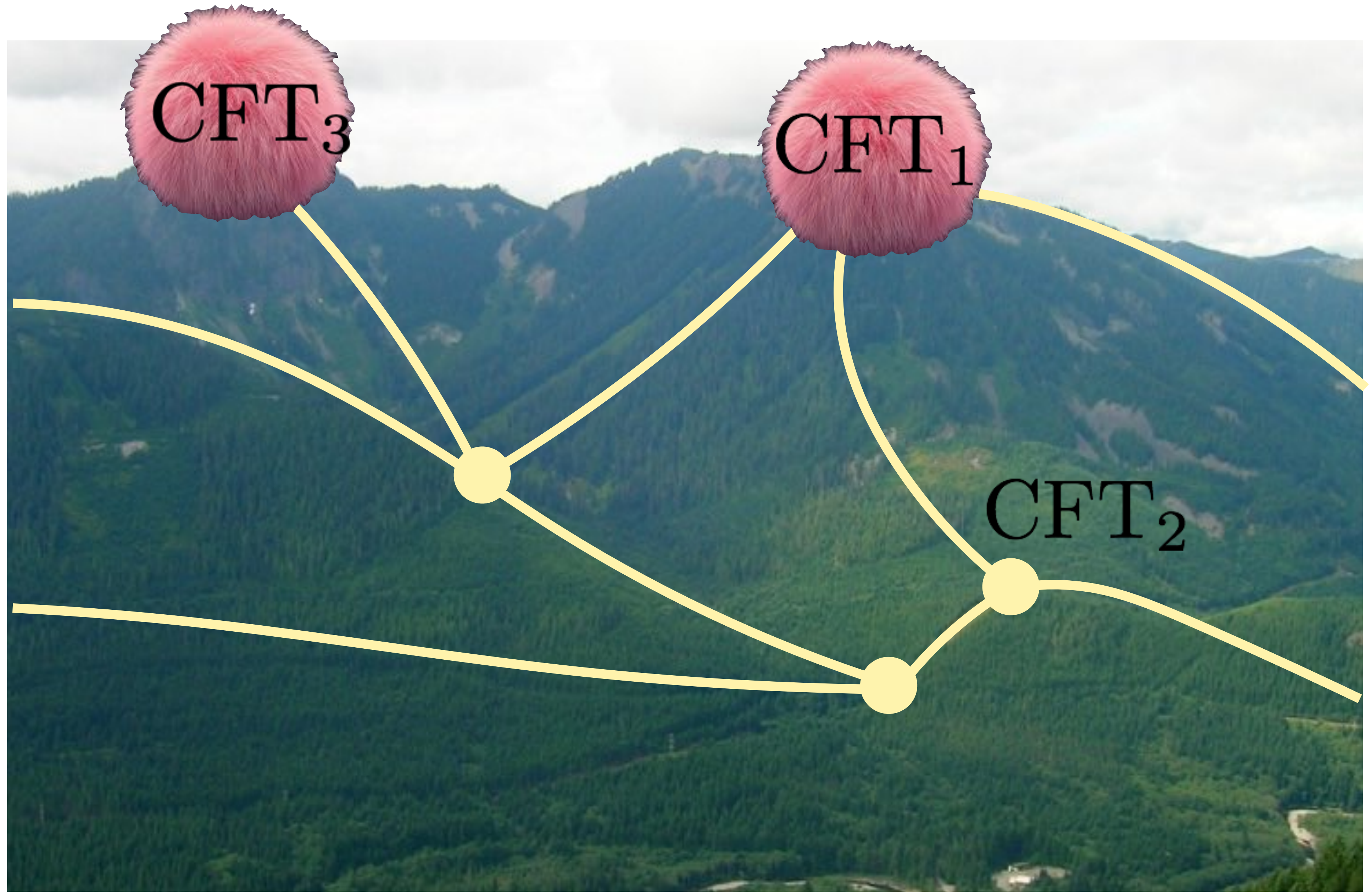
3d Ising Field Theory: Fuzzy Sphere Meets TCSA

Liam Fitzpatrick
Boston University
Bootstrap, August 7, 2026



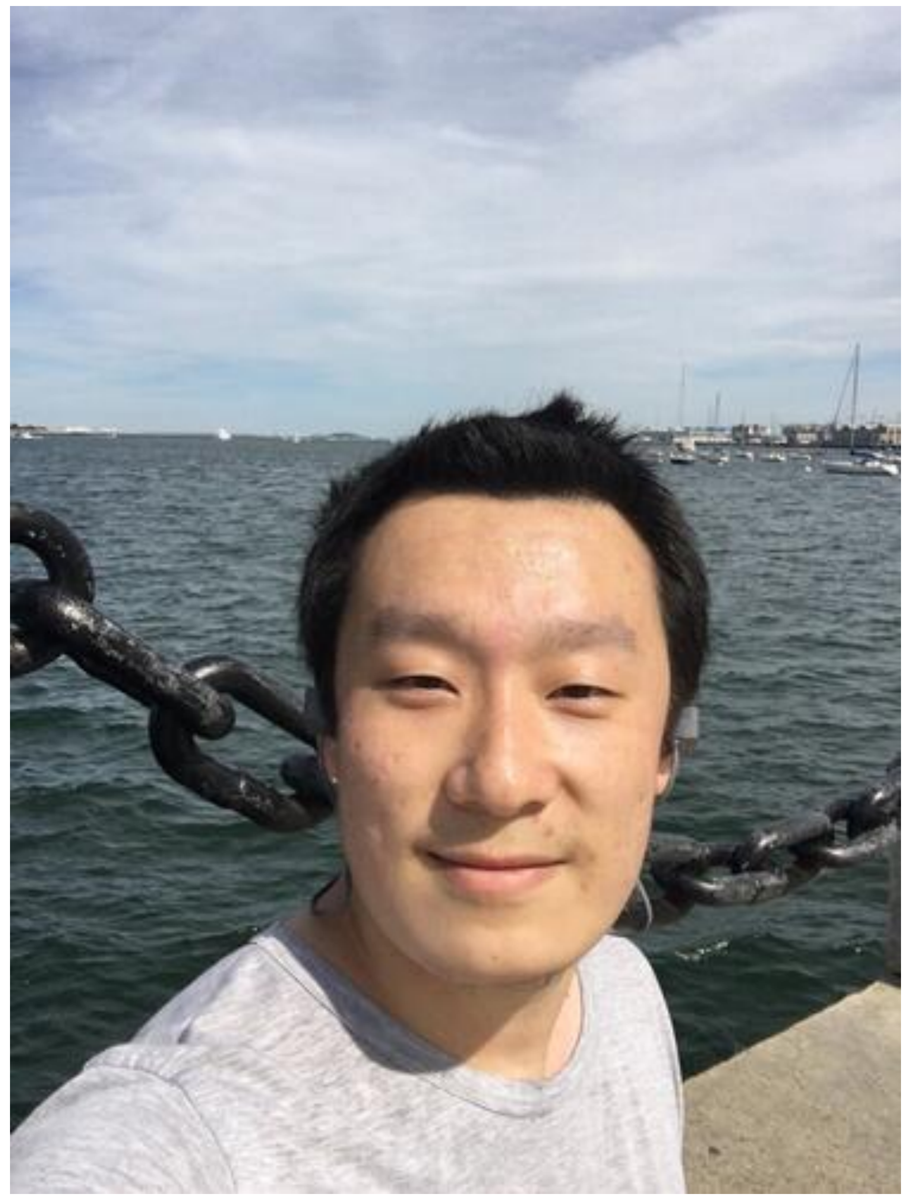
based on 2409.02998, 2602.04958 with Giulia Fardelli, Ami Katz
and WIP with Giulia Fardelli, Ami Katz, and Yuan Xin

