CFTs are lazy

—or how the lazy training of neural networks reveals a useful structure in CFT correlators—

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Based on 2604.18673 [hep-th], 2604.18686 [hep-th], 2607.24913 [hep-th]
with **Kausik Ghosh**, **Sidhaarth Kumar** and **Andreas Stergiou**

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Motivation

The spacetime dependence of CFT correlators is constrained by conformal symmetry

- 2-point functions are completely fixed $\longrightarrow$ *normalization + scaling dimension*

$$\langle \varphi_i(x_1)\varphi_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

- 3-point functions are also fixed $\longrightarrow$ *OPE coefficient*

$$\langle \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_k+\Delta_i-\Delta_j}}$$

4- and higher-point functions are much less constrained ...

Motivation

4-point functions are theory-dependent non-trivial functions of conformal cross-ratios:

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} \equiv z\bar{z}\,, \qquad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} \equiv (1-z)(1-\bar{z})$$

E.g. with four identical scalars,

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \frac{1}{|x_{12}|^{2\Delta_\varphi}|x_{34}|^{2\Delta_\varphi}}\,\boldsymbol{\mathcal{G}(u,v)}$$

Q: How much (and what kind of) input is sufficient to find a 'good' approximation of the reduced correlator $\boldsymbol{\mathcal{G}(u,v)}$? Quantify 'good'...

Motivation

Why?

- $\boldsymbol{\mathcal{G}(u,v)}$ is the main vehicle of the modern conformal bootstrap

  Operator Product Expansion (OPE) + positivity + crossing

$$\mathcal{G}(z,\bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\Delta_\varphi} \mathcal{G}(1-z,\, 1-\bar{z})$$

  are used to constrain CFT data

  $\longrightarrow$ however, $\mathcal{G}(u,v)$ is rarely the main target

Motivation

- In general, the use of crossing alone on a limited, finite number of correlators is not constraining enough

- If some minimal input is able to corner **physical** correlators, then hopefully we can use it to obtain a more constraining, more economical, better guided bootstrap

  Possibly in previously inaccessible situations too, e.g. without positivity

The result in a nutshell

1. Restrict on diagonal kinematics $\bar{z} = z$. For now, single correlator.

- Use 1d crossing

$$\mathcal{G}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\varphi} \mathcal{G}(1-z)$$

- Input a couple of scaling dimensions

- Input an anchor $\mathcal{G}(z_0)$

Our algorithm outputs the full correlator on the line $\mathcal{G}(z)$, $z \in (0,1)$

The result in a nutshell

- No positivity requirements
- The input is minimal
- We are not searching in the space of CFT data.

  We search directly in the space of CFT correlators!
- In all examples below,
  (**4-point**, **thermal 2-point** functions across dimensions & **2d partition functions**)
  the relative error from the exact CFT expression is consistently below 1% !

The result in a nutshell

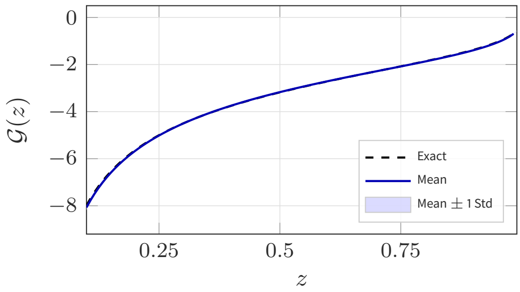
Non-unitary 2d CFT: $\mathcal{M}(2,5)$ (Lee-Yang)

$$\mathcal{G}(z) = \mathcal{G}_1(z)^2 + N(\Delta_\varphi)\,\mathcal{G}_2(z)^2$$

$$\mathcal{G}_1(z) = (1-z)^{2/5}\,{}_2F_1\left(\tfrac{3}{5},\tfrac{4}{5},\tfrac{6}{5};z\right)$$

$$\mathcal{G}_2(z) = (1-z)^{1/5} z^{-1/5}\,{}_2F_1\left(\tfrac{1}{5},\tfrac{2}{5},\tfrac{4}{5};z\right)$$

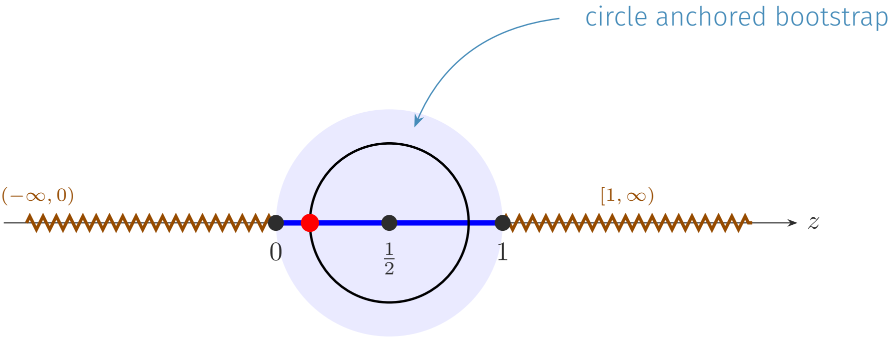
$$N\left(-\tfrac{2}{5}\right) = -\frac{25\cdot 2^{2/5}\sqrt{5}\,\Gamma\left(\frac{1}{5}\right)^2\Gamma\left(\frac{6}{5}\right)^2}{\pi\,\Gamma\left(-\frac{1}{10}\right)^2} \approx -3.65312$$



