New Developments in Five-Point Bootstrap

P. Tadić

University of Oxford

Bootstrap 2026.

2305.08914 [hep-th], 2312.13344 [hep-th], 2507.01223 [hep-th].
with D. Poland and V. Prilepina

Motivation

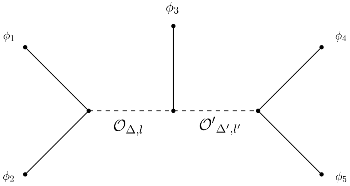
Conformal bootstrap

- Nonperturbative method for solving CFTs.
- Symmetry and consistency conditions $\leftrightarrow$ CFT data.
- All CFT data can be bootstrapped from four-point correlators.
- Bootstrapping correlators with the external spinning operators is hard (large systems, slow block evaluation...).[Chang, Dommes, et al. '24.]
- Use higher-point correlators with external scalars.
- Using five-point correlators, we (approximately) compute the OPE coefficients of two spinning and one scalar operator, e.g. $\lambda^{n_{IJ}}_{T\epsilon T}$, $\lambda^{n_{IJ}}_{T\epsilon' T}$.
- These OPE coefficients are inputs for Hamiltonian truncation technique when studying the RG flow (see eg. [Fitzpatrick, Katz, '22.]).

Five-point correlators

- Using $12-45$ operator product expansion, we have

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\phi_5(x_5)\rangle =$$
$$P(x_i)\sum_{(\mathcal{O},\mathcal{O}')}\sum_{n_{IJ}=0}^{\min(l,l')}\lambda_{12\mathcal{O}}\lambda_{45\mathcal{O}'}\lambda^{n_{IJ}}_{\mathcal{O}3\mathcal{O}'}\mathcal{G}^{(n_{IJ})}_{(\Delta,l,\Delta',l')}(u_1,v_1,u_2,v_2,w).$$



- $n_{IJ}\sim$ conformally invariant tensor structures in $\langle\mathcal{O}_{\mu_1\dots\mu_l}(x_2)\phi_3(x_3)\mathcal{O}'_{\nu_1\dots\nu_{l'}}(x_4)\rangle$.
- Only the comb channel topology.

Five-point conformal blocks

- In the radial coordinates the five-point conformal blocks are given by

$$\mathcal{G}^{(n_{IJ})}_{(\Delta,l,\Delta',l')} = \sum_{n=0}^{\infty} R^{\Delta+\Delta'+n} \sum_{m} \sum_{j_1,j_2} \sum_{k=0}^{\min(j_1,j_2)} c(n,m,j_1,j_2,k)\times$$
$$r^{\Delta-\Delta'+m}\hat{w}^k \eta_1^{j_1-k}\eta_2^{j_2-k},$$

$$m \in [-n, -n+2, \ldots, n-2, n]$$

$$j_1 \in \left[\frac{n+m}{2}+l, \frac{n+m}{2}+l-2, \ldots, \mathrm{Mod}\left(\frac{n+m}{2}+l, 2\right)\right].$$

- Two quadratic Casimir diff. eqs. $\leftrightarrow$ recursion rels. for c-coeffciients
- At order n, one needs to compute $\mathcal{O}(n^4)$ coeffcieints

Five-point conformal blocks

- At order $n = 0$, there are exactly $1 + \min(l, l')$ independent functions of angular coordinates η_1, η_2 and $\hat{w}$, that solve Casimir equations and correspond to different tensor structures.
- Quantum number n_{IJ} enters the given expression through the initial conditions $c(0, 0, l, l', k) \propto \binom{n_{IJ}}{k}$.
- Convergence of the blocks and their derivatives can be accelerated by using the Padé approximant. Schematically

$$\mathcal{G} \sim \sum_{n=0}^{N_{\max}} R^n f_n(r, \eta_1, \eta_2, \hat{w}),$$

and we use Padé approximant

$$G = \left[\frac{N_{\max}}{2} \Big/ \frac{N_{\max}}{2}\right]_{\mathcal{G}}(R).$$

