Moments in the CFT Landscape

David Poland

Yale University

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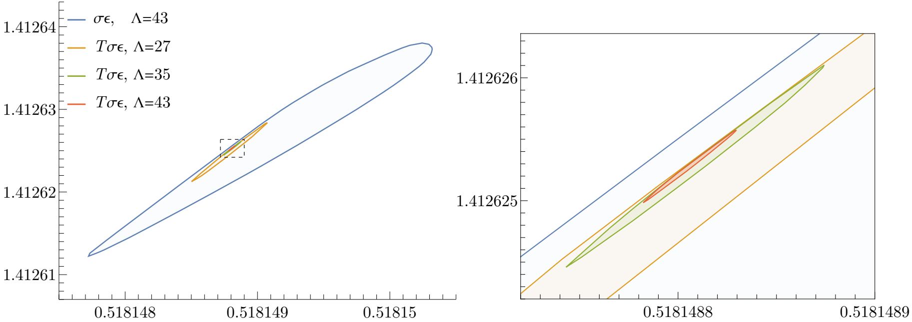
Bootstrap 2026

based on [Li-Yuan Chiang, DP, Gordon Rogelberg, arXiv:2603.18140]
+ [DP, Rogelberg, arXiv:2501.00092] + work in progress

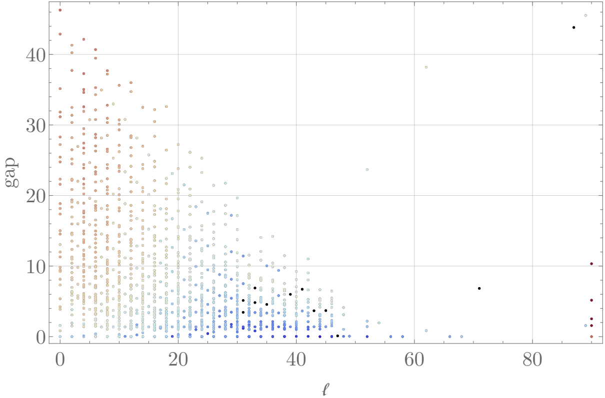
Ising island



[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, DP, Simmons-Duffin '24]

- Success at isolating known CFTs via low-lying data
- But standard approach targets a single operator at a time

Extremal Spectra



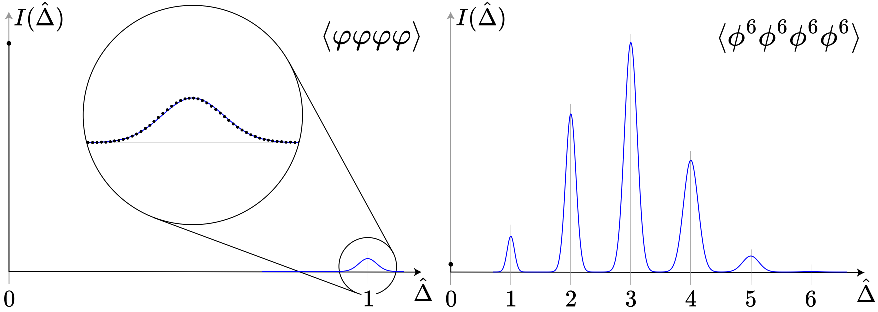
- Extremal spectra can look very messy, with fake operators, missing operators, sharing effects, unresolved operator mixing, ...

What does crossing say about the spectrum as a whole?

Global observables

- In some correlators the OPE is dominated by a dense collection of many states $\rightarrow$ resolving individual operators isn't the right question
- An alternative is global observables: moments of the OPE
- Moments already star in the S-matrix/EFT and modular bootstraps
  [Bellazzini et al '20; Arkani-Hamed, Huang, Huang '20; Chiang et al '21-'23]

Example: the heavy GFF spectrum



- Double-twists in heavy GFF correlators ($\Delta_\varphi = \Delta_{\phi^6} = 200$) organize into Gaussian saddles labeled by operator length [DP, Rogelberg '25]
- A few moments (mean, variance, ...) capture the whole heavy OPE

(for more about $\langle HHHH\rangle$, see Gordon's talk!)

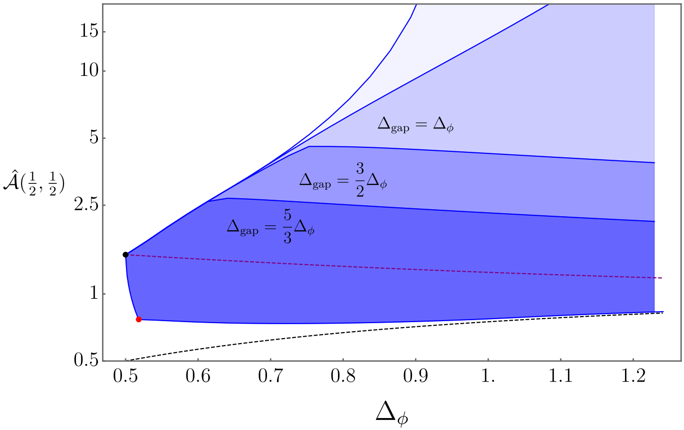
OPE moments

Define OPE moments as weighted averages over the conformal data:

$$\nu_{m,n} = \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}}\, g_{\Delta,\ell}\big(\tfrac{1}{2},\tfrac{1}{2}\big)\, \Delta^m\, \ell^n$$

- Capture statistical shape of OPE distribution
- Larger m probes higher dimension parts of spectrum, etc
- Can be defined in arbitrary kinematics, here focus on $z = \bar{z} = 1/2$

Correlator Bounds



- Zeroth moment is the correlator itself, bounded in 3d in [Paulos, Zheng '21] (see also spectral function bootstrap in 2d [Collier, Kravcuk, Lin, Yin '17])
- Note: $\nu_{0,0} = \hat{\mathcal{A}}\left(\frac{1}{2},\frac{1}{2}\right)$ can be unbounded from above at finite Δ_ϕ

Normalized moments

- We could instead look at normalized moment ratios like:

$$\langle \Delta^m \rangle \equiv \frac{\nu_{m,0}}{\nu_{0,0}}\,, \qquad \langle \Delta^m \rangle_* \equiv \frac{\nu_{m,0}}{\nu_{0,0} - 1} \quad \text{(identity excluded)}$$

- $\langle \Delta \rangle$ is the average scaling dimension contributing to the correlator
- They are analytically bounded at large Δ_ϕ [DP, Rogelberg '25]

Can we bound these numerically using crossing + unitarity + SDP?

Numerical Moment Bounds

Primal:

$$\text{Maximize} \quad \underbrace{\frac{\nu_i}{\nu_0 + \alpha}}_{\text{Moment ratio}} \;, \qquad \nu_i \equiv \sum_\ell \int_{\Delta_{\mathsf{min}}(\ell)}^{\infty} p_\ell(\Delta)\, \underbrace{V_\ell^{(i)}(\Delta)}_{\text{Polynomial}}\, d\Delta$$

$$\text{s.t.} \quad \underbrace{\sum_\ell \int_{\Delta_{\mathsf{min}}(\ell)}^{\infty} p_\ell(\Delta)\vec{N}_\ell(\Delta) d\Delta = \vec{b},}_{\text{Crossing symmetry}} \qquad \underbrace{p_\ell(\Delta) \geq 0}_{\text{Positive OPE density}}$$

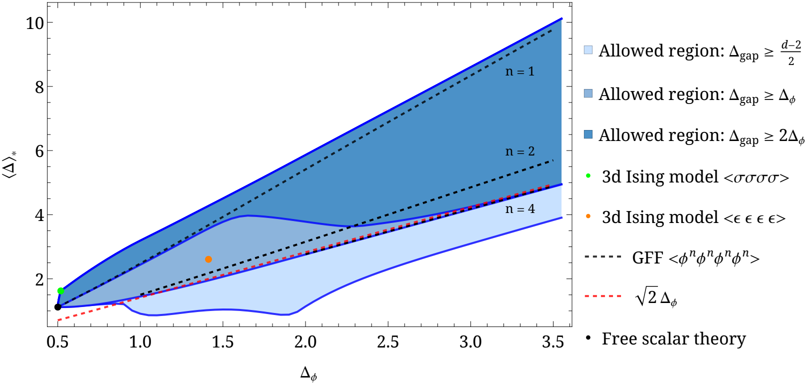
Dual:

$$\text{Minimize} \quad r$$

$$\text{s.t.} \quad \vec{z} \cdot \vec{N}_\ell(\Delta) - V_\ell^{(i)}(\Delta) + r\, V_\ell^{(0)}(\Delta) \geq 0 \quad \forall \Delta \geq \Delta_{\mathsf{min}}(\ell)$$

$$\vec{z} \cdot \vec{b} - r\,\alpha \leq 0$$

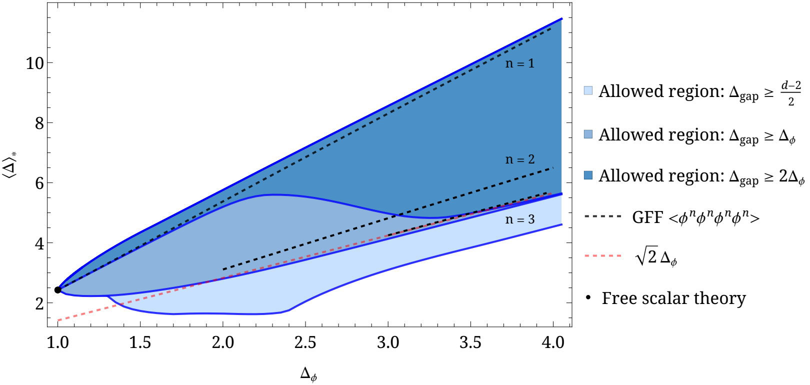
The first-moment landscape ($d = 3$)