We used $\Delta_\phi = -2/5$, the value of the correlator at some point $(z_0 = 0.3)$ and crossing

The result in a nutshell

2. Extend on the plane beyond the diagonal kinematics $\bar{z} = z$



Explicit setup on the line

Problem: Given a scalar operator φ with scaling dimension $\boldsymbol{\Delta_\varphi}$ we want to reconstruct its (crossing-symmetric) 4-point function $\mathcal{G}(z)$ on $(0,1)$

Set
$$\mathcal{G}(z) = L(z) + H(z)$$

$\boldsymbol{L(z)}$ accounts for dominant contributions in the limit $z \to 0$ ($L(z)$ for **low**)

In many examples, $L(z) = 1$ from the identity contribution to OPE

For $H(z)$ we are given ($H(z)$ for **high**)

- its asymptotics close to $z = 0$:
$$H(z) \sim z^{\boldsymbol{\delta}}$$
- its value $\boldsymbol{H(z_0)}$ at an "anchor point" $z_0 \in (0,1)$

Explicit setup on the line

The above input and the crossing equation

$$\mathcal{G}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\varphi} \mathcal{G}(1-z)$$

are **not** enough to determine a unique function. **THIS IS AN ILL-DEFINED PROBLEM!**

Indeed, for any $\mathcal{G}(z)$ satisfying the assumptions, the function

$$\mathcal{G}'(z) = \mathcal{G}(z) + (1-z)^{-2\Delta_\varphi} g(z)$$

also satisfies them for any $g(z)$ with the properties

- $g(z) = g(1-z)$,
- $g(z_0) = 0$,
- $\lim_{z\to 0} z^{-\delta} g(z) = 0$.

A more nuanced search in function space

Although the problem is ill-defined, we know we should be searching for special configurations in some restricted class. **CFT correlators are not generic functions.**

(1) Is there a general convenient parametrization of the relevant function space?

(2) Is there a search algorithm biased towards functions relevant in CFT?
What could be the attracting property towards physical CFT correlators?

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Neural networks (NNs) are a flexible class of function approximators
[Motivated by ***Deep Finite Temperature Bootstrap****, VN-Papageorgakis-Stratoudakis-Woolley '25]*

(1′) $f_{\boldsymbol{\theta}}(\vec{x}) = \cdots \sigma\left[W^{(2)} \cdot \left[\sigma(W^{(1)} \cdot \vec{x} + \vec{b}^{(1)})\right] + \vec{b}^{(2)}\right].$
σ is some non-linear activation function. Parameters $\{W^{(\ell)}, \vec{b}^{(\ell)}\}$ are tunable and are collectively denoted by $\boldsymbol{\theta}$.

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- **Universal Approximation Theorems:** Sufficiently wide NNs can approximate any continuous function to arbitrary accuracy

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  This **implicit bias towards smooth, low-complexity functions** is a key reason why NNs generalize well in many practical situations. In other applications, e.g. in Physics, this is unwanted, it obstructs learning desired high-frequency features

Spectral Bias: N. Rahaman et al., *On the Spectral Bias of Neural Networks*, ICML 2019 [arXiv:1806.08734] (**2,257** cit.); Z.-Q. J. Xu et al., *Frequency Principle*, CiCP 2020 [arXiv:1901.06523] (**732** cit.); A. Jacot, F. Gabriel, C. Hongler, *Neural Tangent Kernel*, NeurIPS 2018 [arXiv:1806.07572] (**3,587** cit.).

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Our experimental discovery: The spectral bias of anchored NNs trained on crossing symmetry aligns very well (universally) with a quantifiable property of smoothness in CFT

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Neural Network details

Represent the unknown function $H(z)$ using a fully connected feed-forward neural network

- Architecture: Lightweight NN with 2 or 3 hidden layers and width of 64 or 128 neurons. Input is $z \in (0,1)$, output is a scalar.
- Ansatz: We compose the network output with fixed prefactors:

$$H(z) = z^{\delta}(1-z)^{-\delta'}\,\mathrm{NN}_{\boldsymbol{\theta}}(z)$$

- Prefactors: z^{δ} hard-codes the correct leading small-z behaviour. $(1-z)^{-\delta'}$ (typically $\delta' = 2\Delta_{\varphi}$) ensures efficiency near $z = 1$.

