Recursion relations for conformal blocks in 4 dimensions

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Based on work with **Petr Kravchuk**

Motivation

- Conformal blocks are the basic ingredients of the conformal bootstrap, but general spinning blocks are difficult to compute.
- Zamolodchikov recursion in generic dimensions reconstructs a conformal block from its large-Δ limit and its singularities:

$$H_{\Delta,\rho} = H_{\infty,\rho} + \sum_i \frac{R_i r^{n_i > 0}}{\Delta - \Delta_i} H_{\Delta_i + n_i, \rho_i} \tag{1}$$

where

$$H_{\Delta,\rho} = r^{-\Delta} G_{\Delta,\rho}, \quad r = r(z, \bar{z}) \tag{2}$$

- They give a systematic method to compute conformal blocks.
- In even dimensions conformal blocks can have double poles

$$H_{\Delta,\rho} = H_{\infty,\rho} + \sum_i \left(\frac{R_i r^{n_i} H_{\Delta_i + n_i, \rho_i}}{(\Delta - \Delta_i)^2} + \frac{???}{\Delta - \Delta_i} \right). \tag{3}$$

Context

- Virasoro blocks **[Zamalodchikov '87]**.
- Scalar blocks **[Kos, Poland & Simmons-Duffin '13]**.
- Theory for spinning blocks and blocks in odd dimensions have simple poles **[Penedones, Trevisani & Yamazaki '16]**.
- Closed form expressions for general spinning blocks in $3d$ **[Erramilli, Iliesiu & Kravchuk '19]**.

Conformal blocks

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} G_{\Delta_{\mathcal{O}},\rho_{\mathcal{O}}}(x_1, \dots x_4). \qquad (4)$$

Let V be the parabolic Verma module generated by $|\mathcal{O}_{\Delta,\rho}\rangle$ and let $\{|e_i\rangle\}$ be a basis of V.

The conformal block associated with the exchange of $|\mathcal{O}_{\Delta,\rho}\rangle$ is

$$G_{\Delta,\rho}(x_i) = \sum_{i,j} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|e_i\rangle (Q_V)^{-1}_{ij} \langle e_j|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \qquad (5)$$

where

$$(Q_V)_{ij} = \langle e_i|e_j\rangle. \qquad (6)$$

Conformal blocks as maps

Let T_{ij} denote the conformal representation carried by the pair $\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)$.
The left and right three-point functions define maps

$$\mathcal{T}_{\mathrm{L}} : V \longrightarrow T_{12}, \qquad |\psi\rangle \longmapsto \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\psi\rangle, \tag{7}$$

$$\mathcal{T}_{\mathrm{R}}^{\vee} : T_{34}^{\vee} \longrightarrow V^{\vee}. \tag{8}$$

The Shapovalov form defines

$$Q_V : V \longrightarrow V^{\vee}, \qquad (Q_V)_{ij} = \langle e_i|e_j\rangle. \tag{9}$$

For generic Δ it is non-degenerate, giving the inverse Shapovalov map

$$Q_V^{-1} : V^{\vee} \longrightarrow V. \tag{10}$$

At a special value $\Delta = \Delta^*$ a null state appears and Q_V becomes degenerate. Regarded as a meromorphic function in Δ, Q_V^{-1} develops a pole.

Conformal blocks as maps

$$\mathcal{G}_{\Delta,\rho} = \mathcal{T}_{\mathrm{L}} \circ Q_V^{-1} \circ \mathcal{T}_{\mathrm{R}}^{\vee}. \tag{11}$$

Equivalently,

$$T_{34}^{\vee} \xrightarrow{\mathcal{T}_{\mathrm{R}}^{\vee}} V^{\vee} \xrightarrow{Q_V^{-1}} V \xrightarrow{\mathcal{T}_{\mathrm{L}}} T_{12}. \tag{12}$$

A conformal block may be viewed as

$$\mathcal{G}_{\Delta,\rho} : T_{34}^{\vee} \longrightarrow T_{12}. \tag{13}$$

Representing an element of $T_{34}^{\vee}$ by a function or distribution $f(x_3, x_4)$, the map is

$$G_{\Delta,\rho} : f(x_3, x_4) \longmapsto \left[(x_1, x_2) \longmapsto \int dx_3\, dx_4\, G_{\Delta,\rho}(x_1, x_2, x_3, x_4) f(x_3, x_4)\right]. \tag{14}$$

For a simple pole,

$$(Q_V)^{-1} = \frac{A}{\Delta - \Delta^*} + \dots . \tag{15}$$

Multiplying by $\Delta - \Delta^*$ we find the residue in the generic situation is given by