Five-point conformal blocks

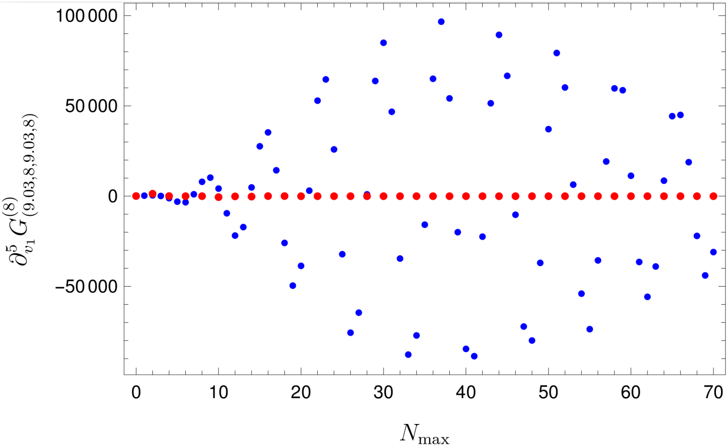


Figure: Conformal block evaluated at $R = 2 - \sqrt{3}$, $r = 1$, $\eta_1 = \eta_2 = \hat{w} = 0$.

3d critical Ising model: $\langle \sigma\sigma\epsilon\sigma\sigma \rangle$

- The associativity of the operator product expansion requires:

$$\langle \sigma\sigma\epsilon\sigma\sigma \rangle = \langle \sigma\sigma\epsilon\sigma\sigma \rangle$$

- The crossing relation is given by

$$\sum_{(\mathcal{O},\mathcal{O}')\neq\hat{1}} \sum_{n_{IJ}} \frac{\lambda_{\sigma\sigma\mathcal{O}}\lambda^{n_{IJ}}_{\mathcal{O}\epsilon\mathcal{O}'}\lambda_{\sigma\sigma\mathcal{O}'}\left(PG_{(\mathcal{O},\mathcal{O}')} - \tilde{P}\tilde{G}_{(\mathcal{O},\mathcal{O}')}\right)}{\lambda_{\sigma\sigma\epsilon}\left(\tilde{P}\tilde{G}_{(\hat{1},\epsilon)} - PG_{(\hat{1},\epsilon)}\right)} = 1,$$

where tilde denotes contributions in the crossed channel obtained by exchanging $x_2 \leftrightarrow x_4$.

3d critical Ising model: $\langle\sigma\sigma\epsilon\sigma\sigma\rangle$

- σ is $\mathbb{Z}_2$ odd and ϵ is $\mathbb{Z}_2$ even scalar.
- We **truncate** the sum over the primary operators to include only operators with $\Delta \leqslant \Delta_{\text{cutoff}} \approx 5.02$. [Gliozzi, '13.] [Gliozzi, Rago, '14.] [Li, '18.]
- $\mathbf{1}$, ϵ, ϵ', $T_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$.
- We fix all conformal dimensions of exchanged operators as those are computed in the bootstrap of the four-point functions.
- We also fix all (known) OPE coefficients of two scalars and one spinning operator.

Disconnected correlator: $\langle\sigma\sigma\epsilon\sigma\sigma\rangle_{\mathrm{d}}$

- Truncated operators are approximated with contributions from the disconnected correlator

$$\langle\sigma\sigma\epsilon\sigma\sigma\rangle_{\mathrm{d}} = \langle\sigma\sigma\rangle\langle\sigma\sigma\epsilon\rangle + (\text{permutations}).$$

- Pairs of operators that contribute to the disconnected correlator are $(\hat{1},\epsilon)$ and $([\sigma,\sigma]_{n,s},[\sigma,\sigma]_{n',s'})$.

- Double-twist sigma operators on the leading Regge trajectory in the 3d critical Ising model $[\sigma,\sigma]_{0,s}^{\text{Ising}} \sim : \sigma\partial_{\mu_1}\ldots\partial_{\mu_s}\sigma :$ have small anomalous dimensions. [Simmons-Duffin, '16.]

3d critical Ising model: $\langle\sigma\sigma\epsilon\sigma\sigma\rangle$

- We take derivatives $\mathcal{D}_i$ of the crossing relation with respect to the cross-ratios and evaluate them at

$$u_1 = v_1 = u_2 = v_2 = 1, \quad w = \frac{3}{2},$$

$$R = 2 - \sqrt{3}, \quad r = 1, \quad \eta_1 = \eta_2 = \hat{w} = 0.$$

- Symmetric w.r.t. $x_2 \leftrightarrow x_4$ exchange.
- We get more constraints than there are unknown OPE coefficients.
- **Cost function** $\mathcal{F}$: sum of squares of the constraints. We minimize it to compute the unknown OPE coefficients.
- <u>What constraints to include in the cost function?</u>

The cost function

- What do we expect from a cost function?
- Cost function should be possible to solve with the least amount of flat directions.
- We select constraints that maximize