Collaborators

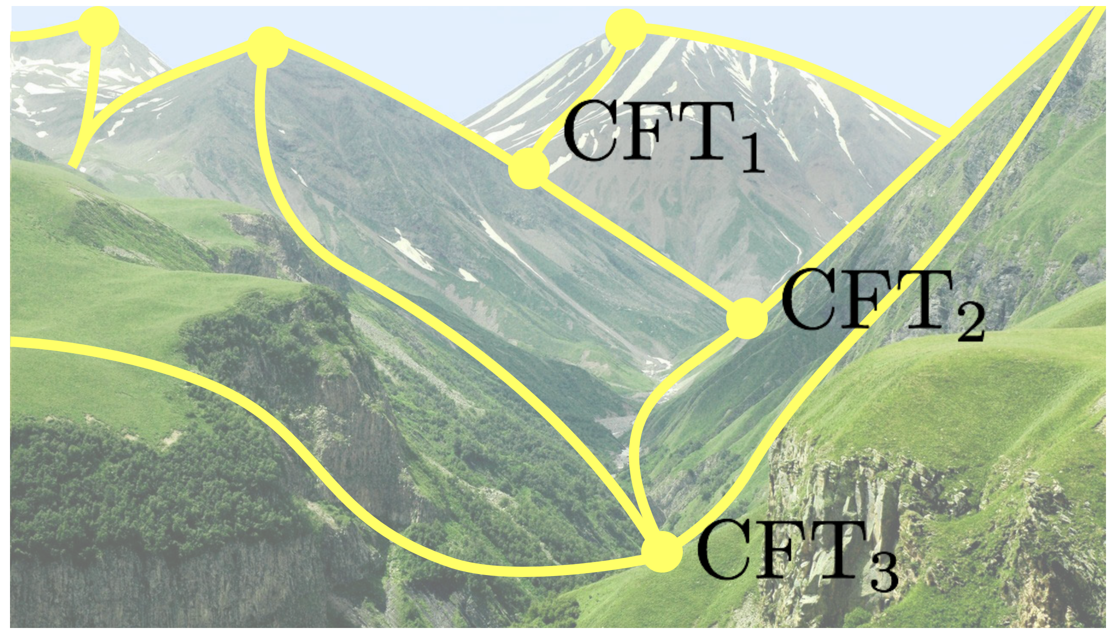
Giulia Fardelli

Ami Katz

Yuan Xin

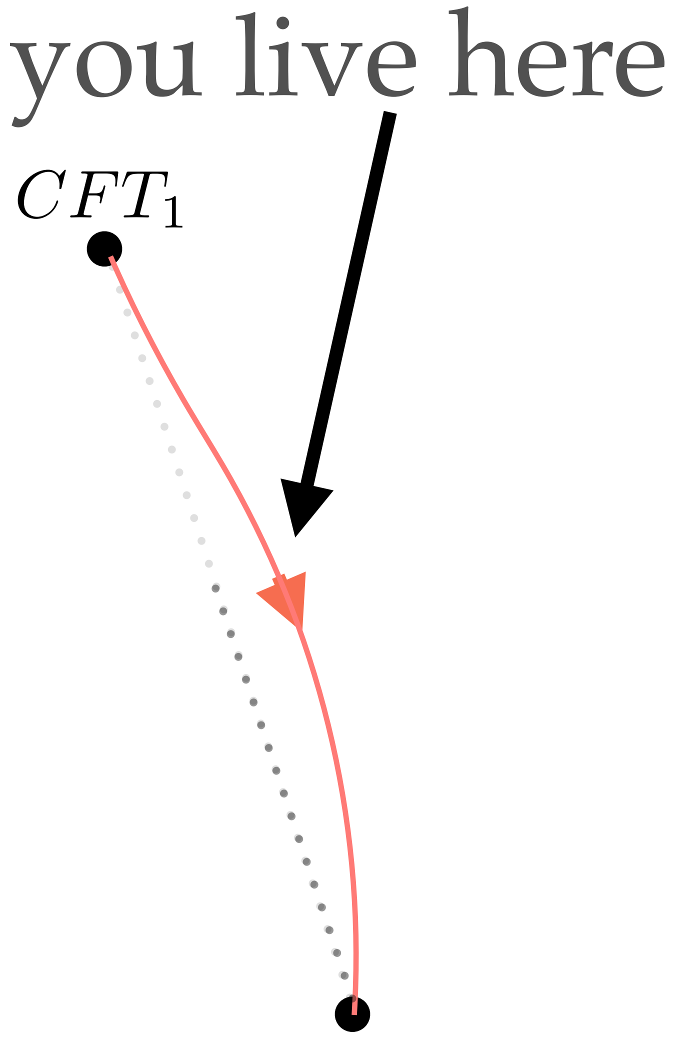


Nonperturbative Formulation of QFT Using Only CFT Data



$$S = S_{\text{CFT}} + g \int d^d x \mathcal{O}_{\text{rel}}(x)$$

$\Delta < d$



QFTs are RG Flows from a UV CFT

Given *just* UV CFT data, compute physics observables in the resulting QFT *nonperturbatively* in parameter g (beyond conformal perturbation theory)

Numerically “Solving” RG flows: Truncated Conformal Space Approach (TCSA)

RG flow triggered by relevant deformation

$$H = H_{\mathrm{CFT}} + \lambda \mathcal{O}_{\mathrm{Relevant}}$$

2d: Yurov & Zamolodchikov ’90, ’91,
+many follow-ups
d>2: Hogervorst, Rychkov, van Rees ’14
d=3: Elias-Miró, Hardy ’20
+ others

Idea: Diagonalize Hamiltonian on Sphere
In Radial Quantization, Hamiltonian Matrix Elements are CFT Data!

$$\langle \mathcal{O}_i | H | \mathcal{O}_j \rangle = \underbrace{\langle \mathcal{O}_i | H_{\mathrm{CFT}} | \mathcal{O}_j \rangle}_{\text{scaling dimensions: } \delta_{ij}\Delta_i} + \lambda \int d^2x \langle \mathcal{O}_i | \mathcal{O}_{\mathrm{relevant}} | \mathcal{O}_j \rangle$$

scaling dimensions: $\delta_{ij}\Delta_i$ 3-point functions (OPE coeffs)

2d Ising Field Theory

UV CFT: 2d Ising model (c=1/2) $S_{\rm CFT} = \int d^2x \psi_L \partial_+ \psi_L + \psi_R \partial_- \psi_R$