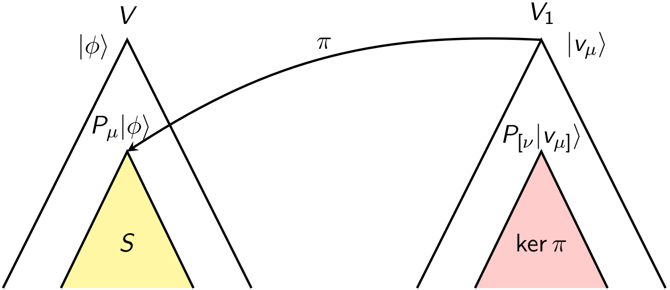
$$\begin{tikzcd}
T^\vee \arrow[r, "\mathcal{T}_{\mathrm{R}}"] \arrow[dr, bend right, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] & V^\vee \arrow[r, "A"] \arrow[d, two heads, "i^\vee"'] & V \arrow[r, "\mathcal{T}_{\mathrm{L}}"] & T \\
& V_1^\vee \arrow[r, "Q_1^{-1}"] & V_1 \arrow[u, hook, "i"'] \arrow[ur, bend right, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}$$

If the Verma module V_1 contains null states, then,

$$
\begin{tikzcd}
T^\vee \arrow[r, "\mathcal{T}_{\mathrm{R}}"] \arrow[ddr, bend right, red, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] & V^\vee \arrow[r, "A"] \arrow[d, two heads, "i^\vee"'] & V \arrow[r, "\mathcal{T}_{\mathrm{L}}"] & T \\
 & S^\vee \arrow[r, "A'"] \arrow[d, hook, "\pi^\vee"'] & S \arrow[u, hook, "i"] & \\
 & V_1^\vee \arrow[r, "Q_1^{-1}"] & V_1 \arrow[u, two heads, "\pi"'] \arrow[uur, bend right, red, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}
$$

Example in $3d$: the singular module as a quotient

Take $|\phi\rangle$ a $\Delta = 0$ scalar and $|v_\mu\rangle$ a $\Delta = 1$ vector.
The null state $P_\mu|\phi\rangle$ generates the singular submodule S.



The quotient map is defined by

$$\pi : V_1 \twoheadrightarrow S, \qquad \pi\big(|v_\mu\rangle\big) = P_\mu|\phi\rangle. \tag{16}$$

We have

$$\pi\big(P_{[\nu}|v_{\mu]}\rangle\big) = P_{[\nu}P_{\mu]}|\phi\rangle = 0. \tag{17}$$

Moving to $4d$

In $4d$, conformal blocks can have second-order poles. Writing $\epsilon = \Delta - \Delta^*$,

$$\mathcal{G}(\epsilon) = \frac{\mathcal{G}^{-2}}{\epsilon^2} + \frac{\mathcal{G}^{-1}}{\epsilon} + O(1), \tag{18}$$

so that

$$\epsilon^2 \mathcal{G}(\epsilon) = \mathcal{G}^{-2} + \epsilon \mathcal{G}^{-1} + O(\epsilon^2). \tag{19}$$

For the $\Delta = 0$ scalar

$$\mathcal{G}^{-1}_{\Delta=0,\ell=0} = \alpha_1 \operatorname{reg} \mathcal{G}_{1,1} + \alpha_2 \operatorname{reg} \mathcal{G}_{3,1} + \alpha_3 \partial_\Delta \mathcal{G}_{4,0} + \alpha_4 \mathcal{G}_{4,0}. \tag{20}$$

reg $G_{\Delta,\ell}$ is the coefficient of ϵ^0 in $G_{\Delta+\epsilon,\ell}$.

Logarithmic Verma modules

Replace

$$\Delta \longrightarrow \Delta^* + \epsilon, \qquad \epsilon^n = 0, \tag{21}$$

and extend all spaces over $\mathbb{C}[\epsilon]/(\epsilon^n)$.
For example, over $\mathbb{C}[\epsilon]/(\epsilon^2)$ we have $|\phi\rangle$ and $\epsilon|\phi\rangle$,

$$D|\phi\rangle = \Delta|\phi\rangle + \epsilon|\phi\rangle, \qquad D\epsilon|\phi\rangle = \Delta\epsilon|\phi\rangle. \tag{22}$$

This retains the required orders in the inverse Shapovalov map and in both three-point structures.