A surprising Neural Network solution: details

- Activation Functions: Use exclusively smooth functions like tanh or GELU.
  Deep-learning theory predicts stronger spectral-bias for these choices.
  This is precisely what we observed to be best for approximating CFT correlators

- Loss Function: Evaluated on a discrete training grid $z_i \in (0, 1)$,

$$\mathcal{L}(\boldsymbol{\theta}) = \mathcal{L}_{\times}(\boldsymbol{\theta}) + \lambda_{\text{⚓}} \mathcal{L}_{\text{⚓}}(\boldsymbol{\theta})$$

- Crossing symmetry: $\mathcal{L}_{\times}$ enforces the crossing equation

- Anchor relation: $\mathcal{L}_{\text{⚓}} = \big(H_{\boldsymbol{\theta}}(z_0) - H_0\big)^2$ anchors the function

A surprising Neural Network architecture: details

- Optimiser: We use the Adam optimiser with a small weight decay for stability
- Learning Rate: Starts at 5×10^{-4}–10^{-3} and decays using a step schedule down to 10^{-5}–10^{-6}
- Training Runs: To collect statistics, we typically execute 100 independent instances per problem
- Convergence: We use early stopping after 5,000 epochs with no improvement
  Optimal loss is typically achieved within a few minutes on a standard laptop

Important note: These choices are fixed and are applied universally to all CFT examples

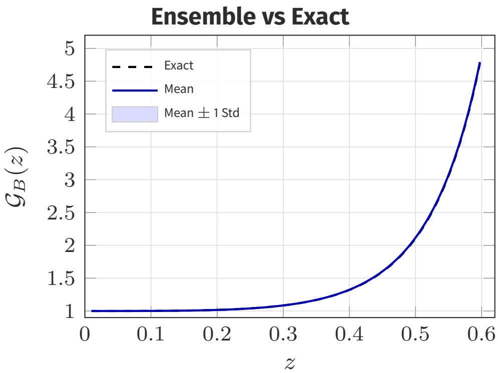
Examples: Generalised Free Fields

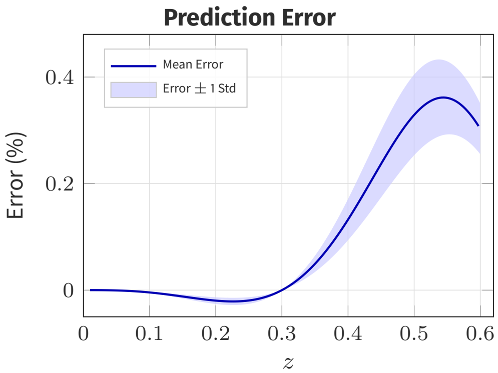
- Bosonic GFF: Tested at $\Delta_\varphi = 1.618$ with an anchor point at $z_0 = 0.3$
- Exact Expression: The exact analytic expression for the reduced four-point function is given by Wick contractions:

$$\mathcal{G}(z) = 1 + z^{2\Delta_\varphi} + \left(\frac{z}{1-z}\right)^{2\Delta_\varphi}$$

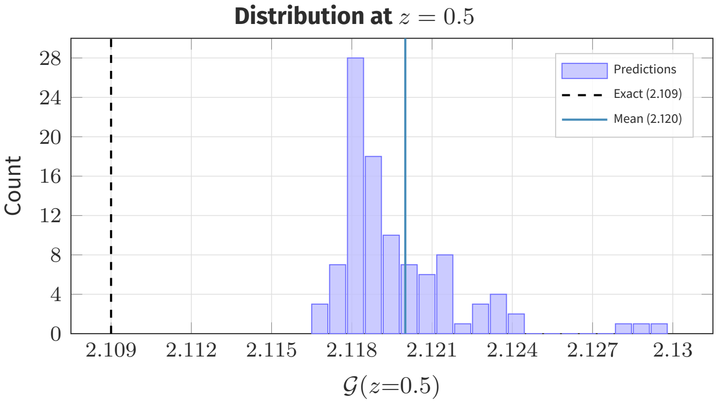
- Performance: The neural network prediction matches the exact analytic expression to within $< 1\%$ relative error

Examples: Generalised Free Fields





Examples: Generalised Free Fields (Histogram)



Examples: CFT_1 Witten Diagrams in AdS_2 **(1d)**

A ϕ^4 scalar field theory in AdS_2 captures holographically the correlators of a dual 1d CFT.

These examples are distinctive: the leading behaviour near $z = 0$ involves logarithmic (not power-law) dependence (tests beyond the generic OPE setting)

- Contact diagram ($\Delta_\phi = 1$, anchor $z_0 = 0.4$):

$$\mathcal{G}_{\text{contact}}(z) = 2z^2 \left(\frac{\log(1-z)}{z} + \frac{\log z}{1-z} \right)$$

$$L(z) = 2(-1 + \log z)\, z^2$$

- One-loop (bubble) diagram: exact expression involves polylogarithms up to Li_4 [Mazac-Paulos '18, Ferrero et al '19].
  Leading behaviour $H_{\text{bubble}}(z) \sim z^3 (\log z)^2$ near $z = 0$