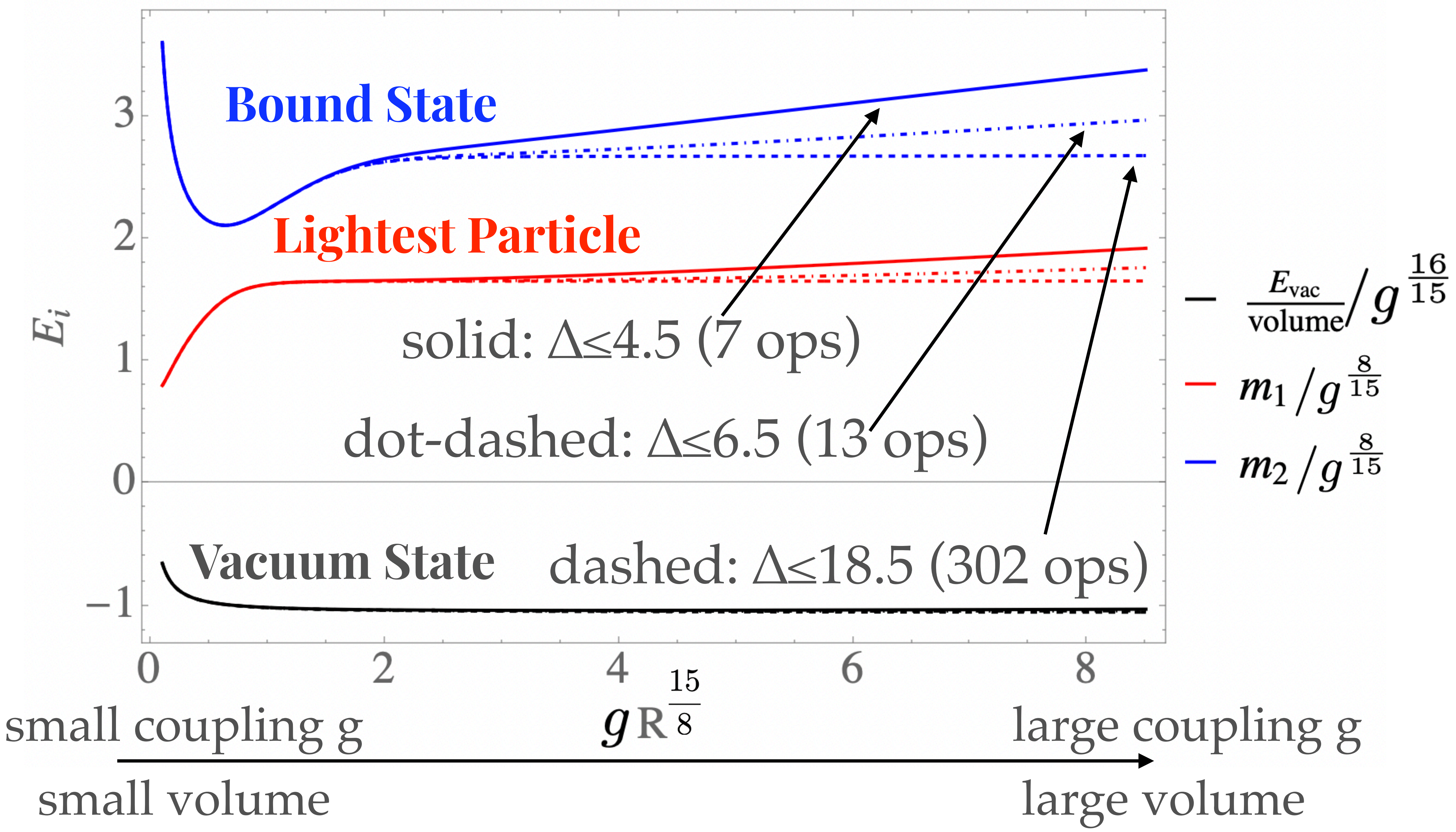
$$S_{\rm IFT} = S_{\rm CFT} + \frac{1}{2\pi}\int d^2x \,(m\epsilon(x) + g\sigma(x))$$

$$H = \begin{array}{c} \left(\begin{array}{ccccccc} \frac{1}{8}+\frac{m}{2} & m & m & g & \frac{g}{2} & \frac{g}{8} & \frac{g}{64} \\ m & \frac{17}{8}+\frac{m}{2} & 0 & \frac{g}{8} & \frac{9g}{16} & \frac{25g}{64} & \frac{225g}{512} \\ m & 0 & \frac{33}{8}+\frac{m}{2} & \frac{9g}{128} & \frac{25g}{256} & \frac{441g}{1024} & \frac{1225g}{8192} \\ g & \frac{g}{8} & \frac{9g}{128} & 0 & m & m & 0 \\ \frac{g}{2} & \frac{9g}{16} & \frac{25g}{256} & m & 1 & 0 & m \\ \frac{g}{8} & \frac{25g}{64} & \frac{441g}{1024} & m & 0 & 3 & m \\ \frac{g}{64} & \frac{225g}{512} & \frac{1225g}{8192} & 0 & m & m & 4 \end{array}\right) \end{array} \begin{array}{l} |\sigma\rangle \\ |\partial^2\sigma\rangle \\ |\partial^4\sigma\rangle \\ |1\rangle \\ |\epsilon\rangle \\ |\partial^2\epsilon\rangle \\ |T\bar{T}\rangle \end{array}$$

m=0

$$E_{\rm vac} = \mathcal{E}_{\rm vac} R = g^{\frac{16}{15}} R f_0(g^{\frac{8}{15}} R) \overset{g\to\infty}{\longrightarrow} A_0 g^{\frac{16}{15}} R$$

$$m_i = g^{\frac{8}{15}} f_i(g^{\frac{8}{15}} R) \overset{g\to\infty}{\longrightarrow} A_i g^{\frac{8}{15}}$$



only using operators with Δ≤4.5 (7 states)

Zamolodchikov + Fonseca + Yurov + al.

Our Goal Today:

3d Ising Field Theory on $\mathbb{R} \times S^2$

(radial quantization)

$$H = H_{3d\ \text{Ising CFT}} + h_z \int d^2\Omega\, \sigma(\Omega)$$

"Magnetic" Ising Field Theory, i.e. Ising CFT plus σ deformation

Spoiler: at large radius R

Vacuum: $E_{\text{vac}} = \mathbf{-1.37} \times h_z^{\frac{3}{3-\Delta_\sigma}} R^2$

1-particle state: $m_1 = \mathbf{1.65} \times h_z^{\frac{1}{3-\Delta_\sigma}}$

Bound state: $m_2/m_1 \approx \mathbf{1.95} < 2$

Our Goal Today: 3d Ising Field Theory on R x S²

Two Approaches:

Fuzzy Sphere: Use **<u>fuzzy sphere</u>** microscopic description

$$H = H_{\mathrm{FS}} + h\int d^2\Omega \psi^\dagger \sigma^z \psi$$

N non-rel fermions in Lowest Landau Level

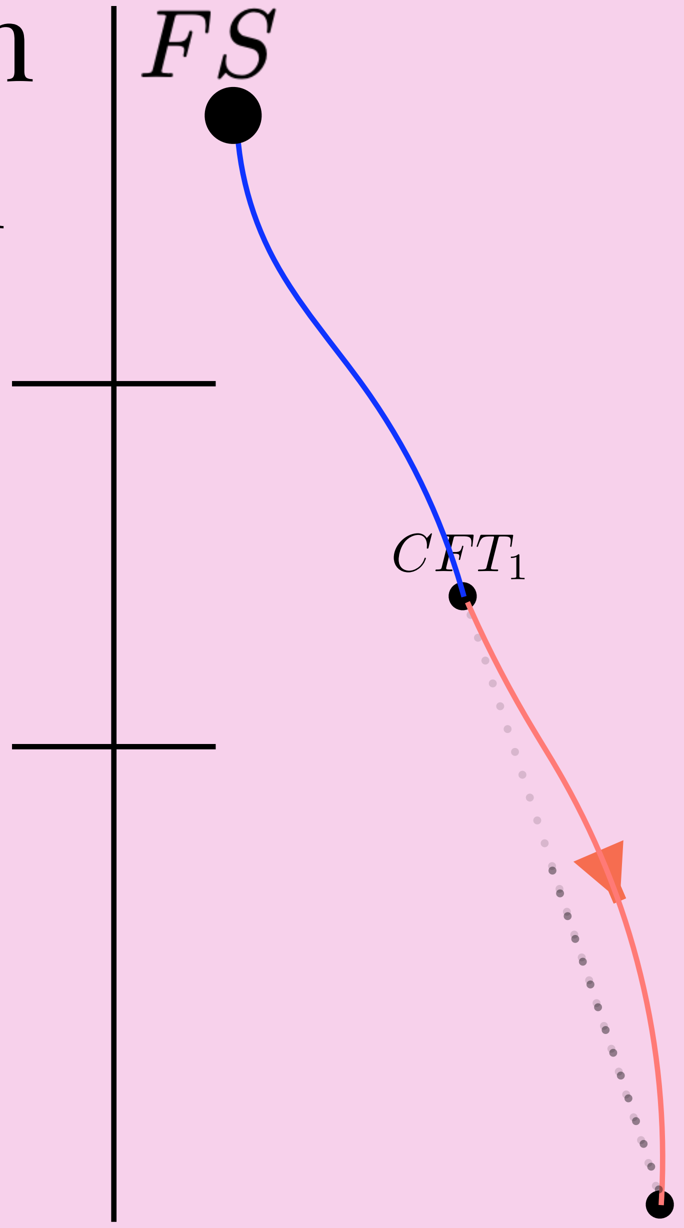
$\Lambda_{\mathrm{UV}} \sim \sqrt{B} \sim \sqrt{N}$

