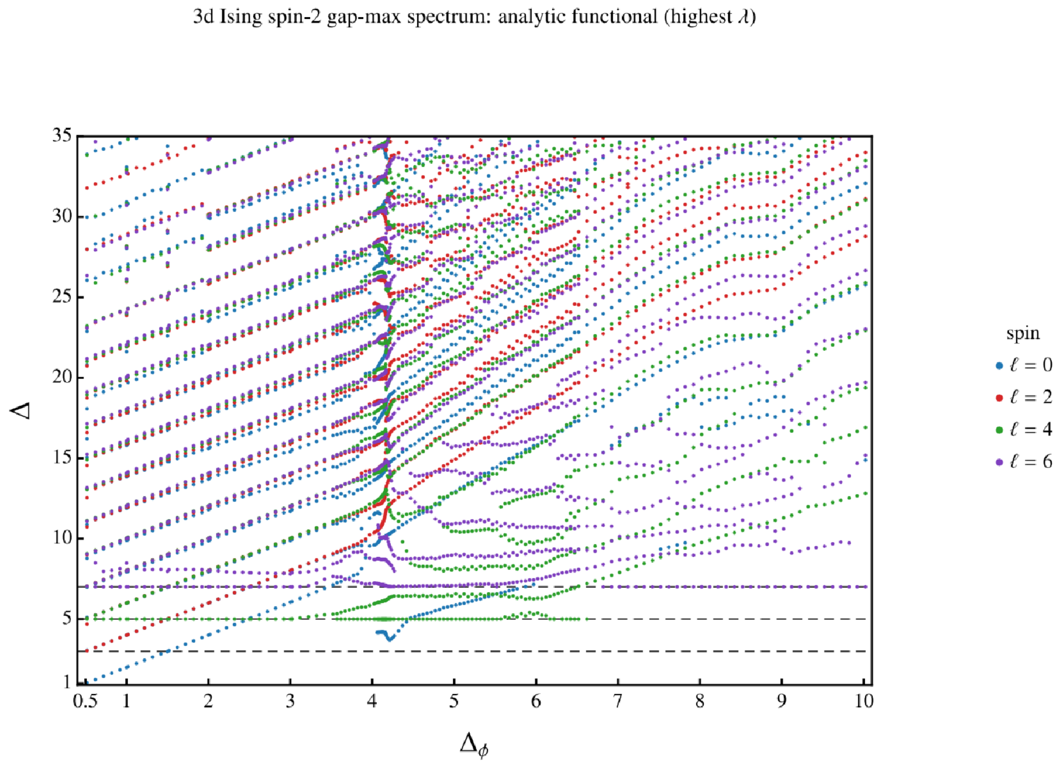
free-Majorana-fermion CFT ($\Delta_\epsilon = 2$, $\Delta_{\epsilon'} = 6$)?

schematic operator	free (Δ, ℓ)	free $\sum_i f_i^2$	extremal (Δ, ℓ)	extremal f^2
$T = J_2 \sim \psi\gamma\,\partial\psi$	(3, 2)	$\frac{3}{2}$ (1.500)	(3.000, 2)	1.515
$J_4 \sim \psi\gamma\,\partial^3\psi$	(5, 4)	$\frac{35}{128}$ (0.2734)	(5.082, 4)	0.2748
$\epsilon' = \epsilon\,\Box\,\epsilon$	(6, 0)	2 (2.000)	(6.090, 0)	2.047
$\epsilon\;\partial\partial\,\epsilon$	(6, 2)	$\frac{3}{4}$ (0.7500)	(5.861, 2)	0.7109
$J_6 \sim \psi\gamma\,\partial^5\psi$	(7, 6)	$\frac{2079}{65536}$ (0.03172)	(7.000, 6) (7.354, 6)	0.01962 0.01231
$\epsilon\,\Box^2\,\epsilon$	(8, 0)	$\frac{9}{88}$ (0.1023)	not resolved	not resolved
$\epsilon\;\partial^2\Box\,\epsilon$	(8, 2)	$\frac{81}{154}$ (0.5260)	(7.956, 2)	0.5725
$\epsilon\;\partial^4\epsilon$	(8, 4)	$\frac{35}{192}$ (0.1823)	(7.938, 4)	0.1727
$J_8 \sim \psi\gamma\,\partial^7\psi$	(9, 8)	$\frac{6435}{2097152}$ (0.003068)	(9.022, 8) (9.603, 8)	0.002253 0.0008500