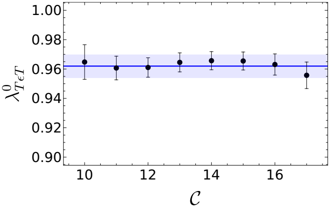
$$\mathcal{I} = \frac{\det\left(\frac{\partial^2 \mathcal{F}}{\partial_{\lambda_i}\partial_{\lambda_j}}\Big|_{\min}\right)}{\mathcal{F}\big|_{\min}}.$$

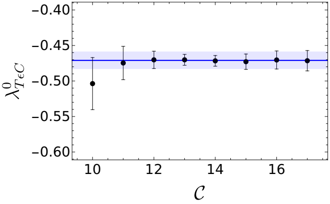
- We do the calculation with different numbers of constraints $\mathcal{C}$'s in the cost function and for each $\mathcal{C}$ we use several sets with largest value of $\mathcal{I}$.

3d critical Ising model: $\langle\sigma\sigma\epsilon\sigma\sigma\rangle$

- Without spin-6 contributions, combining data from 20 sets of constraints with largest $\mathcal{I}$ for $12 \leqslant \mathcal{C} \leqslant 17$, we get:

	est. values
$\lambda^0_{T\epsilon T}$	0.962(8)
$\lambda^0_{T\epsilon C}$	-0.471(12)
$\lambda^0_{T\epsilon S}$	-0.196(8)
$\lambda^4_{C\epsilon C}$	-0.25(4)
$\lambda^3_{C\epsilon C}$	-0.6(2)
$\lambda^2_{C\epsilon C}$	1.8(8)
$\lambda^1_{C\epsilon C}$	-8(2)
$\lambda^0_{C\epsilon C}$	-2.9(9)
$\lambda_{\epsilon'\epsilon C}$	0.75(20)

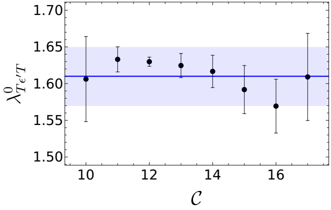


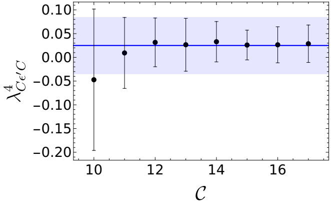


3d critical Ising model: $\langle \sigma\sigma\epsilon'\sigma\sigma \rangle$

- No OPE is taken with middle operator $\rightarrow$ We can use irrelevant scalars as the middle operator, ex. ϵ'.
- $\mathbf{1}, \epsilon, \epsilon', T_{\mu\nu}, C_{\mu\nu\rho\sigma}$ plus the disconnected contributions.

	est. values
$\lambda^0_{T\epsilon' T}$	1.61(4)
$\lambda^0_{T\epsilon' C}$	-1.2(3)
$\lambda^4_{C\epsilon' C}$	0.02(6)
$\lambda^3_{C\epsilon' C}$	2(2)
$\lambda^2_{C\epsilon' C}$	3(5)
$\lambda^1_{C\epsilon' C}$	-4(10)
$\lambda^0_{C\epsilon' C}$	3(8)
$\lambda_{\epsilon'\epsilon C}$	0.58(11)
$\lambda_{\epsilon'\epsilon' C}$	2.8(7)





3d critical Ising model: conformal collider bounds

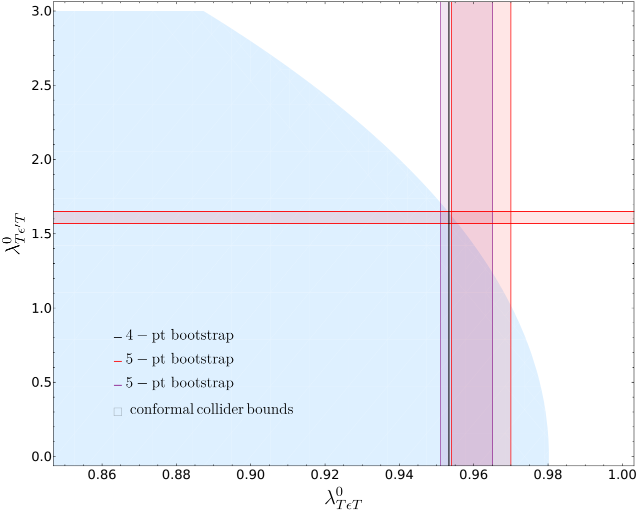


Figure: Conformal collider bounds from [Cordova, Maldacena, Turiaci, '17.]