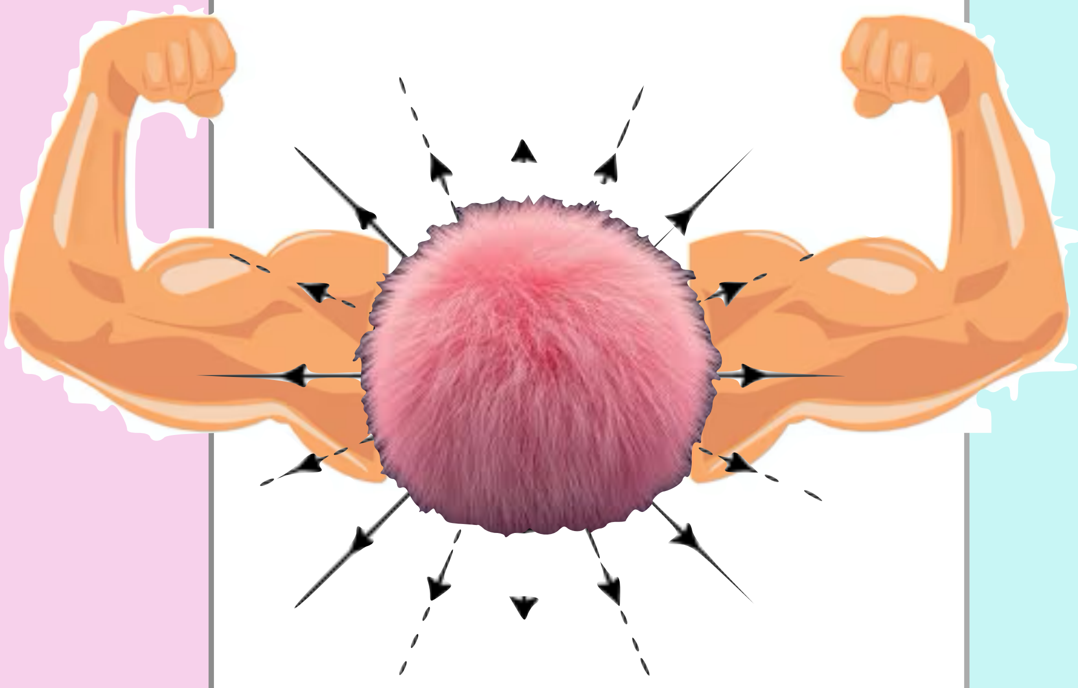
two scale transitions

UV CFT

$\mu \equiv h_z^{\frac{1}{3-\Delta_\sigma}}$

IR massive QFT





TCSA: Using only CFT Data

(get CFT data from **<u>Fuzzy Sphere</u>**)

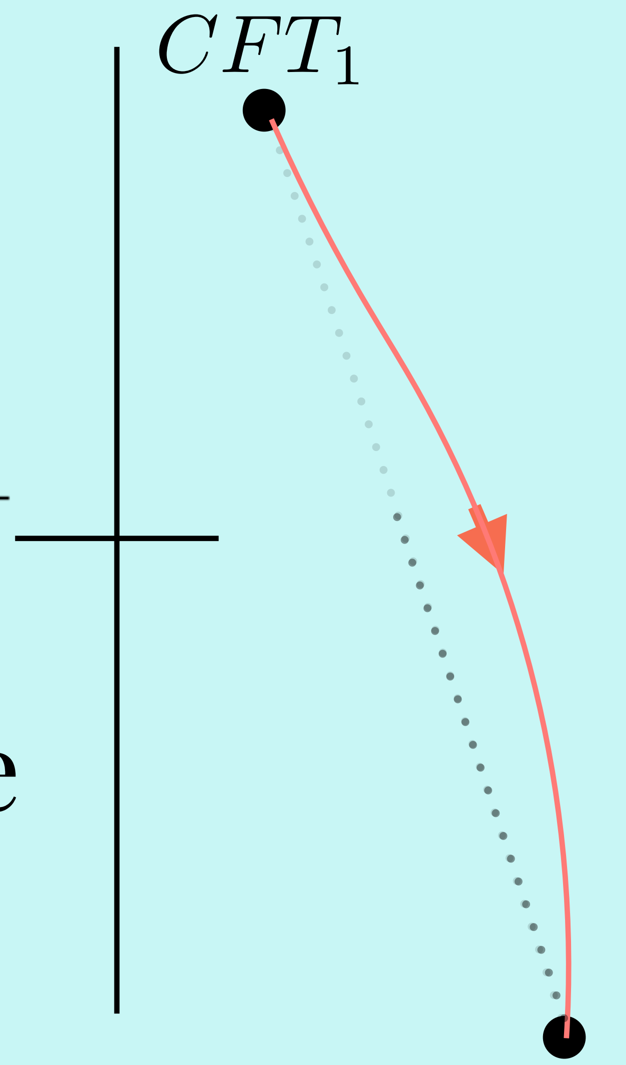
$$H = H_{\text{3d Ising CFT}} + h_z \int d^2\Omega\ \sigma(\Omega)$$

UV CFT

one scale transition

$\mu \equiv h_z^{\frac{1}{3-\Delta_\sigma}}$

IR massive QFT

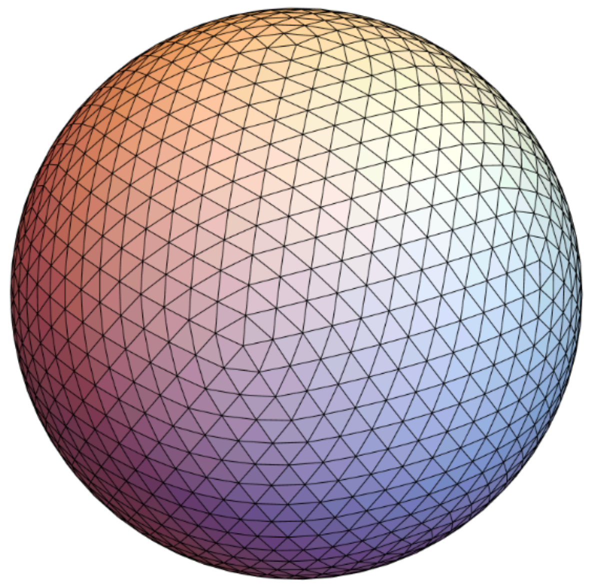


Fuzzy sphere: Rotationally Invariant UV Microscopic Description

Idea: instead of using a lattice, find a regulator that preserves spherical symmetry