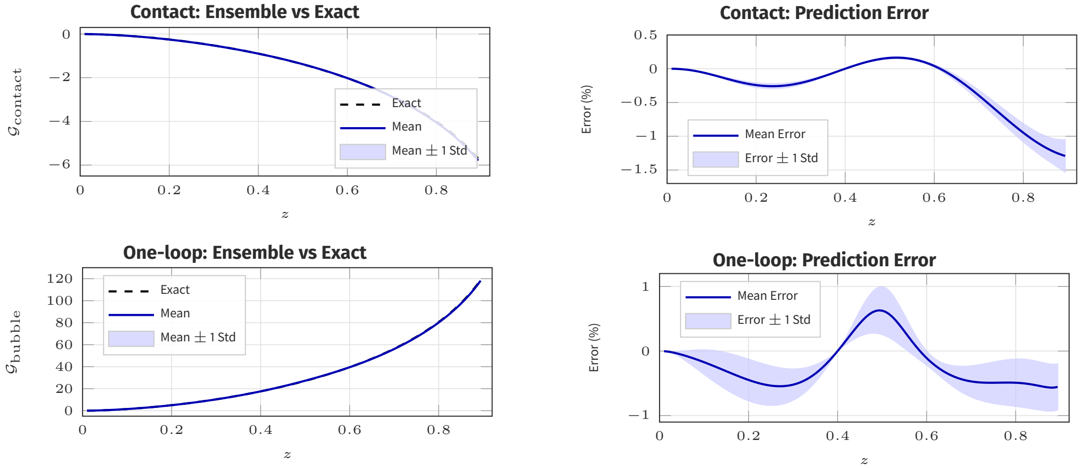
Examples: CFT_1 One-Loop (Bubble) Diagram — Exact Correlator

$$\mathcal{G}_{\text{bubble}}(z) = L(z) + H_{\text{bubble}}(z)$$

$$L(z) = \frac{27}{250}\left(95 + 2\pi^4 - 80\log z + 30(\log z)^2 + 80\zeta(3) - 200\zeta(3)\log z\right) z^2$$

$$\begin{aligned}
H_{\text{bubble}}(z) = &-\frac{27}{250}\, z^2\left(95 + 2\pi^4 - 80\log z + 30\log^2 z + 80\zeta(3) - 200\zeta(3)\log z\right) \\
&+ \frac{9}{250(z-1)^2}\Big[\pi^4 z^2(6-(z-2)z) + 15\big(4z^3\,\text{arctanh}(1-2z) \\
&\quad + (2z-1)\log^4(1-z) + 2z^2\log z + 4(1-2z)\log^3(1-z)\log z \\
&\quad + 2\log(1-z)\big(z - 2z^2 + \pi^2(z-1)^2(2+z^2)\log z\big) + 2\log^2(1-z)\big) \\
&\quad - \big(\pi^2(2z-1) - 3(z-1)^2(1+z^2)\log^2 z\big) \\
&\quad - 180(z-1)^2\big((z^2-1)\log(1-z) + (2+z^2)\log z\big)\,\text{Li}_3(1-z) \\
&\quad - 180z^2\big((3+(z-2)z)\log(1-z) + (z-2)z\log z\big)\,\text{Li}_3(z) \\
&\quad + 360(z-2)z^3\,\text{Li}_4(1-z) + 360(z-1)^3(1+z)\,\text{Li}_4(z) + 360(2z-1)\,\text{Li}_4\left(\tfrac{z}{z-1}\right)\Big]
\end{aligned}$$

Examples: CFT_1 Witten Diagrams in AdS_2



Examples: Tricritical Ising Model $\mathcal{M}(4,5)$ **(2d)**

Unitary minimal model with $c = 7/10$ and $\Delta_\varphi = 1/5$: $\langle \varphi_{1,2}\varphi_{1,2}\varphi_{1,2}\varphi_{1,2} \rangle$

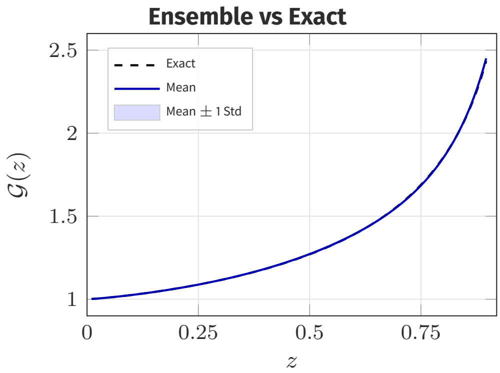
$$\mathcal{G}(z) = \mathcal{G}_1(z)^2 + N(\Delta_\varphi)\,\mathcal{G}_2(z)^2$$

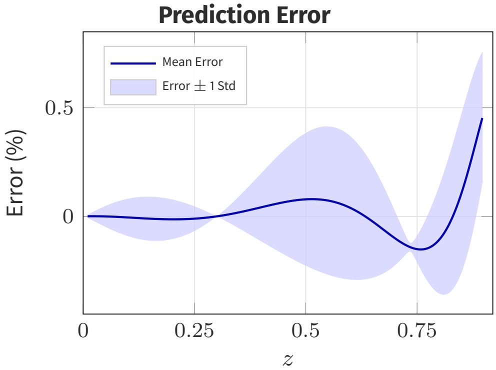
$$\mathcal{G}_1(z) = (1-z)^{-1/5}\,{}_2F_1\left(\tfrac{1}{5},\, -\tfrac{2}{5};\, \tfrac{2}{5};\, z\right)$$

$$\mathcal{G}_2(z) = (1-z)^{2/5}\,z^{3/5}\,{}_2F_1\left(\tfrac{4}{5},\, \tfrac{7}{5};\, \tfrac{8}{5};\, z\right)$$

$$N\left(\tfrac{1}{5}\right) = \frac{\sqrt{5}\,\Gamma\left(\frac{2}{5}\right)^2\Gamma\left(\frac{7}{5}\right)^2}{8\sqrt{2}\,\pi\,\Gamma\left(\frac{13}{10}\right)^2} \approx 0.372469$$