$\Delta_\phi = 2$ gap maximization extremal spectrum

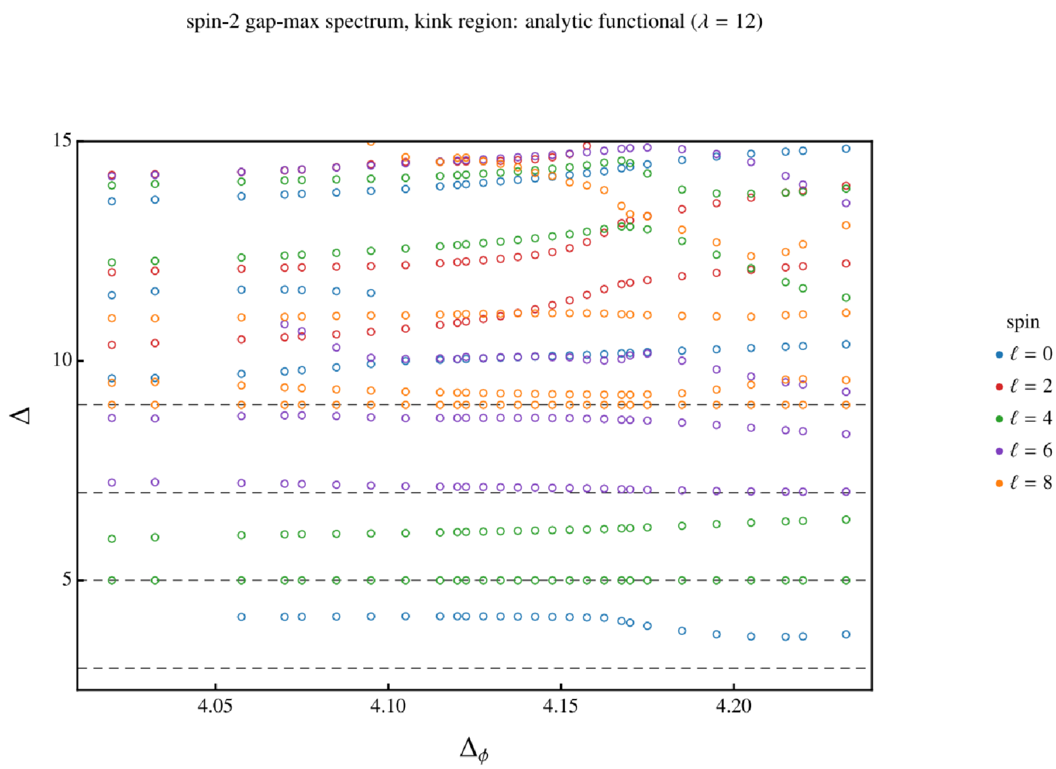


GFF?



- $\Delta_\phi \leq 1$, guaranteed to be saturated by GFF (Caron-Huot, Mazac, Rastelli, Simmons-Duffin, 2020)

Kink


spin-2 gap-max spectrum, kink region: analytic functional (λ = 12)
15
10
5
Δ
4.05
4.10
4.15
4.20
Δ_ϕ
spin
$\ell = 0$
$\ell = 2$
$\ell = 4$
$\ell = 6$
$\ell = 8$

Performance

Derivative basis best at Ising:

Λ=43, bound 1.412639144, takes around ~10 hours, but for analytic functional:

λ	$\Delta_\epsilon(\lambda)$	$\Delta_\epsilon - \Delta_\epsilon^{ref}$	$f_{\sigma\sigma\epsilon}^2$
1	1.4436410201	3.10×10^{-2}	1.0758441560
2	1.4133371669	7.12×10^{-4}	1.1056881030
3	1.4127680141	1.43×10^{-4}	1.1062579050
4	1.4126700862	4.51×10^{-5}	1.1063516460
5	1.4126389395	1.39×10^{-5}	1.1063826010
6	1.4126349333	9.93×10^{-6}	1.1063865840
7	1.4126329977	8.00×10^{-6}	1.1063885280
8	1.4126321207	7.12×10^{-6}	1.1063894120
9	1.4126315183	6.52×10^{-6}	1.1063900150
10	1.4126312814	6.28×10^{-6}	1.1063902530
ref.	1.412625	–	1.106396

The analytic functional equivalent, λ=5 takes around 30 seconds on a laptop .

Prospect

- Global Symmetry, supersymmetry, Long-Range Ising
- Mixed correlator
- Dimension higher than 3, especially even dimension