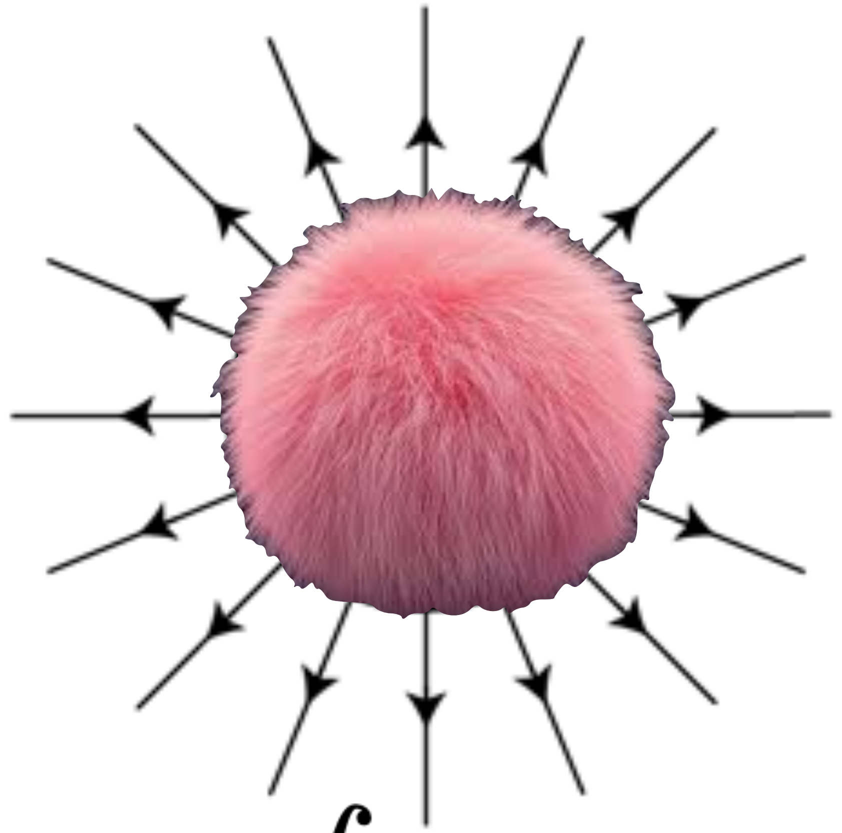
$$H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - g\sum_i \sigma_i^x \longrightarrow H = -\int d^2\Omega\,\left(J(\psi^\dagger\sigma^z\psi)(\psi^\dagger\sigma^z\psi) + g\psi^\dagger\sigma^x\psi\right)$$

Lattice Spin System



$\longrightarrow$

Fuzzy Sphere System



Zhu, Han, Huffman, Hofmann, He ’23 + …

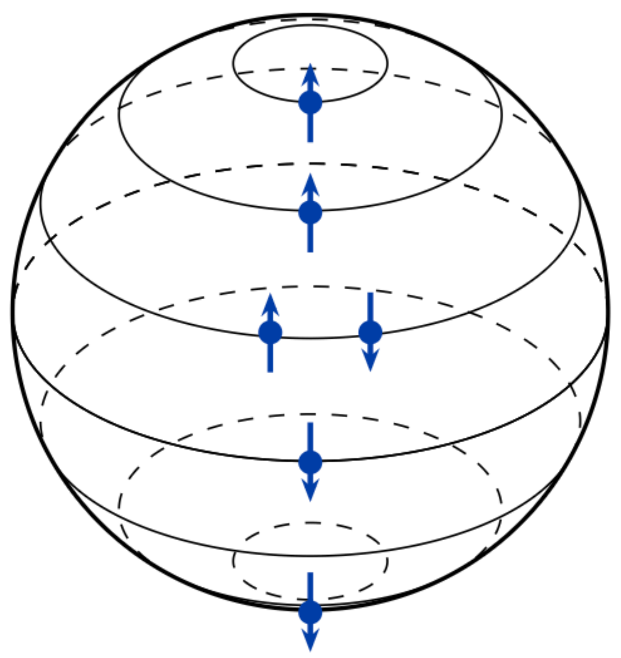
$$\text{Flux} = \int dA = 4\pi s$$

[Haldane (83), Greiter (11)]

Lowest Landau Level $\Longrightarrow$

$N_e = 2s + 1$

Half filling



$N = 2s + 1$ Lowest Landau Level modes (from constant radial Magnetic flux)

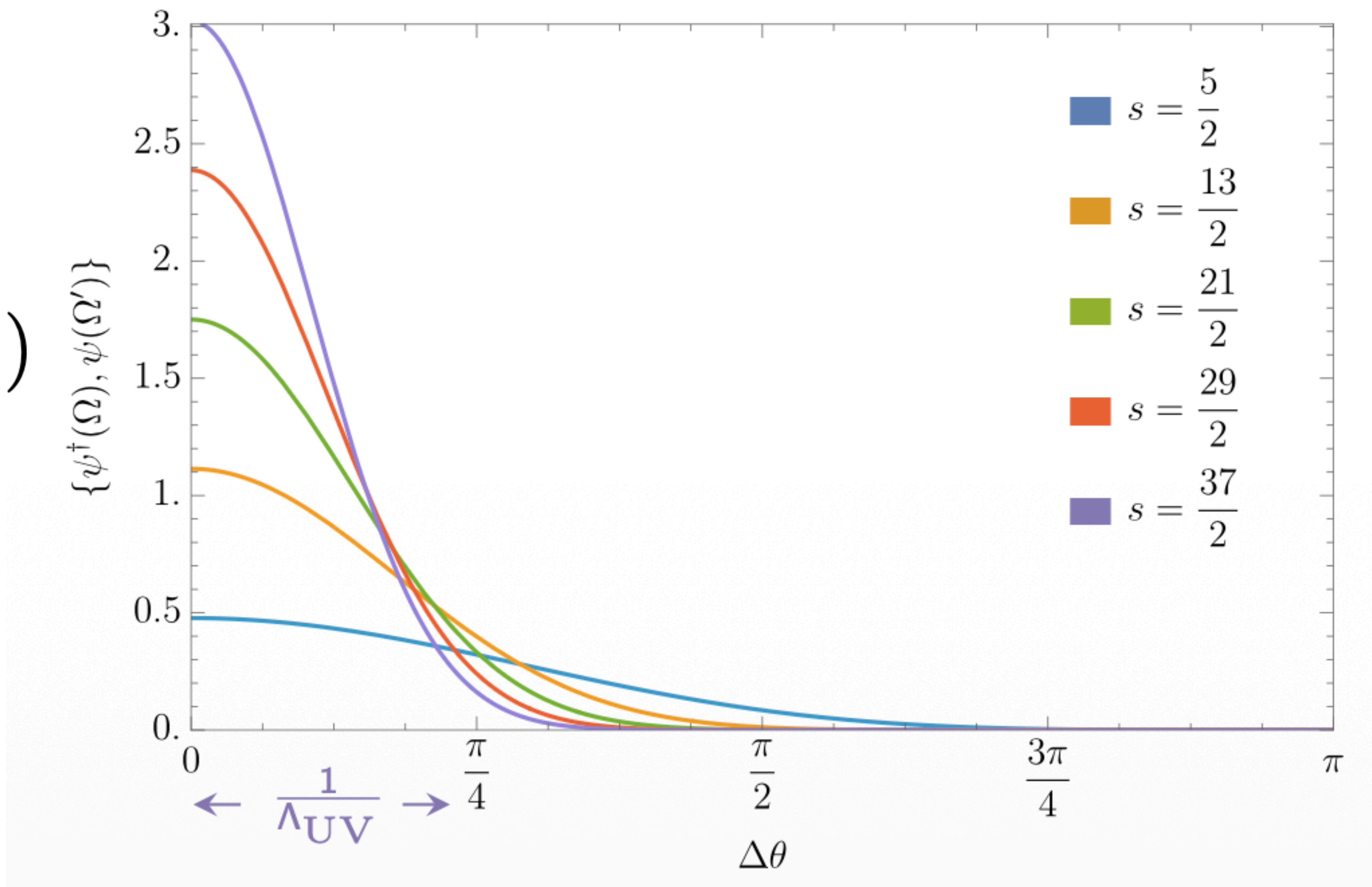
$$(\vec{\nabla} + \vec{A})^2 f = E_{\mathrm{LLL}} f$$