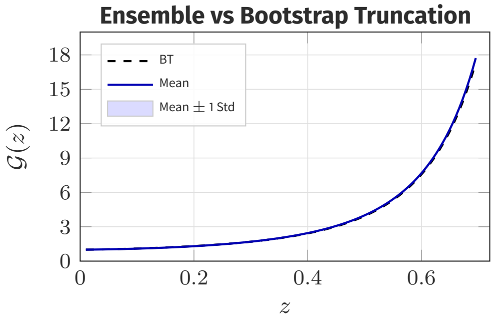
Examples: Tricritical Ising Model $\mathcal{M}(4,5)$

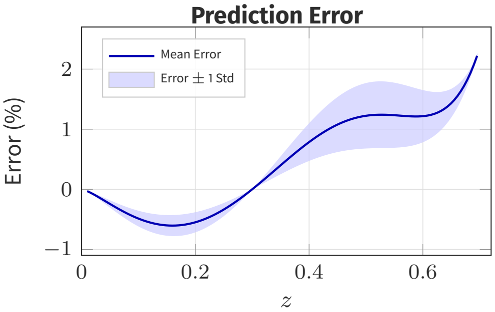




Examples: 3d Ising Model (ϵ Correlator) **(3d)**

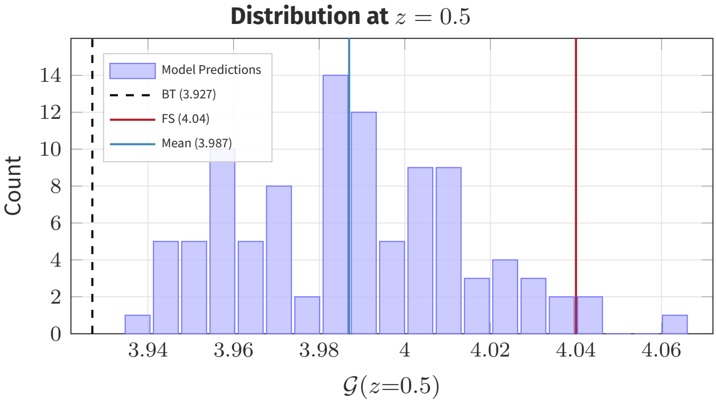
High-precision scaling dimensions from numerical bootstrap: $\Delta_\sigma = 0.5181$, $\Delta_\epsilon = 1.4126$.
No closed-form expression for the correlator exists.
The anchor value at $z_0 = 0.3$ is evaluated from bootstrap truncation (BT) of the OPE





Examples: 3d Ising Model (ϵ Correlator, Histogram)

FS denotes the **Fuzzy Sphere** [Zhu et al, '22, Han et al, '23] prediction



Examples: CFT$_4$ Half-BPS Operators in 4d $\mathcal{N}=4$ SYM **(4d)**

4-point function $\langle S_2 S_2 S_2 S_2\rangle$ of half-BPS operators S_2 in the $\mathbf{20}'$ representation of $SO(6)_R$. Superconformal Ward identities imply

$$\mathcal{G} = \mathcal{G}_{\text{free}} + (\alpha z - 1)(\alpha\bar{z} - 1)(\bar{\alpha} z - 1)(\bar{\alpha}\bar{z} - 1)\,\mathcal{H}(u,v)\,.$$

In the large-c limit, $\mathcal{H} = \mathcal{H}^{(0)} + \frac{1}{c}\mathcal{H}^{(1)} + \cdots$, with $\mathcal{H}^{(0)} = v^{-2} + 1$,
$\mathcal{H}^{(1)} = v^{-1} - u^2 \bar{D}_{2422}(u,v)$.
$\mathcal{H}^{(1)}(z)$ satisfies

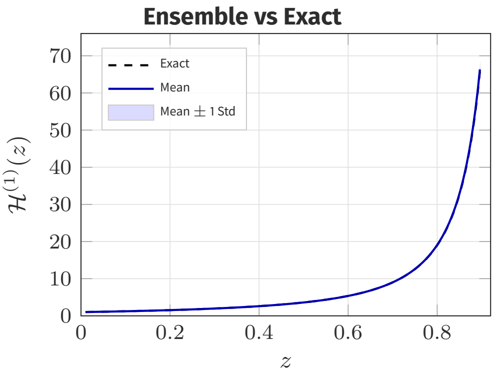
$$(1-z)^4\mathcal{H}^{(1)}(z) - z^4\mathcal{H}^{(1)}(1-z) - (1-2z) = 0$$