[Chiang, DP, Rogelberg '26]

- $\langle\Delta\rangle_*$ at $\Lambda = 23$ under increasing scalar gaps; dashed $=$ GFF
- Upper bound kinks at 3d Ising; linear behavior at large Δ_ϕ

The first-moment landscape ($d = 4$)



[Chiang, DP, Rogelberg '26]

- Similar qualitative structure, with the free theory at $\Delta_\phi = 1$

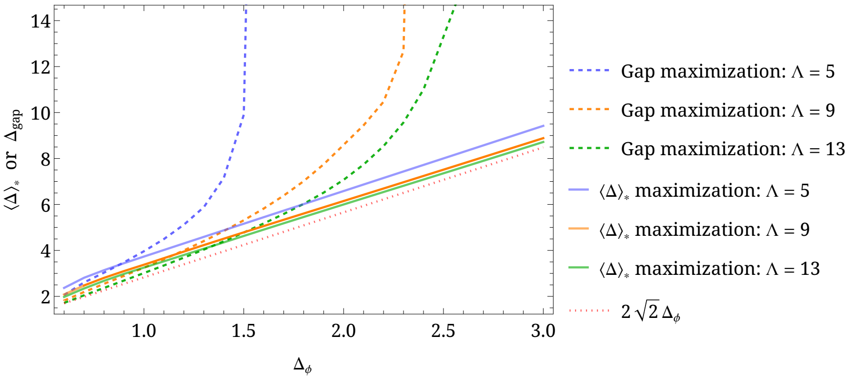
The heavy correlator limit

At large Δ_ϕ, moments obey analytical two-sided bounds [DP, Rogelberg '25]:

$$2^{\frac{k}{2}} \leq \lim_{\Delta_\phi \to \infty} \frac{\langle \Delta^k \rangle}{\Delta_\phi^k} \leq 2^{\frac{3k}{2}-1}$$

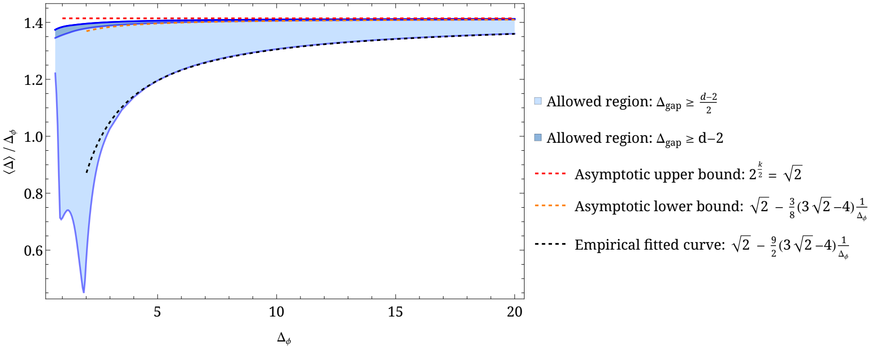
- Saturated by generalized free fields $\langle \varphi^n \varphi^n \varphi^n \varphi^n \rangle$:
  - lower bound: $n \to \infty$ at fixed Δ_φ
  - upper bound: $n = 1$ with $\Delta_\varphi \to \infty$
- Can the numerical bootstrap reach this regime?

Moments vs gap maximization



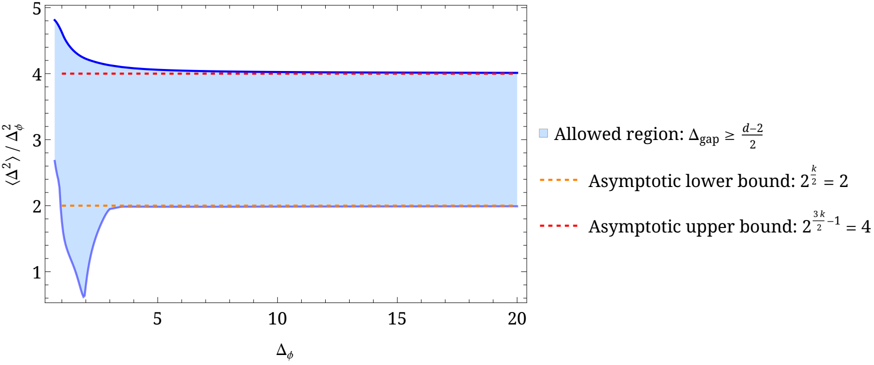
- At fixed Λ, gap maximization stops producing bounds at large Δ_ϕ (However, see Zechuan's talk...)
- Moments are tightly bounded already at $\Lambda = 3$!

Numerical bounds: first moment



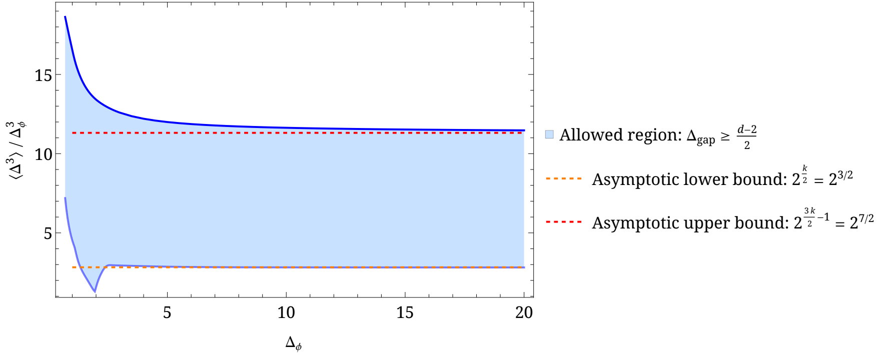
- Rapid convergence to $\langle\Delta\rangle \to \sqrt{2}\,\Delta_\phi$ (strong bounds already at $\Lambda = 3$)
- With a gap $\Delta_{\text{gap}} \geq d-2$: band shrinks onto $\sqrt{2}\,\Delta_\phi - \frac{d}{8}(3\sqrt{2}-4)$ (derived analytically in [DP, Rogelberg '25])

Numerical bounds: second moment



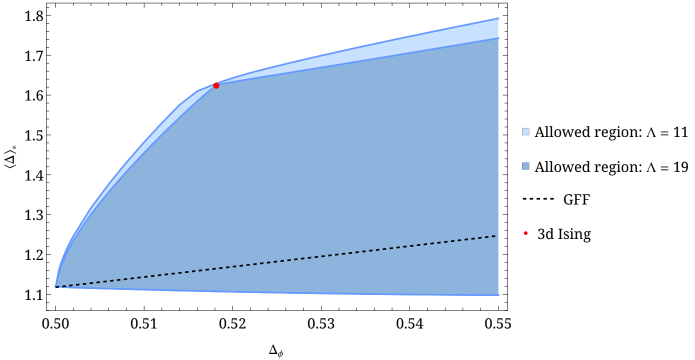
- Two-sided bounds converge to the analytic power laws (dashed)

Numerical bounds: third moment



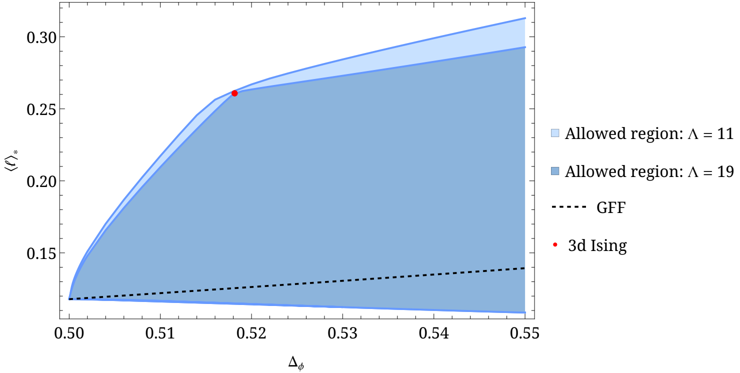
- Same story for $\langle\Delta^3\rangle$ — even deep in the heavy regime

First Δ-moment: the Ising kink



- Kink at $\Delta_\phi \approx 0.518$ with $\langle\Delta\rangle_* \approx 1.6$
- Extremal solution dominated by ϵ, with corrections from $T^{\mu\nu}, \epsilon', \ldots$

Spin moments



- Upper bound on $\langle\ell\rangle_*$, also with 3d Ising kink around $\langle\ell\rangle_* \approx 0.26$
- Positive lower bound $\rightarrow$ spinning operators *must* appear

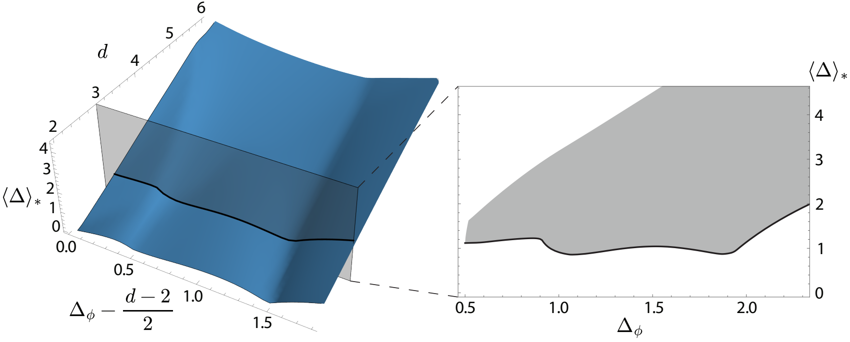
Precision Ising moments

Fix $(\Delta_\sigma, \Delta_\epsilon)$ to Ising values [Chang et al '24] and bound moments at $\Lambda = 27$:

Moment	Bounds	From stable spectrum of [Simmons-Duffin '16]
$\nu_{0,0}$	0.768530(25)	0.7685483(69)
$\nu_{1,0}$	1.247745(15)	1.247736(21)
$\nu_{0,1}$	0.2004013(17)	0.200381(14)
$\nu_{2,0}$	2.2725170(20)	2.272380(91)
$\nu_{1,1}$	0.6255126(41)	0.625408(68)
$\nu_{0,2}$	0.4246333(36)	0.424550(54)

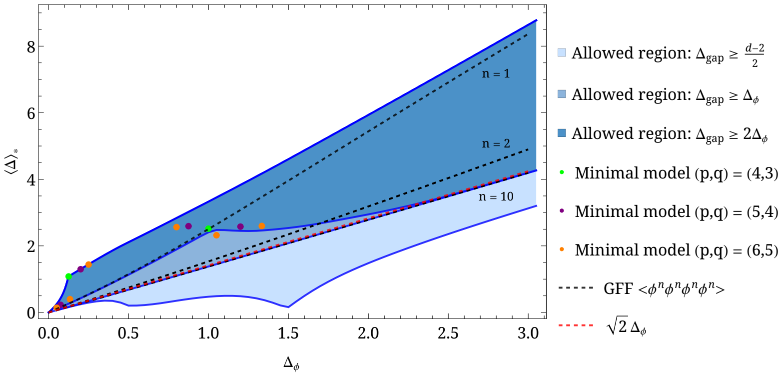
- Some bounds exclude the values computed from the "stable operators" $\rightarrow$ real sensitivity to missing contributions!

The other side of the landscape



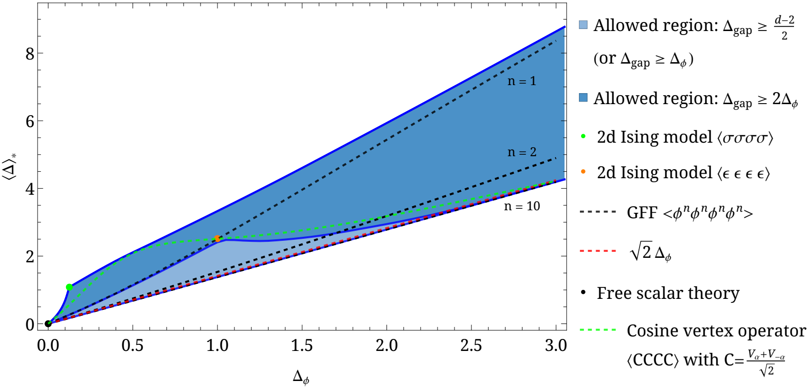
- Lower bound on $\langle\Delta\rangle_*$ over (d, Δ_ϕ): two continuous kink families
- Show interesting operator transitions, but no physical interpretation yet

The limit $d \to 2^+$



- In $d = 2 + 10^{-6}$: sharp kinks at $\Delta_\phi = \frac{1}{2}$ and $\frac{3}{2}$
- Absent in strict $d = 2$: a discontinuity, related to the behavior of the scalar block at the unitarity bound: $g_{\Delta,0} \propto \frac{(d/2-1)^3}{\Delta-(d/2-1)} g_{d/2+1,0}$

The first-moment landscape ($d = 2$)



[Chiang, DP, Rogelberg '26]

- 2d Ising moment sits at kink of the upper bound
- Vertex operator correlators snake through region in an interesting way

Next step: Virasoro moments in $d = 2$

- In 2d we should organize the OPE into Virasoro primaries at fixed c:

$$\nu_{m,n} = \sum_{\text{Vir. primaries}} \lambda^2_{\phi\phi\mathcal{O}} \, \mathcal{V}^c_{\Delta,\ell}\big(\tfrac{1}{2}, \tfrac{1}{2}\big) \, \Delta^m \, \ell^n$$

- Obstacle: `SDPB` wants blocks in the form

(positive prefactor) $\times$ (low-degree polynomial of Δ)

Zamolodchikov recursion

Virasoro blocks are computed efficiently in the elliptic nome $q = e^{i\pi\tau(z)}$ [Zamolodchikov '84, '87; Perlmutter '15; Cho, Collier, Yin '17]:

$$
\begin{aligned}
\mathcal{F}(h_i, h; q) &= \text{(prefactor)} \times (16q)^{h-\frac{c-1}{24}} H(c, h, h_i; q) \\
H(c, h, h_i; q) &= 1 + \sum_{r,s\geq 1} \frac{q^{rs} R_{r,s}(c, h_i)}{h - h_{r,s}(c)} H(c, h_{r,s} + rs, h_i; q)
\end{aligned}
$$

- Poles at the Kac dimensions $h_{r,s}$
- $q_* = e^{-\pi} \approx 0.043$ at the self-dual point $\to$ rapid convergence
- But keeping all poles at order q^N $\to$ huge-degree approximations

Interlude: Optimal interpolation

Which approximation errors matter for a bootstrap bound?
[Chang, Dommes, Kravchuk, DP, Simmons-Duffin '25]

- The bound is set by the optimal functional $\leftrightarrow$ solution to crossing
- Known asymptotics $\rightarrow$ we want a good approximation of:

$$4^{-\Delta} g_{\Delta,\ell} \approx r^{\Delta} \frac{P_\ell(\Delta)}{\prod_{p=1}^{n_{\text{poles}}}(\Delta - \Delta_p)}\,,$$

matched to the full block at some set of interpolation nodes

Interlude: Optimal interpolation

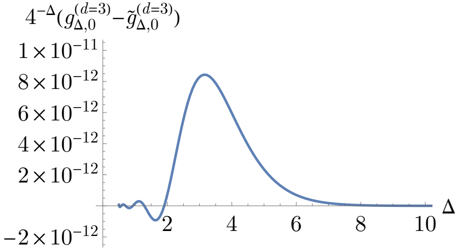
- Use "measure" $\mu(x) = \frac{r^x}{\prod_p (x+k_p)}$ to find optimal interpolation nodes
- Minimize error norm $\to$ find nodes x_k s.t. $\mu(x) | \prod_k (x - x_k) | \sim O(1)$
- Leads to simple formula for optimal density:

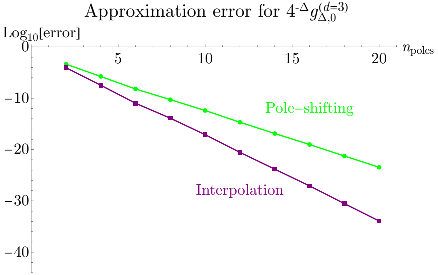
$$\rho(x) = \frac{1}{(n+1)\pi} \sqrt{\frac{b-x}{x}} \left(-\log r + \sum_p \sqrt{\frac{k_p}{b+k_p}} \frac{1}{x+k_p} \right)$$

where b is the effective range of support and satisfies

$$-\frac{b}{2} \log r + \sum_p \left(1 - \sqrt{\frac{k_p}{b+k_p}} \right) = n+1$$

Interlude: Optimal interpolation





- Block errors exponentially small in the number of nodes
  $\rightarrow$ accurate bounds from much smaller SDPs, with better conditioning

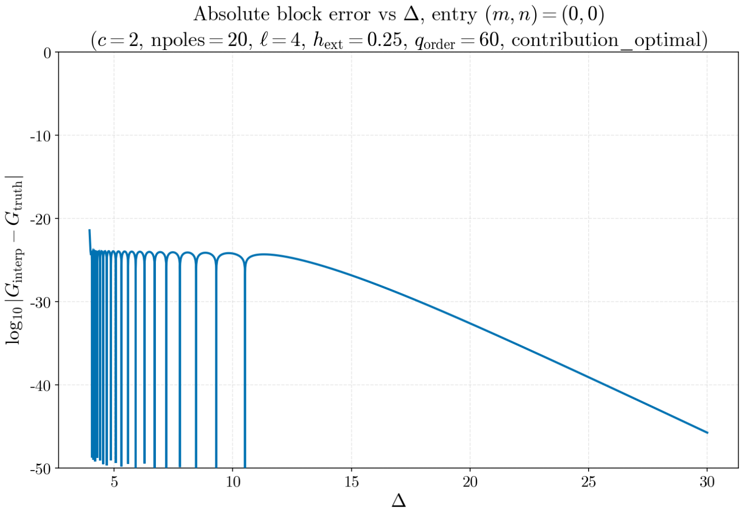
virasoro_blocks

New code: Zamolodchikov recursion + optimal interpolation [DP, in progress]