$$\psi(\hat{n}) = \sum_{i=1}^{N} f_i(\hat{n}) c_i^\dagger$$

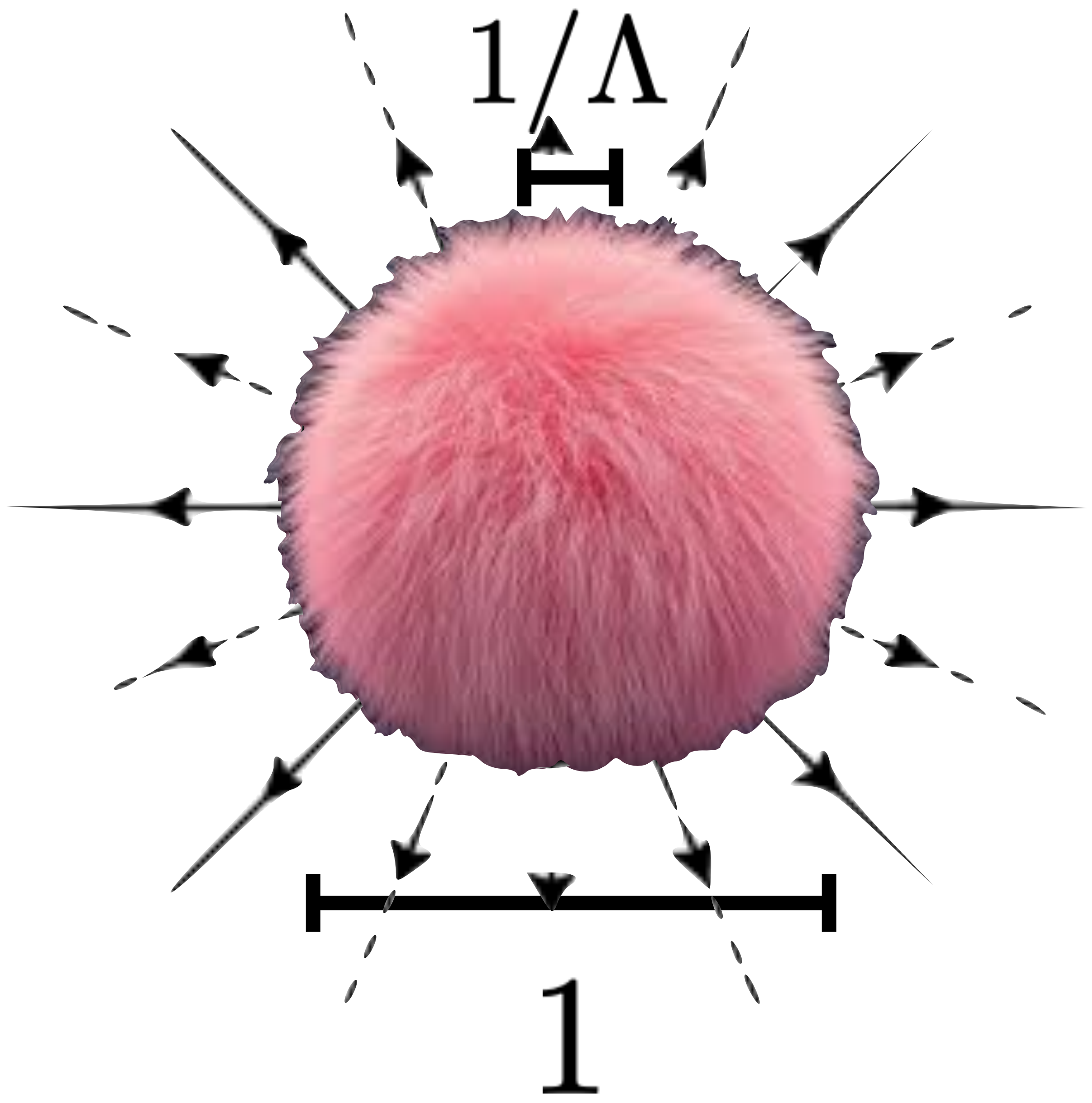
Fuzzy sphere

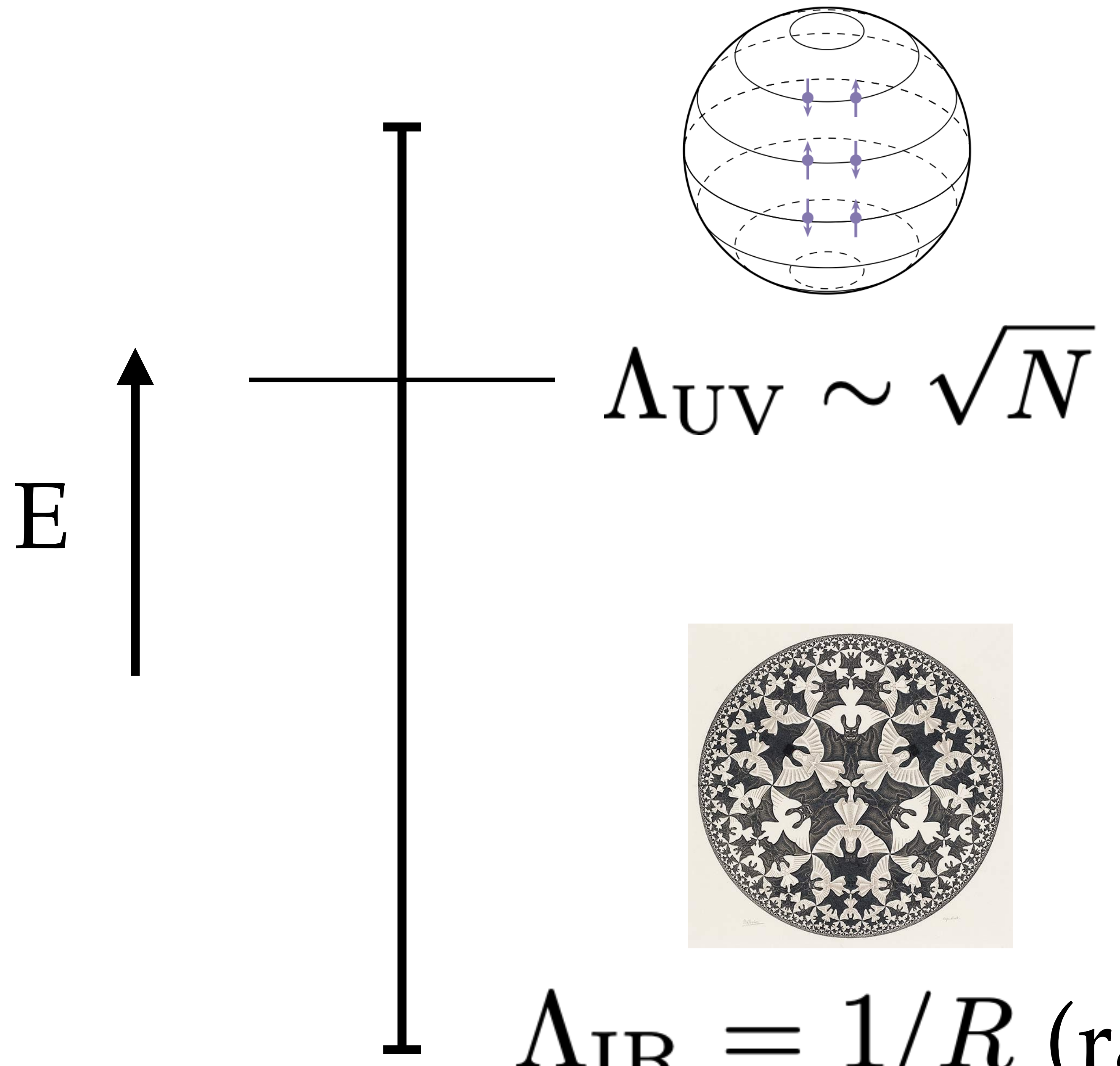
Key property: completeness relation of modes is a spherically symmetric regulated delta function