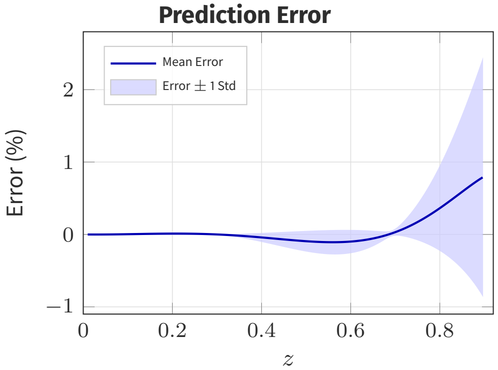
with exact solution

$$\bar{D}_{2422}(z) = \frac{1}{35(z-1)^5 z^5}\Big[4\big(z(2z-7)+7\big)z^5\tanh^{-1}(1-2z) \\ + 4(z-1)\big((z-1)z+1\big)^2 z - 2\big(7(z-1)z+2\big)\log(1-z)\Big]$$

NN setup: $L(z) = 1,\ \ H(z) = z(1-z)^{-2}\,\mathrm{NN}_{\boldsymbol{\theta}}(z),$ anchor at $z_0 = 0.3$.

Examples: CFT$_4$ Half-BPS Operators in 4d $\mathcal{N} = 4$ SYM





Examples: Mixed Correlators – 2d Ising Model

2d Ising: $\Delta_\sigma = \frac{1}{8}, \quad \Delta_\epsilon = 1$ | mixed σ-ϵ system: two coupled correlators

General parametrisation :

$$\mathcal{G}^{ijkl}(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^{\Delta_i+\Delta_j} \, x_{34}^{\Delta_k+\Delta_l}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{ij}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{kl}} \mathcal{G}(z)\,, \qquad \Delta_{ji} = \Delta_j - \Delta_i$$

Crossing equation :

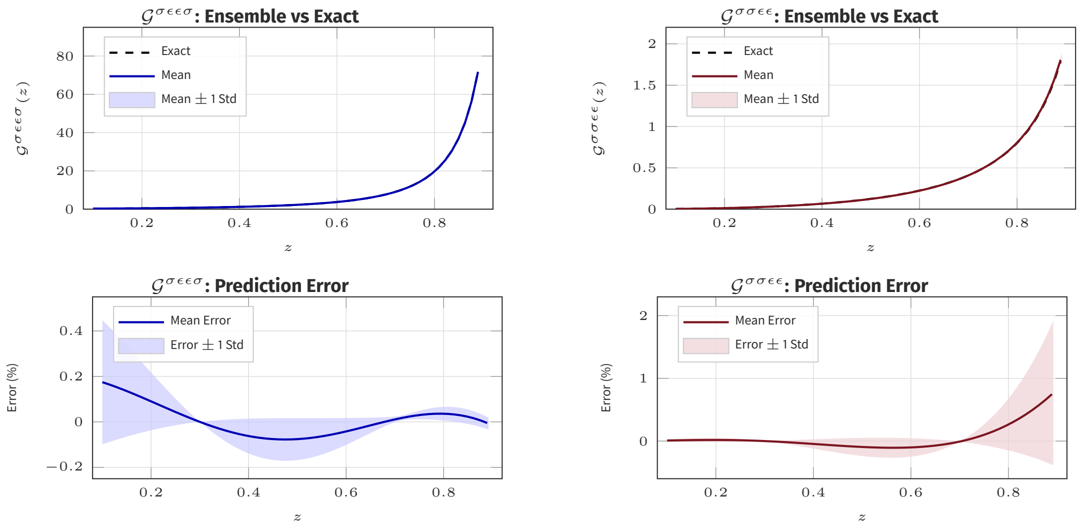
$$\mathcal{G}^{\sigma\epsilon\epsilon\sigma}(z) = \frac{z^{\Delta_\sigma+\Delta_\epsilon}}{(1-z)^{2\Delta_\epsilon}} \, \mathcal{G}^{\sigma\sigma\epsilon\epsilon}(1-z)$$

Exact line-restricted correlators :

$$\mathcal{G}^{\sigma\sigma\epsilon\epsilon}(z) = \frac{(z-2)^2}{4(1-z)}\,, \qquad \mathcal{G}^{\sigma\epsilon\epsilon\sigma}(z) = \frac{z^{1/8}(z+1)^2}{4(z-1)^2}$$

NN setup: two independent networks, each one parametrising $\mathcal{G}^{\sigma\epsilon\epsilon\sigma}(z)$, $\mathcal{G}^{\sigma\sigma\epsilon\epsilon}(z)$, with its own gap prefactor and anchor constraint, coupled through crossing equations.