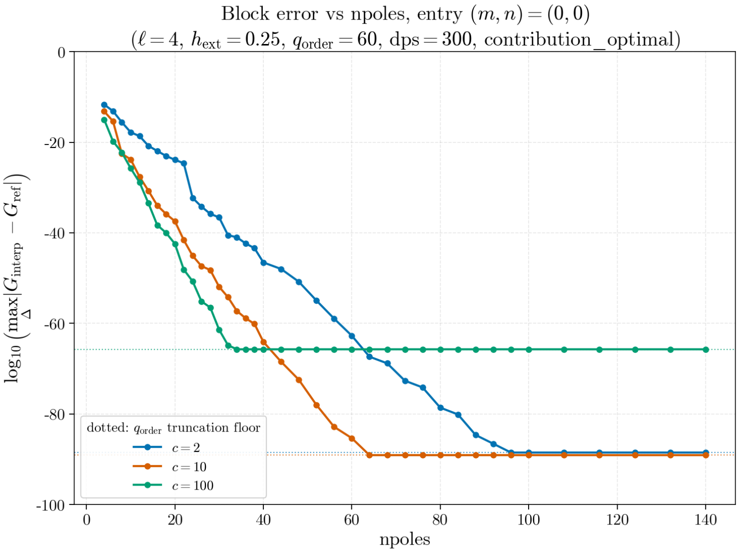
$$\partial^m \overline{\partial}^n \, \mathcal{V}^c_{\Delta,\ell}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \approx (q_*)^\Delta \frac{P_\ell^{m,n}(\Delta)}{\prod_p (\Delta - \Delta_p)}$$

- Keeps the n_{poles} most relevant Kac poles $\rightarrow$ low-degree approximations
- Note: complex Δ-poles for $1 < c < 25$, but can be folded into measure
- Output plugs into existing SDPB pipelines

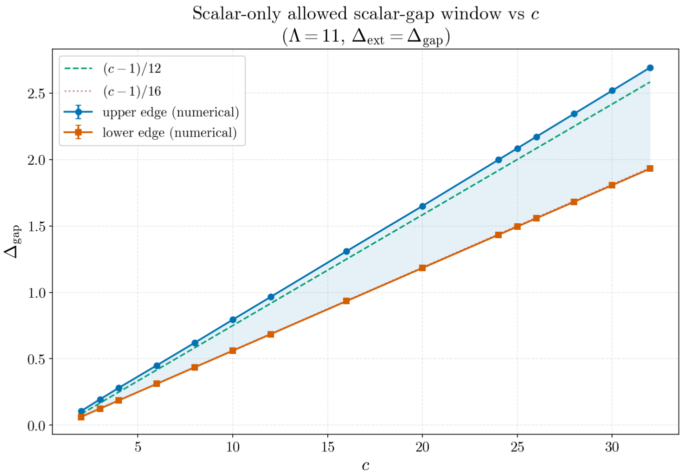
Virasoro block interpolation error



Virasoro block interpolation error

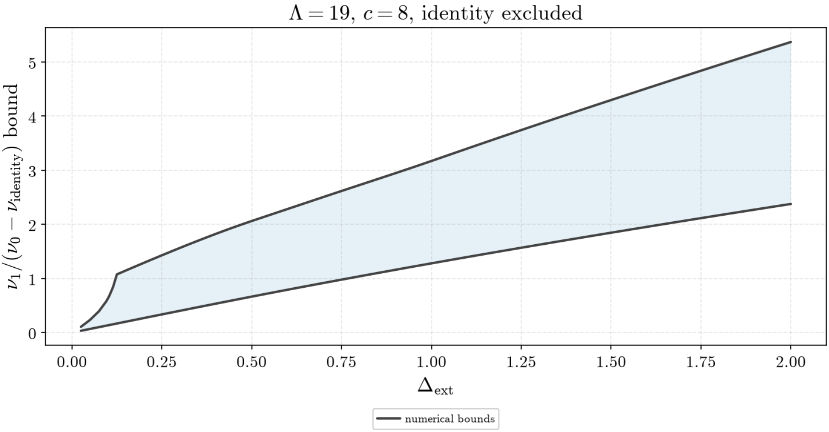


Virasoro gap bounds assuming only scalars (preliminary!)



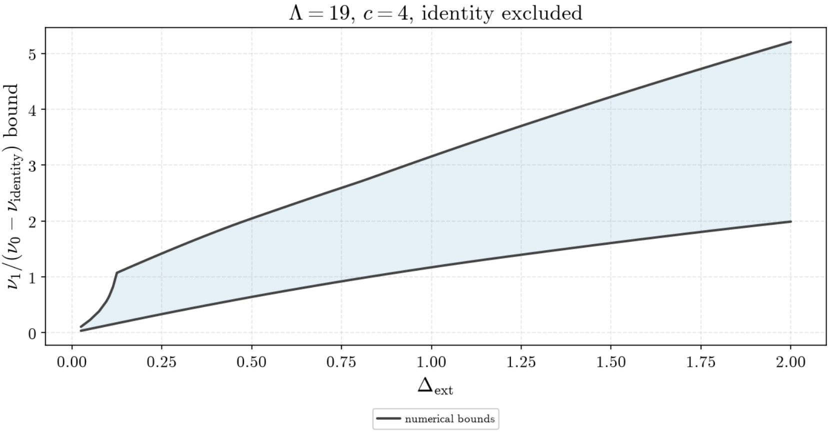
- Gap bounds assuming only scalars reproduce "Liouville window" between $(c-1)/16$ and $(c-1)/12$ [Collier, Kravchuk, Lin, Yin '17]

Virasoro moment bounds (preliminary!)



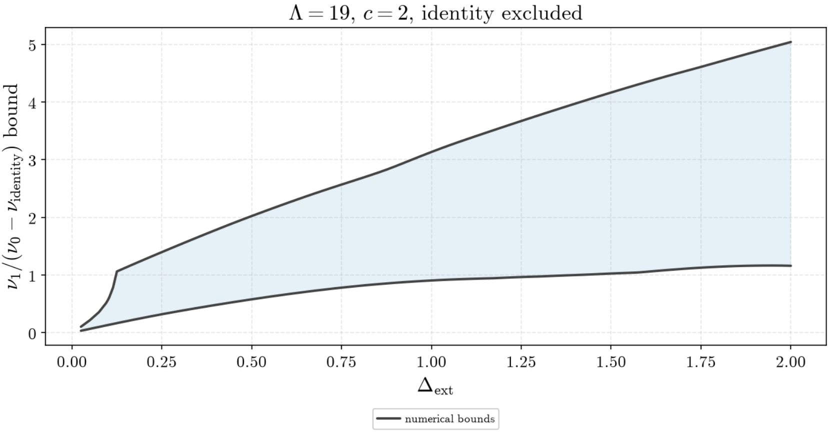
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 8$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



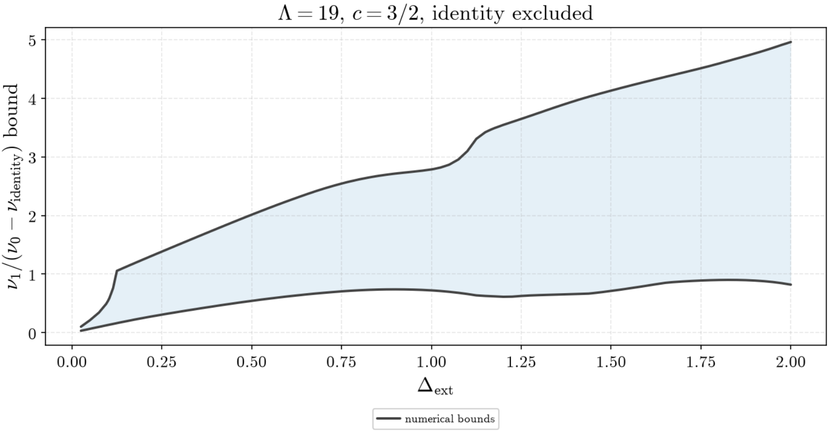
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 4$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



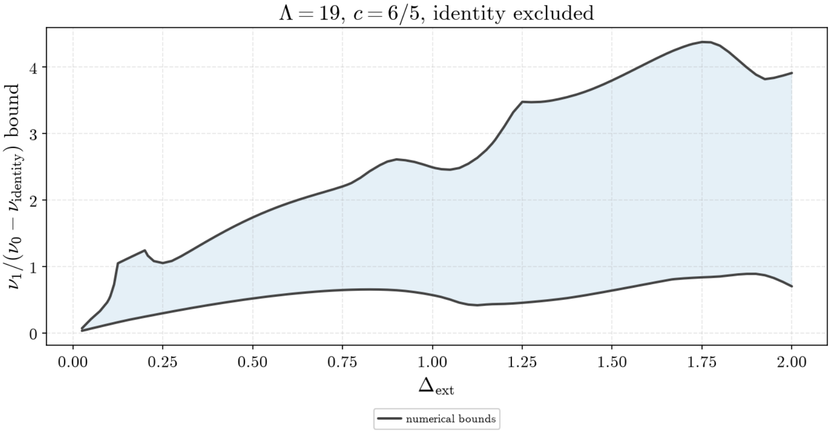
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 2$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



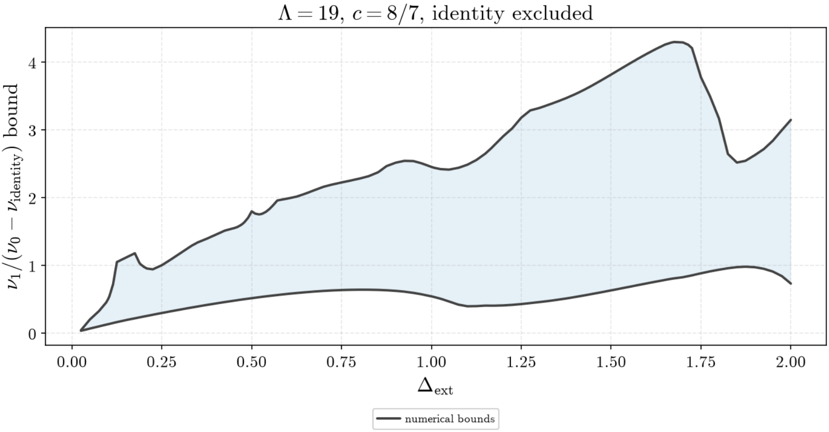
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 3/2$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