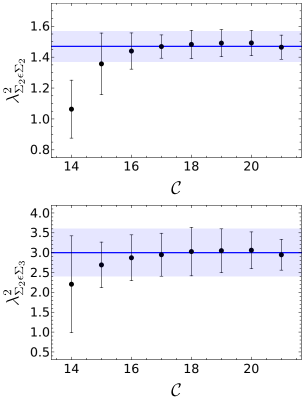
3d critical Ising model: $\langle\sigma\epsilon\epsilon\epsilon\sigma\rangle$ ($\langle\sigma\epsilon\epsilon'\epsilon\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\epsilon\rangle$)

- Same method works for other five-point correlators:

$$\langle\sigma\epsilon\epsilon\epsilon\sigma\rangle_{\mathrm{d}} = \langle\sigma\sigma\rangle\langle\epsilon\epsilon\epsilon\rangle + \langle\epsilon\epsilon\rangle\langle\sigma\sigma\epsilon\rangle + \text{perm.}$$

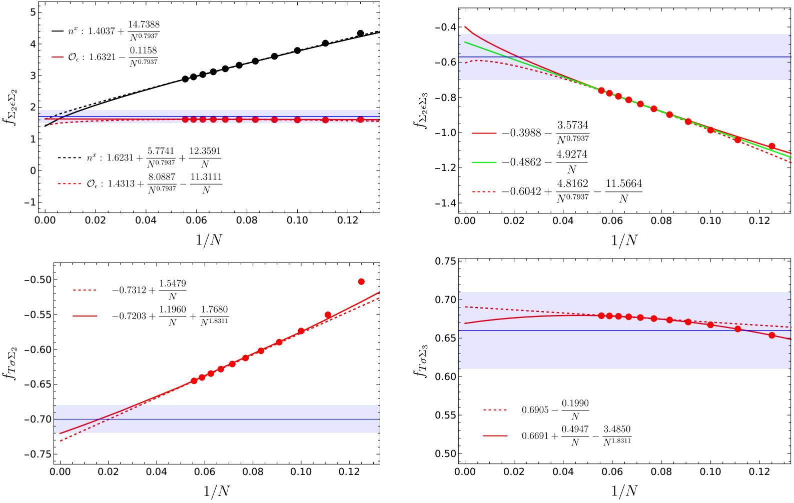
$[\sigma,\epsilon]_{0,0}$, $[\sigma,\epsilon]_{1,0}$, $[\sigma,\epsilon]_{0,1}$ have to be subtracted since no analogs exist in Ising.

	est. values
$\lambda^2_{\Sigma_2\epsilon\Sigma_2}$	1.47(10)
$\lambda^1_{\Sigma_2\epsilon\Sigma_2}$	-0.2(8)
$\lambda^0_{\Sigma_2\epsilon\Sigma_2}$	1.5(8)
$\lambda^2_{\Sigma_2\epsilon\Sigma_3}$	3.0(6)
...	...
$\lambda^3_{\Sigma_3\epsilon\Sigma_3}$	0.8(4)
...	...



3d critical Ising model: $\langle\sigma\epsilon\epsilon\epsilon\sigma\rangle$, $\langle\sigma\epsilon\epsilon'\epsilon\sigma\rangle$, $\langle\sigma\sigma\sigma\sigma\epsilon\rangle$

- Comparison with fuzzy sphere predictions [Hu, He, Zhu, '23.]



Summary and future developments

- Numerical evaluation of five-point conformal blocks (and their derivatives).
- OPE coefficients from five-point correlators in 3d Ising:

$$\langle\sigma\sigma\epsilon\sigma\sigma\rangle,\ \langle\sigma\sigma\epsilon'\sigma\sigma\rangle,\ \langle\sigma\epsilon\epsilon\epsilon\sigma\rangle,\ \langle\sigma\epsilon\epsilon'\epsilon\sigma\rangle,\ \langle\sigma\sigma\sigma\sigma\epsilon\rangle$$

- Other CFTs, $O(N)$ models?
- Six-point conformal blocks[1] and correlators. [Antunes et al. '23.]
- Analytic bootstrap? see: [Antunes, Costa, Goncalves, Harris, Kaviraj, Kravchuk, Mann, Quintavalle, Salgarkar, Schomerus, Vilas Boas . . .]

[1] in the snowflake channel ✓

Thank you!