Examples: Mixed Correlator Results – 2d Ising Model



Thermal 2-Point Functions at Zero Spatial Separation

Setup: CFT on $S^1_\beta \times \mathbb{R}^{d-1}$, Euclidean time $\tau \in [0, \beta)$

Set $\beta = 1$, focus on zero spatial separation ($z = \tau \in (0, 1)$)

Thermal 2-pt function :

$$g(\tau, |x|) = \langle \phi(x)\phi(0) \rangle_\beta \,, \qquad \mathcal{G}(z) = z^{2\Delta_\phi}\, g(z, z)$$

KMS/crossing equation :

$$\mathcal{G}(z) = \left(\frac{z}{1-z} \right)^{2\Delta_\phi} \mathcal{G}(1-z)$$

NN setup: $L(z) = 1$ (captures identity contribution), $\mathcal{G}(z) = 1 + H(z)$

Identical problem and NN architecture to the four-point function case

Examples: 3d Ising Thermal $\langle\epsilon\epsilon\rangle_\beta$

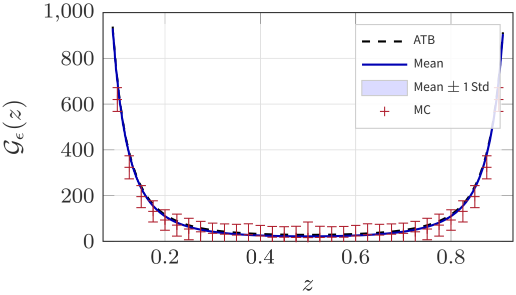
Exact thermal correlators are not known for the 3d Ising model

Anchor at $z_0 = 0.05$ from **Monte-Carlo** (MC) input for a_{Δ_ϵ}

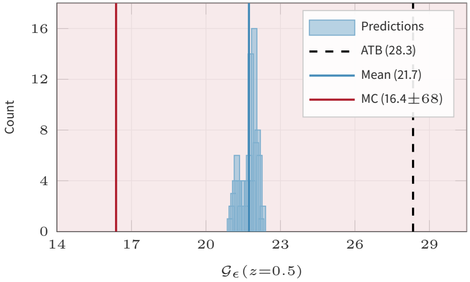
Results compared with the **Analytic Thermal Bootstrap** (ATB, Barrat et al. '25)

Ansatz: $\mathcal{G}_\epsilon = 1 + H_\epsilon,\ \ H_\epsilon(z) = z^{\Delta_\epsilon}(1-z)^{-2\Delta_\epsilon}\mathrm{NN}_{\theta,\epsilon}(z)$

Ensemble vs ATB/MC



Distribution at $z = 0.5$



Modular and Annulus Bootstrap

Neural networks can also be used to study torus/annulus partition functions of 2d CFTs by mapping them to 4-point functions on the plane

- Torus Partition Function

  Modular invariance becomes crossing symmetry of a four-point function of $\mathbb{Z}_2$ twist fields on the plane
  [..., Hartman-Mazac-Rastelli '19,...]

- Annulus Partition Function

  The Cardy open/closed duality becomes crossing symmetry of four-point functions of defect-changing operators
  [Collier-Mazac-Wang '21]

Modular and Annulus Bootstrap

Torus

$$Z(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0-\frac{c}{24}}\right)$$

$$Z(\tau,\bar\tau) = Z\left(-\frac{1}{\tau},-\frac{1}{\bar\tau}\right)$$

$$\Big\downarrow\; z(\tau) = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}$$

On thermal line $\tau = -\bar\tau \quad (z=\bar z)$

$$G(z) = \left(\frac{z}{1-z}\right)^{\frac{c}{4}} G(1-z)$$

Annulus

$$Z^{\text{open}}_{\alpha\beta}(t) = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}\left(e^{-2\pi t(L_0-\frac{c}{24})}\right)$$

$$||$$

$$Z^{\text{closed}}_{\alpha\beta}(1/t) = \langle\beta|e^{-\frac{\pi}{t}(L_0+\bar L_0-\frac{c}{12})}|\alpha\rangle$$

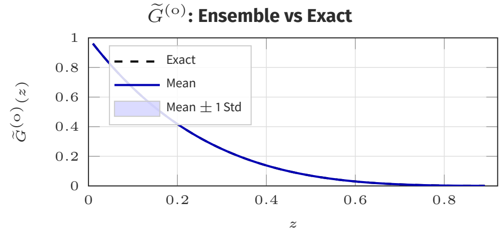
$$\Big\downarrow$$

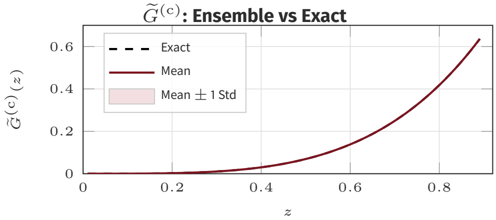
$$G^{(\text{open})}_{\alpha\beta}(z) = \left(\frac{z}{1-z}\right)^{\frac{c}{8}} G^{(\text{closed})}_{\alpha\beta}(1-z)$$

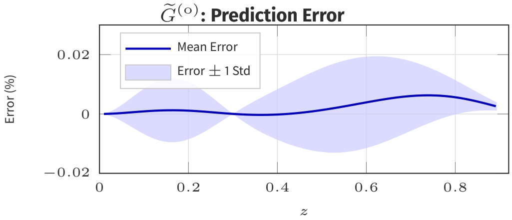
The same anchored NN framework can be applied to the 4-point functions $G^{(\bullet)}(z)$
[Ghosh-Kumar-VN-Stergiou, 2607.24913] tested in minimal models, WZW models and Liouville