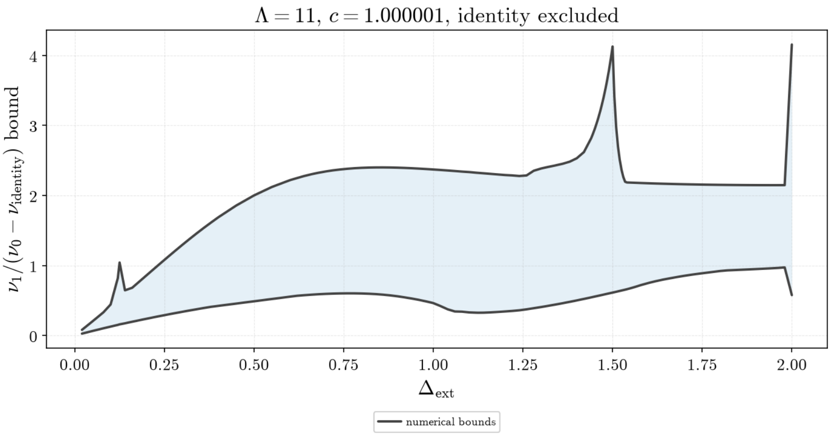
- $\langle\Delta\rangle_*$ over Virasoro primaries at $c = 6/5$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)

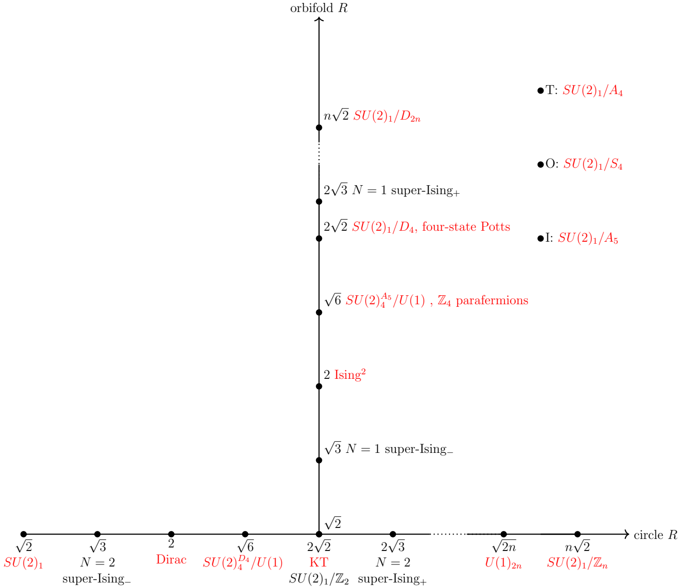


- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 8/7$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)

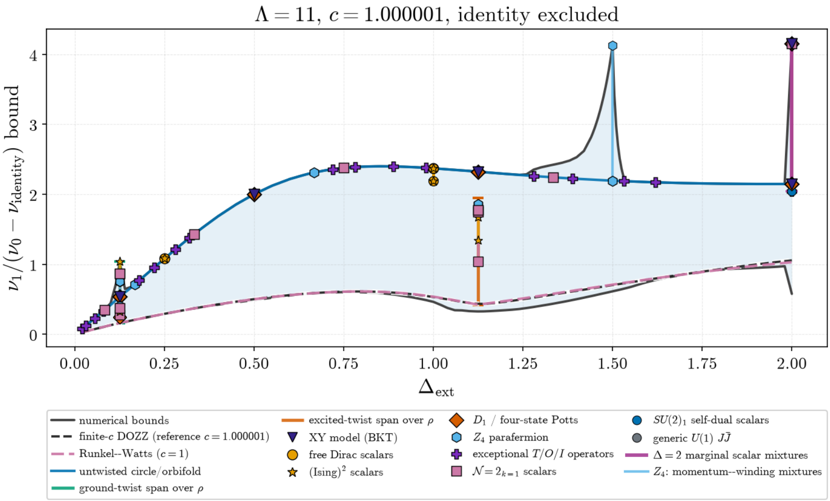


- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 1.000001,\ \Lambda = 11$



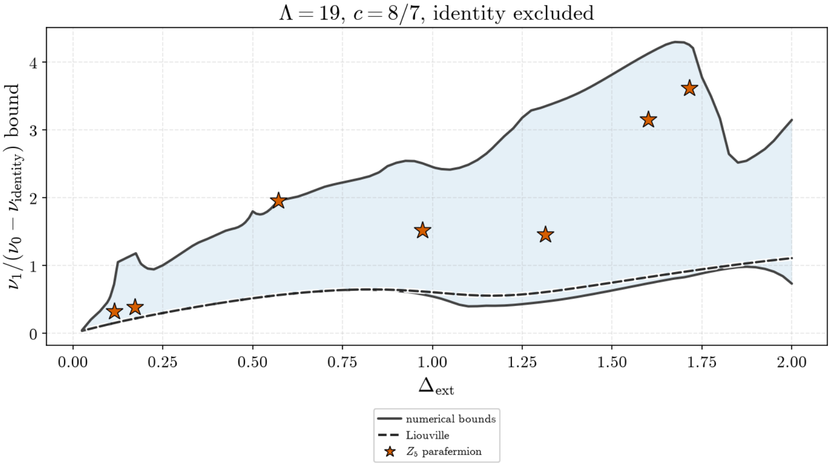
- Rational $c = 1$ CFTs (fig. from [Thorngren, Wang '21])

Virasoro moment bounds (preliminary!)



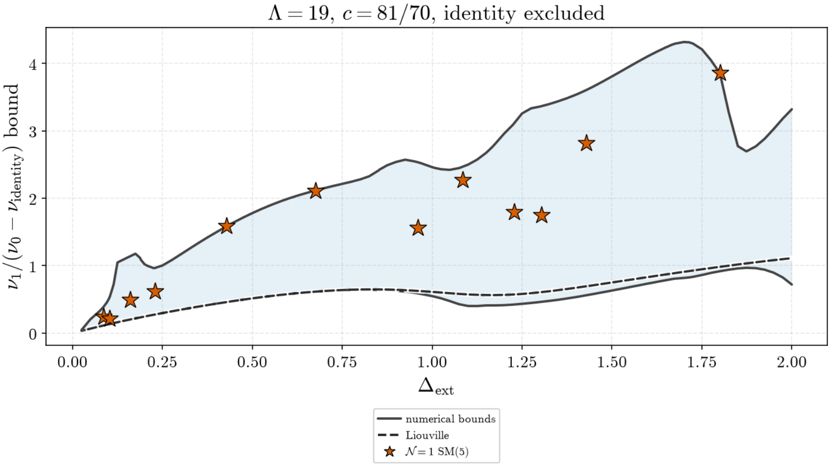
- Known $c = 1$ CFTs: circle branch, orbifold branch, exceptional points, Liouville/Runkel-Watts

Virasoro moment bounds (preliminary!)



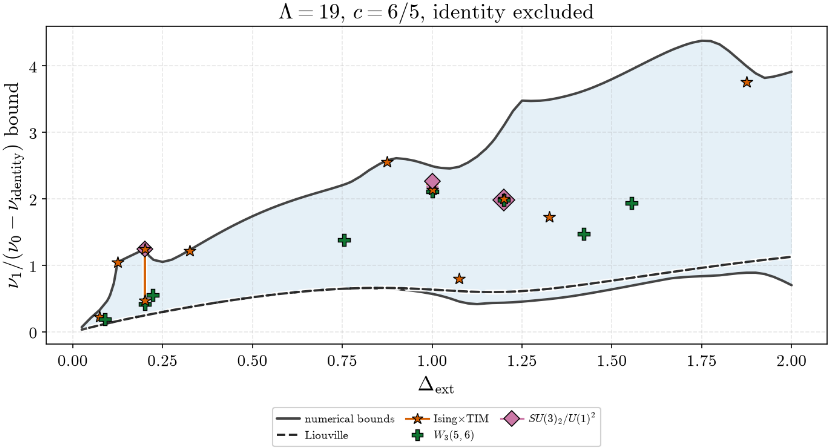
- Known $c = 8/7$ CFTs: Z_5 parafermion, Liouville

Virasoro moment bounds (preliminary!)



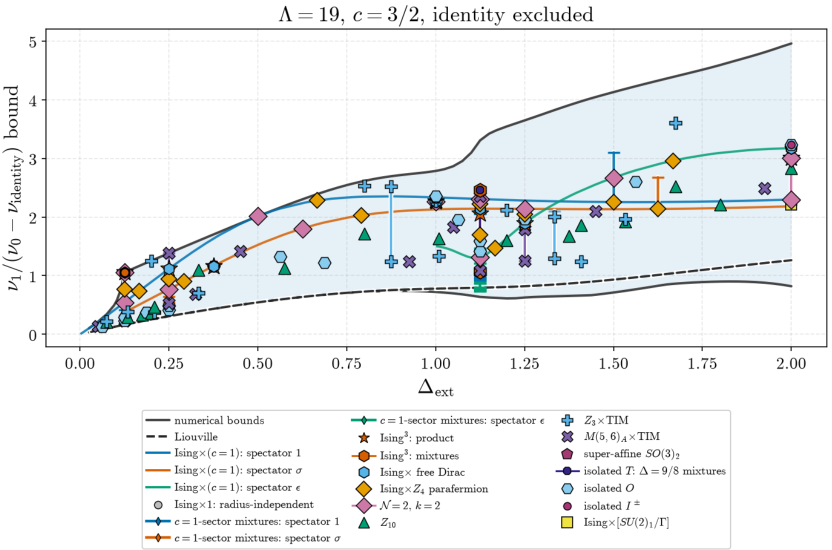
- Known $c = 81/70$ CFTs: $\mathcal{N} = 1$ SM(5) minimal model, Liouville

Virasoro moment bounds (preliminary!)