$$\{\psi^\dagger(\hat{n}), \psi(\hat{n}')\} = \sum_{i=1}^{N} f_i(\hat{n}) f_i^*(\hat{n}') = \delta_\Lambda(\hat{n}\cdot\hat{n}')$$



"UV cut-off" set by number of modes $\Lambda \sim \sqrt{N}$





at UV "lattice" scale, theory is non-relativistic fermions

in IR, theory is 2+1d Ising CFT (on sphere)

$\Lambda_{\rm IR} = 1/R$ (radius of sphere = R)

UV Effects from Fuzzy Sphere Regulator

When you tune to critical point, you are tuning the coefficients $g_{\mathcal{O}}$ to zero

$$H_{\rm UV} = \int d^2\Omega \mathcal{H}(\hat{n})$$

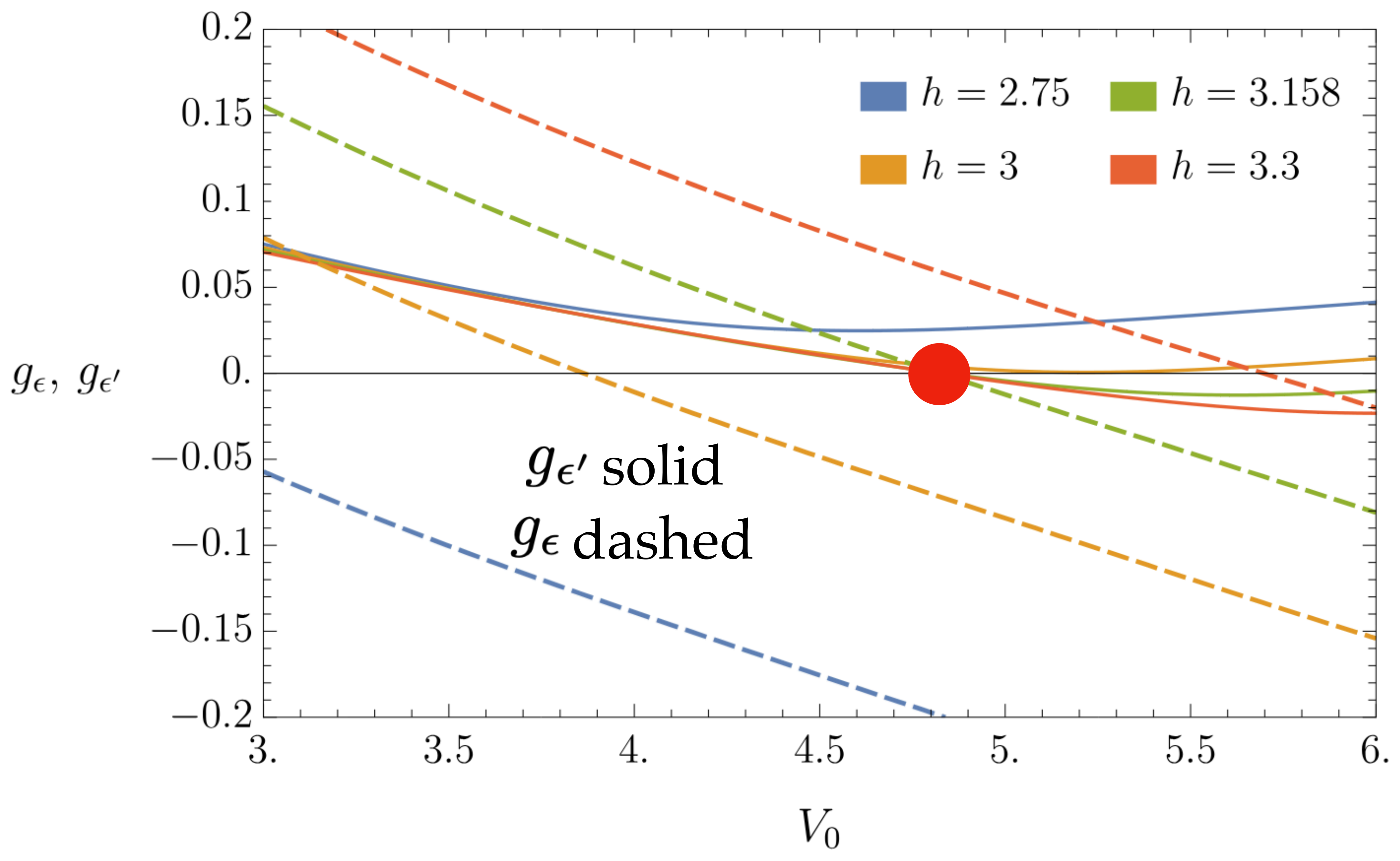
$$g_{\mathcal{O}} = g_{\mathcal{O}}(V_0, V_1, h)$$

$$\mathcal{H} = \overbrace{-(\psi^\dagger\sigma^z\psi)(V_0 + V_1\nabla^2)(\psi^\dagger\sigma^z\psi) + h\psi^\dagger\sigma^x\psi}^{\text{UV operator - set by hand}} = \overbrace{\gamma T^0_0 + g_\epsilon \epsilon + g_{\epsilon'}\epsilon'}^{\text{IR expansion}} + g_C C_4 + \dots$$

IR expansion: Wilson Coefficients suppressed by powers of $\Lambda_{\rm UV} = \sqrt{N}$ plus <u>additional fine-tuning by hand</u>

determine IR parameters as a function of UV parameters ($g_\epsilon = g_\epsilon(V_0, h)$) numerically by comparing to Conformal Perturbation Theory

Rychkov & Lao '23,
ALF, Fardelli, Katz '25
Läuchli, Herviou, Wilhelm, Rychkov '25