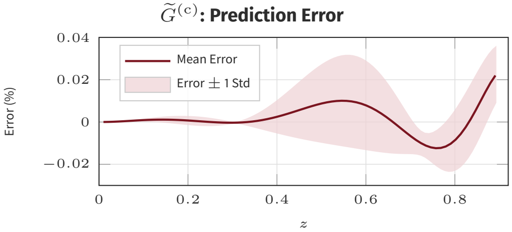
Example: ZZ annulus in Liouville theory

The open/closed duality for annulus PFs involves different correlators in the two channels:
a mixed-correlator bootstrap problem — ZZ[1,1]-ZZ[1,1] annulus $c = 31.5$ $(b \sim 0.6)$









From the Line to the Plane

The full kinematic domain $(z, \bar{z})$ can also be accessed by decomposing the 2D problem into 1D crossing problems on concentric circles

- **Setup:** Circles of radius $R < \frac{1}{2}$ centred at $z = \frac{1}{2}$:

$$z = \tfrac{1}{2} + Re^{i\alpha}, \quad \bar{z} = \tfrac{1}{2} + Re^{-i\alpha}, \quad \alpha \in [0, 2\pi)$$

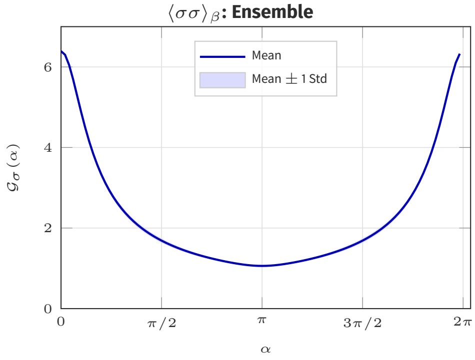
- **Crossing:** $z \to 1 - z$ translates to $\alpha \to \alpha + \pi$, leaving each circle invariant. The crossing equation becomes

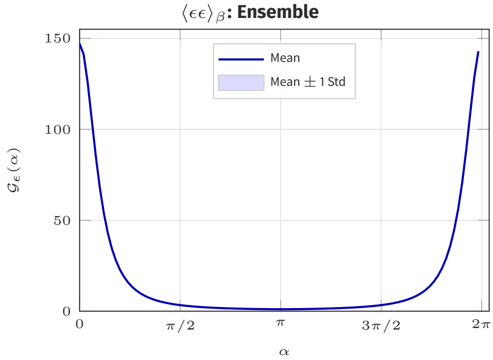
$$\mathcal{G}(\alpha) = \left| \frac{z(\alpha)}{1 - z(\alpha)} \right|^{2\Delta_\phi} \mathcal{G}(\alpha + \pi)$$

- **No New Input needed:** Anchor on the real-line using the $z = \bar{z}$ reconstruction
- **Architecture:** Replace the scalar α input by a Fourier embedding $\beta(\alpha) = (\sin\alpha, \cos\alpha, \ldots, \sin N\alpha, \cos N\alpha) \in \mathbb{R}^{2N}$ to encode the periodic structure

Examples: 3d Ising Thermal Two-Point Functions on the Plane

New predictions — no previous analytic or numerical results in the literature for $\langle\sigma\sigma\rangle_\beta$, $\langle\epsilon\epsilon\rangle_\beta$ on the plane (see however very recent 2607.24919 by Barrat et al). Here, $R = 0.35$, anchor at $\alpha_0 = \pi$





CFT correlator smoothness

We trained NNs in the **LAZY** regime and found that they learn efficiently CFT correlators

Evidence:

- Bosonic/fermionic GFF
- Witten diagrams in AdS_2
- 2d CFTs
- 3d Ising
- Wilson-Fisher in $4-\varepsilon$
- 1/2-BPS operators in 4d $\mathcal{N}=4$ SYM
- Thermal 2-point functions in various dims
- Torus/annulus PFs: 2d minimal models, WZW models, Liouville CFT
- + extensions on the plane

Why?

CFT correlator smoothness

- The Neural Tangent Kernel theory (NTK) in the infinite-width limit reveals the minimization of a norm that measures smoothness

- We do not expect this minimization to be an exact principle of CFT, but it appears to provide a useful ‘relaxation’ with a universal sub-percent error

  This requires a proof with a rigorous bound on the error

- NTK yields naturally an expansion in Chebyshev polynomials

  One can use this insight to remove the NNs, make the whole process deterministic and improve the quality of the above results

Outlook

Where is this going?

- Explore further the requirements for smoothness and its implications
  (Higher external scaling dimensions, structure of OPE spectrum, spinning correlators...)
- Bootstrap the anchors...
- Develop a wider framework that brings together the existing numerical and analytical bootstrap techniques
- More input means more reliability and greater accuracy
- Go beyond the 4-point functions. Go beyond positivity
- ...

Aside: it is very nice that an interpretable feature of deep-learning finds a concrete role in CFT...