- Known $c = 6/5$ CFTs: Ising $\times$ tricritical Ising, $W_3(5,6)$ minimal model, $SU(3)_2/U(1)^2$ parafermion, Liouville

Virasoro moment bounds (preliminary!)



- Known $c = 3/2$ CFTs: Many!

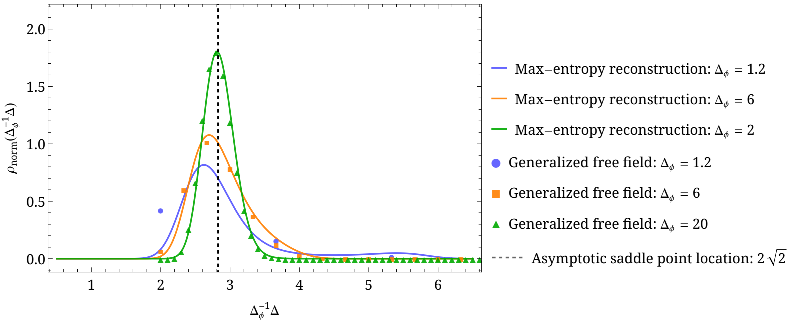
Future directions

- Map the 2d Virasoro moment landscape and hunt for new CFTs
- Mixed-correlator moments over the 3d Ising island
- Interval moments: certify operator counts/distributions in a window
- Hybrid analytic + numerical bootstrap approaches built on moments
- Moments in other kinematics: lightcone, Regge, ...

Thank you!

Backup Slides

Max-Entropy Reconstruction

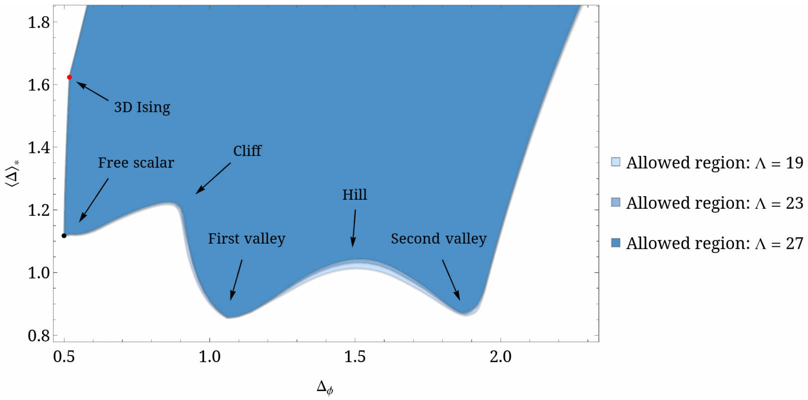


[Chiang, DP, Rogelberg '26]

- Taking a finite # of upper bound moments (here first 5), can reconstruct continuous Gaussian density as a max-entropy distribution:

$$\text{maximize } S[\rho] = -\int \rho(\Delta)\log\rho(\Delta)d\Delta \quad \rightarrow \quad \rho(\Delta) \propto e^{-\sum_{n=1}^{N}\lambda_n\Delta^n}$$

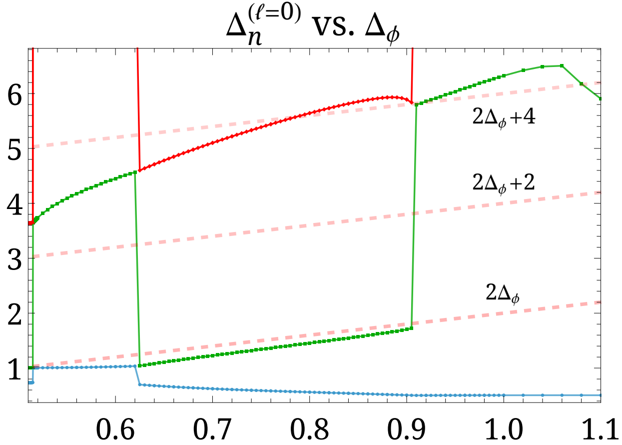
A W-shaped bound in $d = 3$



cliff $\to$ first valley $\to$ hill $\to$ second valley $\to$ linear ramp

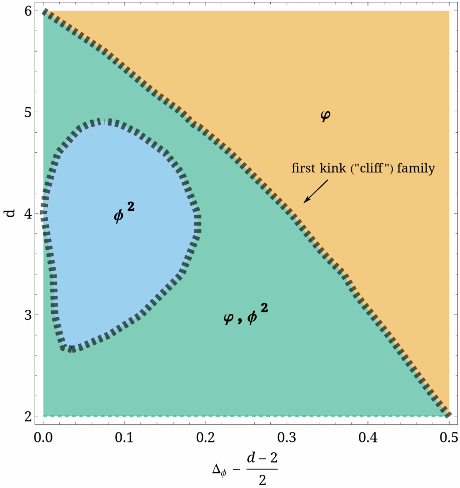
- Stable under increasing Λ, $\ell_{\max}$ $\to$ genuine features of crossing

Spectral reorganization along the lower bound



- A light scalar φ decouples at $\Delta_\phi \approx 0.514$ and reappears at 0.62; at the cliff ($\Delta_\phi \approx 0.91$) the second-lightest scalar decouples

A map of the "CFT park"



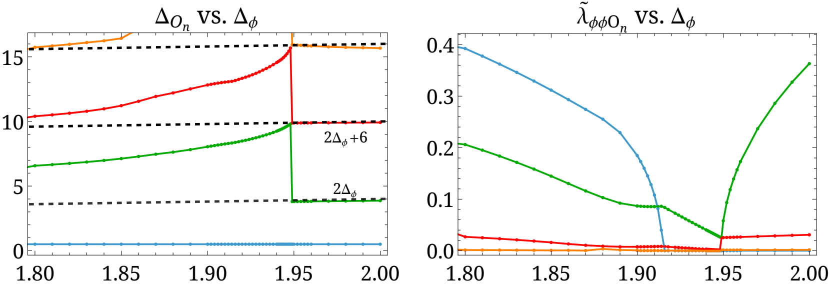
- The "pond": gap state is $\phi^2 \approx 2\Delta_\phi$, no light φ $(2.6 \lesssim d \lesssim 4.9)$

First valley: identity-free solutions

Just past the cliff, the moment reaches its first local minimum:

- The reconstructed extremal correlator grows without bound
- Interpretation: unitary, crossing-symmetric solutions without the identity exist for $\Delta_\phi \gtrsim d-2$, and can be rescaled arbitrarily
- The stress tensor decouples at the first valley in $d \leq 3$

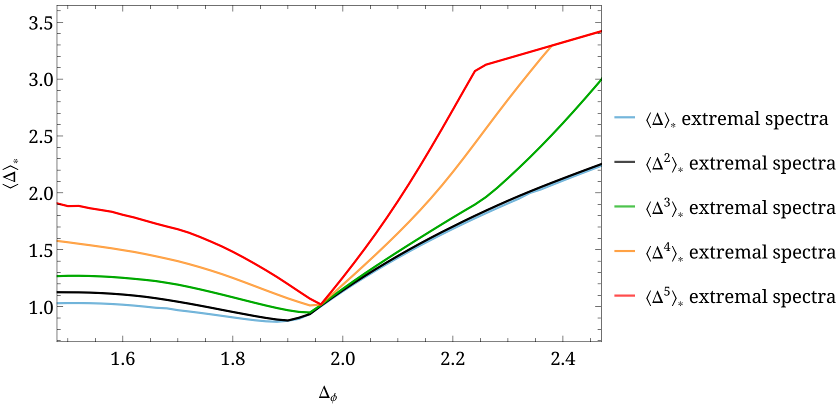
Second valley



Three events in close succession:

- lightest scalar saturates the unitarity bound, with $\lambda^2 \to 0$
- second scalar drops back to $2\Delta_\phi$
- a spectrum simultaneously minimizing the leading moments...

Simultaneous extremization



- Spectra minimizing different $\langle\Delta^k\rangle_*$ all coincide at $\Delta_\phi \approx 1.96$
- Some kind of distinguished solution — we should understand it!