$$\widetilde{T}^\vee \xrightarrow{\widetilde{\mathcal{T}}_{\mathrm{R}}^\vee} \widetilde{V}^\vee \xrightarrow{(\widetilde{Q}_V)^{-1}} \widetilde{V} \xrightarrow{\widetilde{\mathcal{T}}_{\mathrm{L}}} \widetilde{T}. \tag{23}$$

$$
\begin{tikzcd}
\widetilde{T}^\vee \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{R}}"] \arrow[ddr, bend right, red, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] & \widetilde{V}^\vee \arrow[r, "A"] \arrow[d, two heads, "i^\vee"'] & \widetilde{V} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{L}}"] & \widetilde{T} \\
 & S^\vee \arrow[r, "A'"] \arrow[d, hook, "\pi^\vee"'] & S \arrow[u, hook, "i"] & \\
 & \widetilde{V}_1^\vee \arrow[r, "Q_1^{-1}"] & \widetilde{V}_1 \arrow[u, two heads, "\pi"'] \arrow[uur, bend right, red, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\widetilde{T}^{\vee} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{R}}"] \arrow[ddr, bend right, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] & \widetilde{V}^{\vee} \arrow[r, "A"] \arrow[d, two heads, "i^{\vee}"'] & \widetilde{V} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{L}}"] & \widetilde{T} \\
& S^{\vee} \arrow[r, "A'"] \arrow[d, hook, "\pi^{\vee}"'] & S \arrow[u, hook, "i"'] & \\
& \widetilde{V}_1^{\vee} \arrow[r, "Q_1^{-1}"] & \widetilde{V}_1 \arrow[u, two heads, "\pi"'] \arrow[uur, bend right, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}
$$

$\widetilde{\mathcal{O}}_0^{\mathfrak{p}}$

$$
\begin{tikzcd}
\widetilde{T}^{\vee} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{R}}"] \arrow[drr, phantom] \arrow[dd, bend right, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] & \widetilde{V}^{\vee} \arrow[r, "A"] \arrow[d, twoheadrightarrow, "i^{\vee}"'] & \widetilde{V} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{L}}"] & \widetilde{T} \\
 & S^{\vee} \arrow[r, "A'"] \arrow[d, hookrightarrow, "\pi^{\vee}"'] & S \arrow[u, hookrightarrow, "i"'] \arrow[r, phantom, "\widetilde{\mathcal{O}}_0^{\mathfrak{p}}"] & \\
 & \widetilde{V}_1^{\vee} \arrow[r, "Q_1^{-1}"] & \widetilde{V}_1 \arrow[u, twoheadrightarrow, "\pi"'] \arrow[uur, bend right, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}
$$

- Morita theory gives an equivalence

$$\widetilde{\mathcal{O}}_0^{\mathfrak{p}} \simeq \operatorname{mod} A, \tag{24}$$

  where A is a finite-dimensional algebra and $\operatorname{mod} A$ is the category of finite-dimensional A-modules.
- Hence morphisms between logarithmic parabolic Verma modules can be computed by studying morphisms between finite-dimensional A-modules.

Conformal blocks and subquotients

$$
\begin{tikzcd}
\widetilde{T}^\vee \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{R}}"] \arrow[dddr, bend right, "\widehat{\mathcal{T}}_{\mathrm{R}}"'] \arrow[ddr, dashed, "?"'] & \widetilde{V}^\vee \arrow[r, "A"] \arrow[d, two heads, "i^\vee"'] & \widetilde{V} \arrow[r, "\widetilde{\mathcal{T}}_{\mathrm{L}}"] & \widetilde{T} \\
 & S^\vee \arrow[r, "A'"] & S \arrow[u, hook, "i"] \arrow[d, hook, "j"'] & \\
 & X^\vee \arrow[r, "A''"] \arrow[u, two heads, "j^\vee"] \arrow[d, hook, "\pi^\vee"'] & X \arrow[uur, dashed, "?"'] & \\
 & D^\vee \arrow[r, "\widetilde{Q}_D^{-1}"] & D \arrow[u, two heads, "\pi"'] \arrow[uuur, bend right, "\widehat{\mathcal{T}}_{\mathrm{L}}"'] &
\end{tikzcd}
$$

$$\widetilde{Q_D}^{-1} = \bigoplus_i C_i(\epsilon)\widetilde{Q}_i^{-1}(\epsilon). \qquad (25)$$

Summary

- Introduced categorical language for conformal blocks.
- Logarithmic Verma modules allow us to work with Taylor series to keep track of singularities.
- Morita theory lets one systematically look for maps between objects in $\widetilde{\mathcal{O}}_0^{\mathfrak{p}}$.
- In $4d$ the singular submodule S is a subquotient of a direct sum of logarithmic Verma modules.
- We obtain a systematic description of the pole structure of conformal blocks in $4d$.

Thank you!