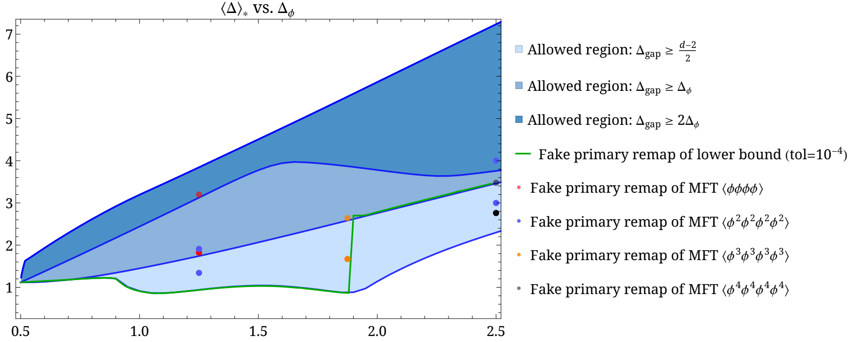
Shadow blocks

An operator at the unitarity bound looks paradoxical ($\lambda^2 \to 0$)...
but the block diverges there, with residue $\propto$ the shadow block:

$$g_{\Delta,0}(z,\overline{z}) \to \frac{R}{\Delta - \frac{d-2}{2}}\, g_{\frac{d+2}{2},0}(z,\overline{z}) \qquad \left(\Delta \to \tfrac{d-2}{2}\right)$$

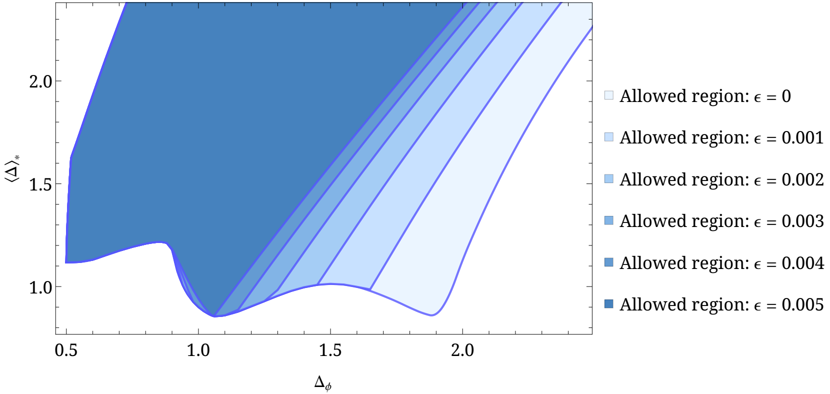
- $\lambda^2\times$ block stays finite: the SDP sees only the total measure
- "Fake primary": the contribution mimics a scalar at $\tilde{\Delta} = \frac{d+2}{2}$
  [Karateev, Kravchuk, Serone, Vichi '19; Erramilli, Iliesiu, Kravchuk, Landry, DP, SD '20]

Remapping the lower bound



- Swap unitarity-bound scalars for their shadows, keeping the measure
- The remapped bound lands on the $\Delta_{\text{gap}} \geq \Delta_\phi$ lower bound

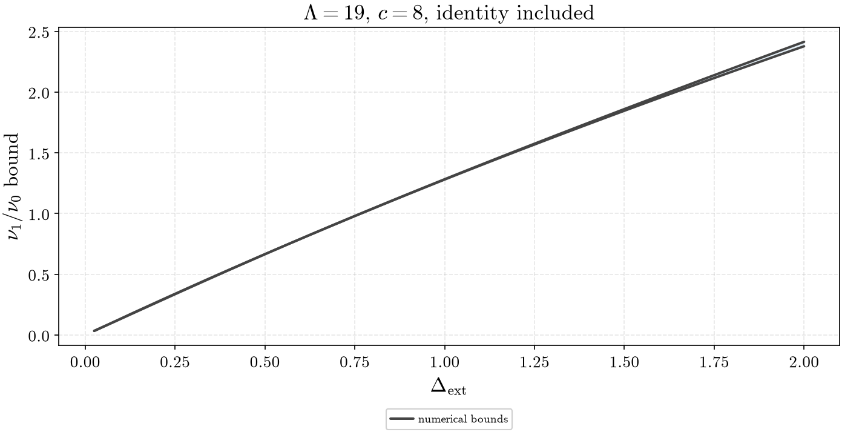
How robust are the kinks?



Impose $\Delta_{\mathsf{gap}} \geq \frac{d-2}{2} + \epsilon$:

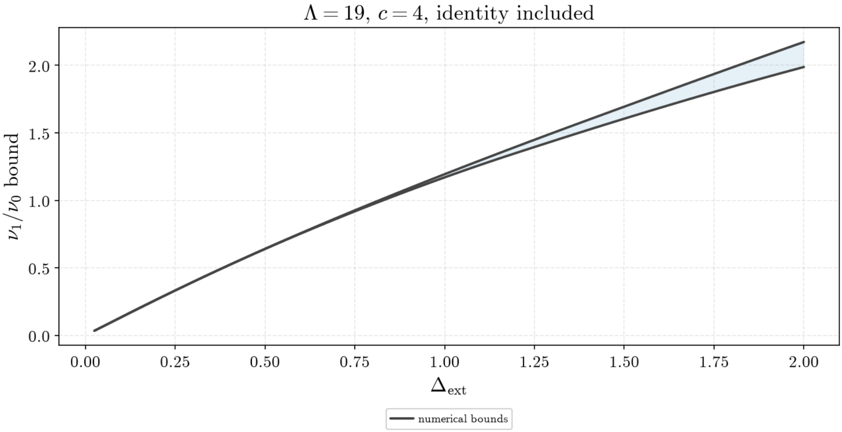
- $\epsilon = 0.001$ already removes the second valley
- The cliff survives larger gaps $\rightarrow$ can't be mapped to an interior solution

Virasoro moment bounds (preliminary!)



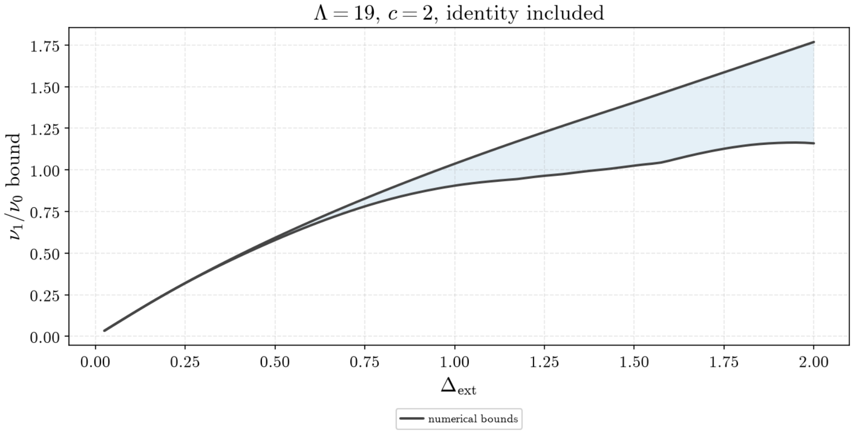
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 8$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



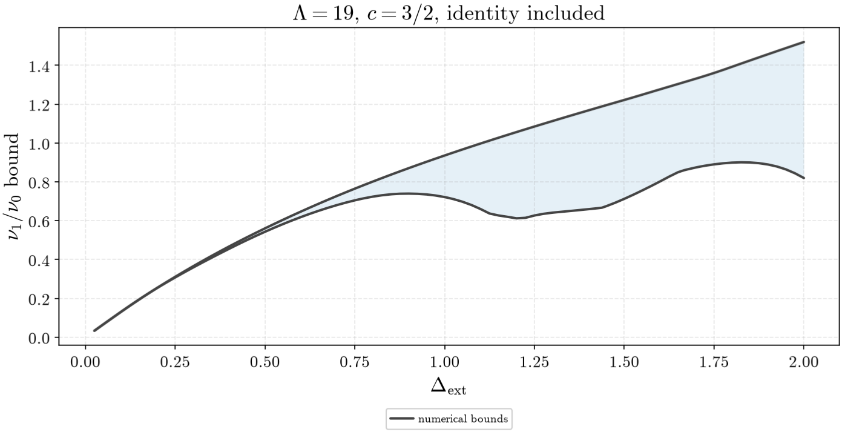
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 4$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



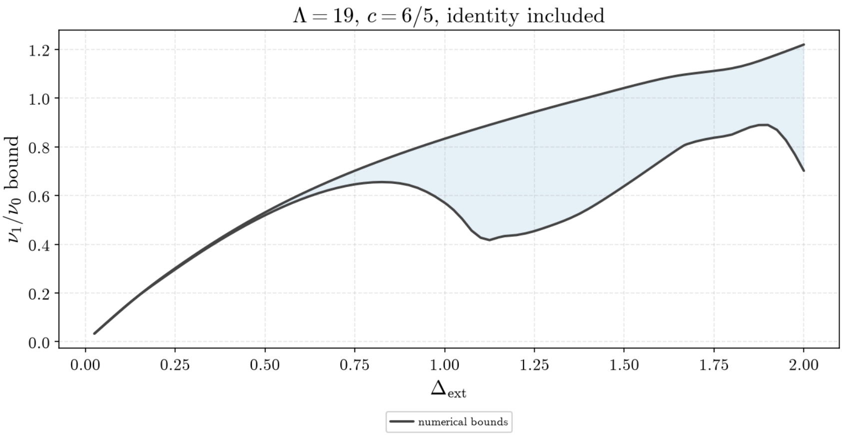
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 2$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



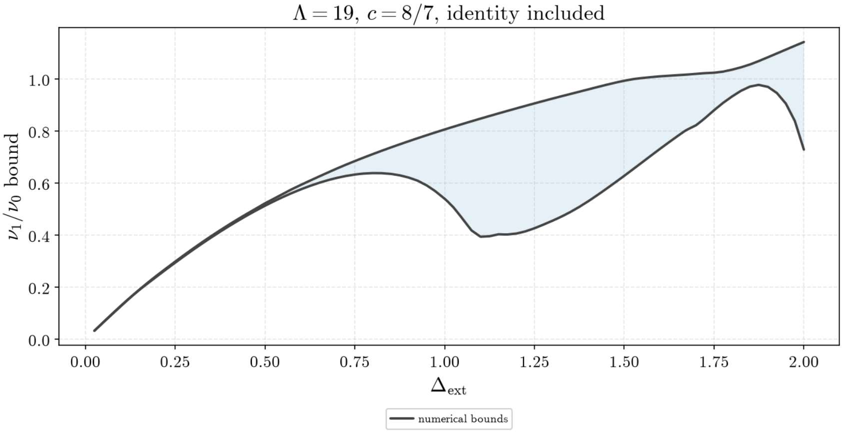
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 3/2$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



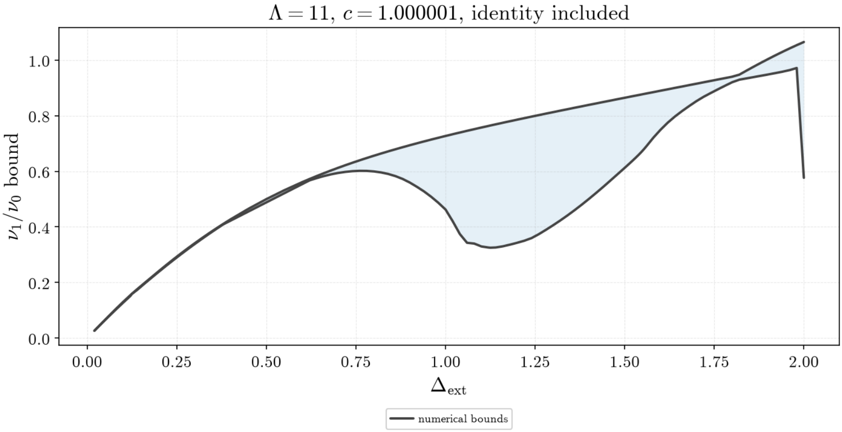
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 6/5$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



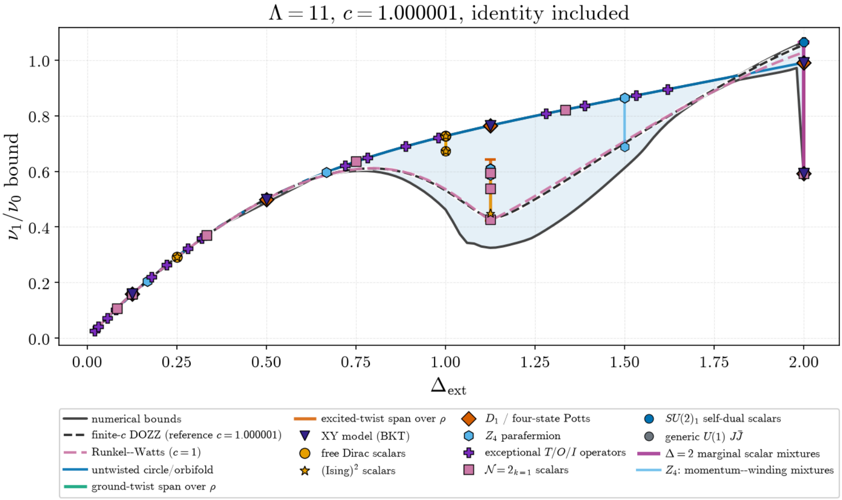
- $\langle \Delta \rangle_*$ over Virasoro primaries at $c = 8/7$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



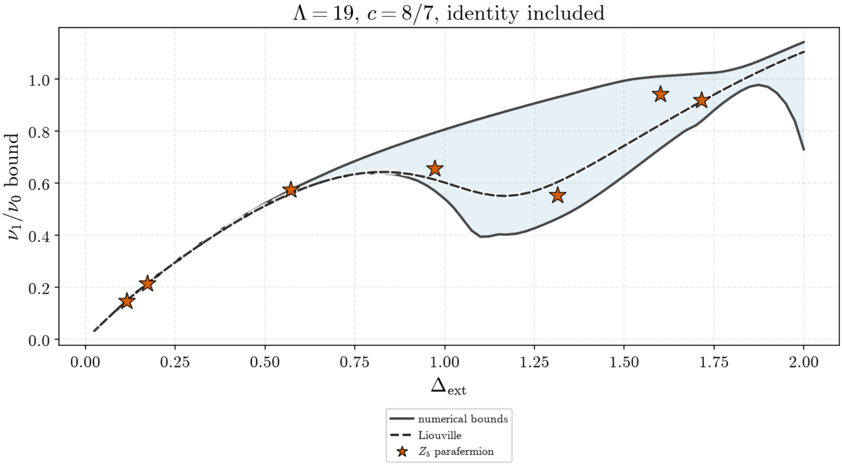
- $\langle \Delta \rangle$ over Virasoro primaries at $c = 1.000001$, $\Lambda = 11$

Virasoro moment bounds (preliminary!)



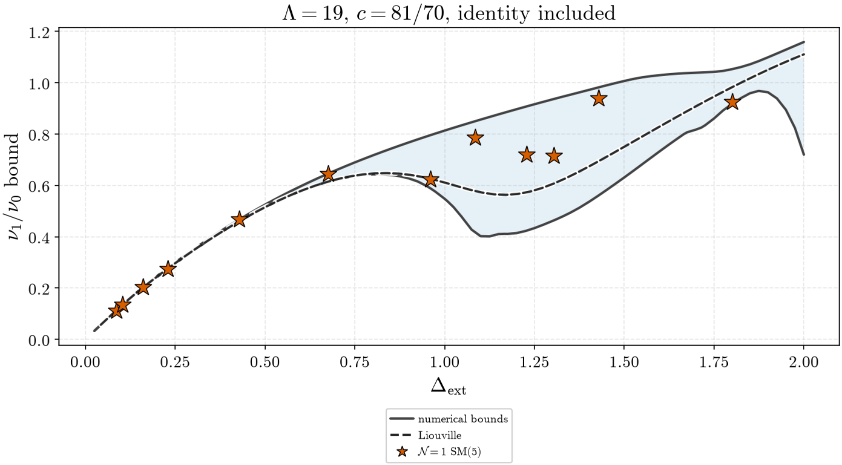
- $\langle \Delta \rangle$ over Virasoro primaries at $c = 1.000001$, $\Lambda = 11$

Virasoro moment bounds (preliminary!)



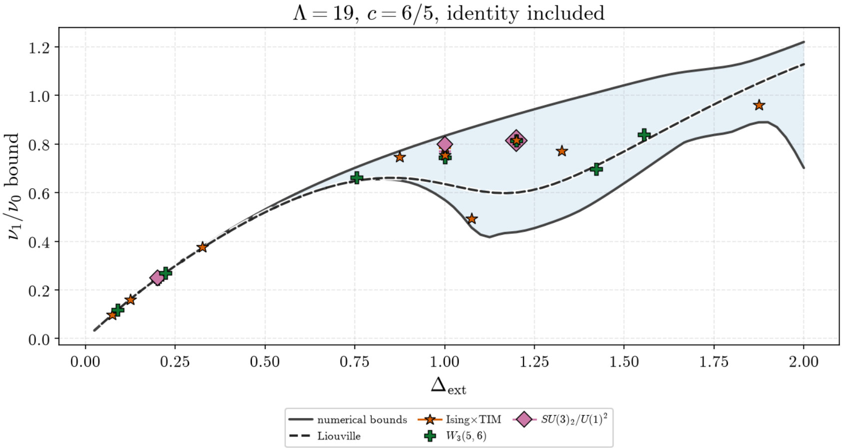
- $\langle \Delta \rangle$ over Virasoro primaries at $c = 8/7$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



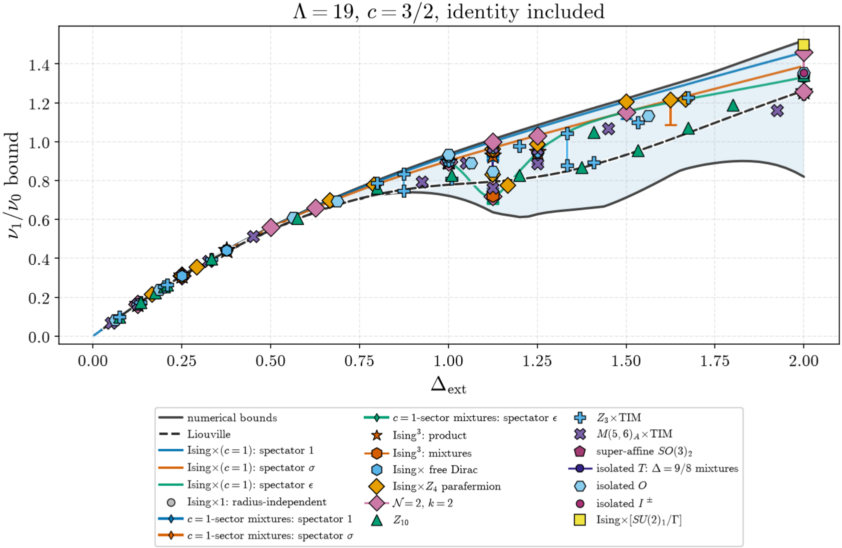
- $\langle \Delta \rangle$ over Virasoro primaries at $c = 81/70,\ \Lambda = 19$

Virasoro moment bounds (preliminary!)



- $\langle \Delta \rangle$ over Virasoro primaries at $c = 6/5$, $\Lambda = 19$

Virasoro moment bounds (preliminary!)



- $\langle\Delta\rangle$ over Virasoro primaries at $c = 3/2$, $\Lambda = 19$