UV Effects from Fuzzy Sphere Regulator

At "critical" point:

dilatation generator

(can set γ=1 by rescaling H_{UV})

$$H_{\rm UV} = \int d^2\Omega \mathcal{H}(\hat{n}) = \gamma D + \sum_{\substack{\mathcal{O} \\ \text{irrelevant}}} g_{\mathcal{O}} \int d^2\Omega \mathcal{O}(\hat{n})$$

removed by tuning V_0, h

	~~ϵ~~	~~ϵ'~~	ϵ''	$T_{\mu\nu}$	$T'_{\mu\nu}$	$C^{\mu\nu\rho\sigma}$
Bootstrap	~~1.413~~	~~3.830~~	6.896	3	5.509	5.023

$$\stackrel{N\gg 1}{\longrightarrow} D + g_{\mathcal{O}} O(\Lambda_{\rm UV}^{3-\Delta_{\mathcal{O}}})$$

$$g_C \Lambda_{\rm UV}^{3-\Delta_C} \approx g_C N^{\frac{3-\Delta_C}{2}} \approx g_C N^{-1.011}$$

errors scale
like ~1/N

Spectrum of
3d Ising Field Theory on S^2

Large coupling and Dimensional Analysis

$$H = H_{\mathrm{FS}} + h \int d^2\Omega \psi^\dagger \sigma^z \psi$$

Fuzzy sphere at Finite N:
Two dimensionless ratios

$$\mu R = h^{\frac{1}{3-\Delta_\sigma}} R$$

$$\sqrt{N} = \Lambda_{\mathrm{UV}} R$$

N non-rel fermions in Lowest Landau Level

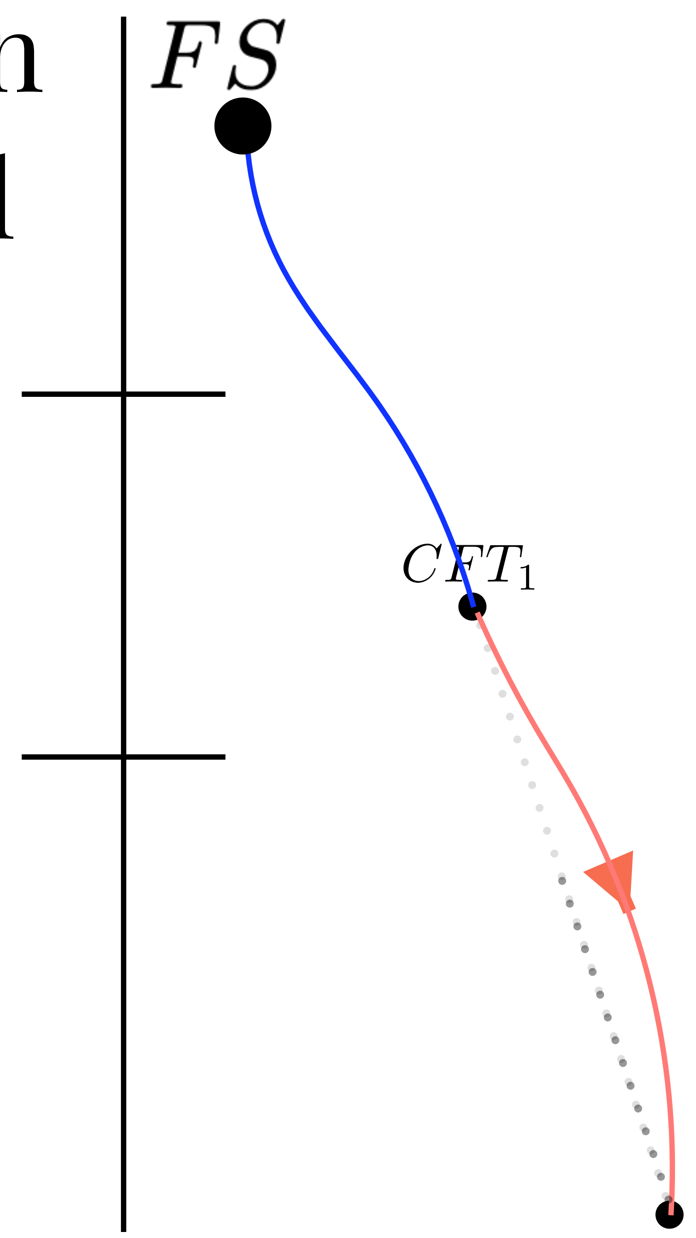
$\Lambda_{\mathrm{UV}} \sim \sqrt{B} \sim \sqrt{N}$

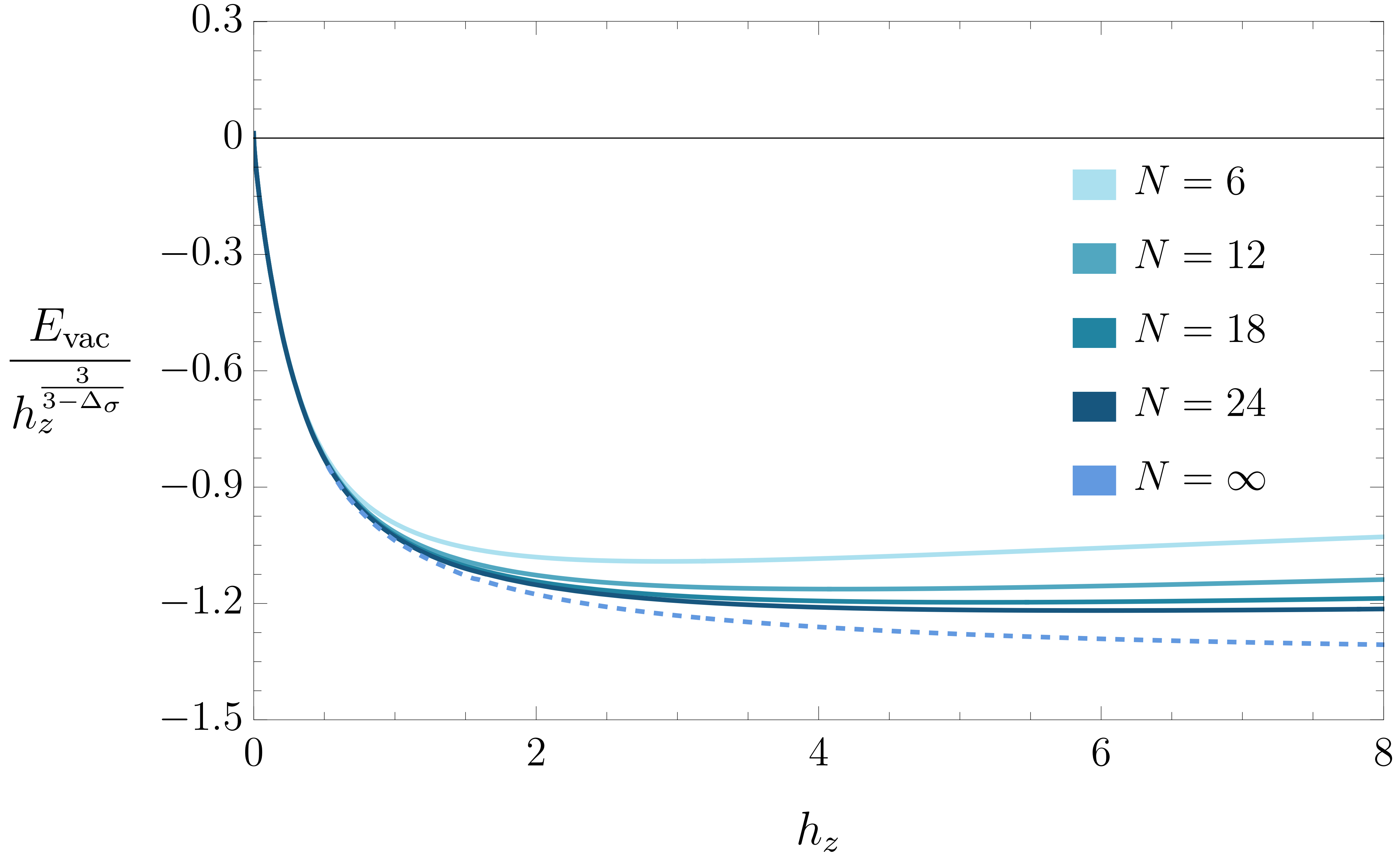
two scale transitions

UV CFT

$\mu \equiv h_z^{\frac{1}{3-\Delta_\sigma}}$

IR massive QFT





$$N \to \infty$$

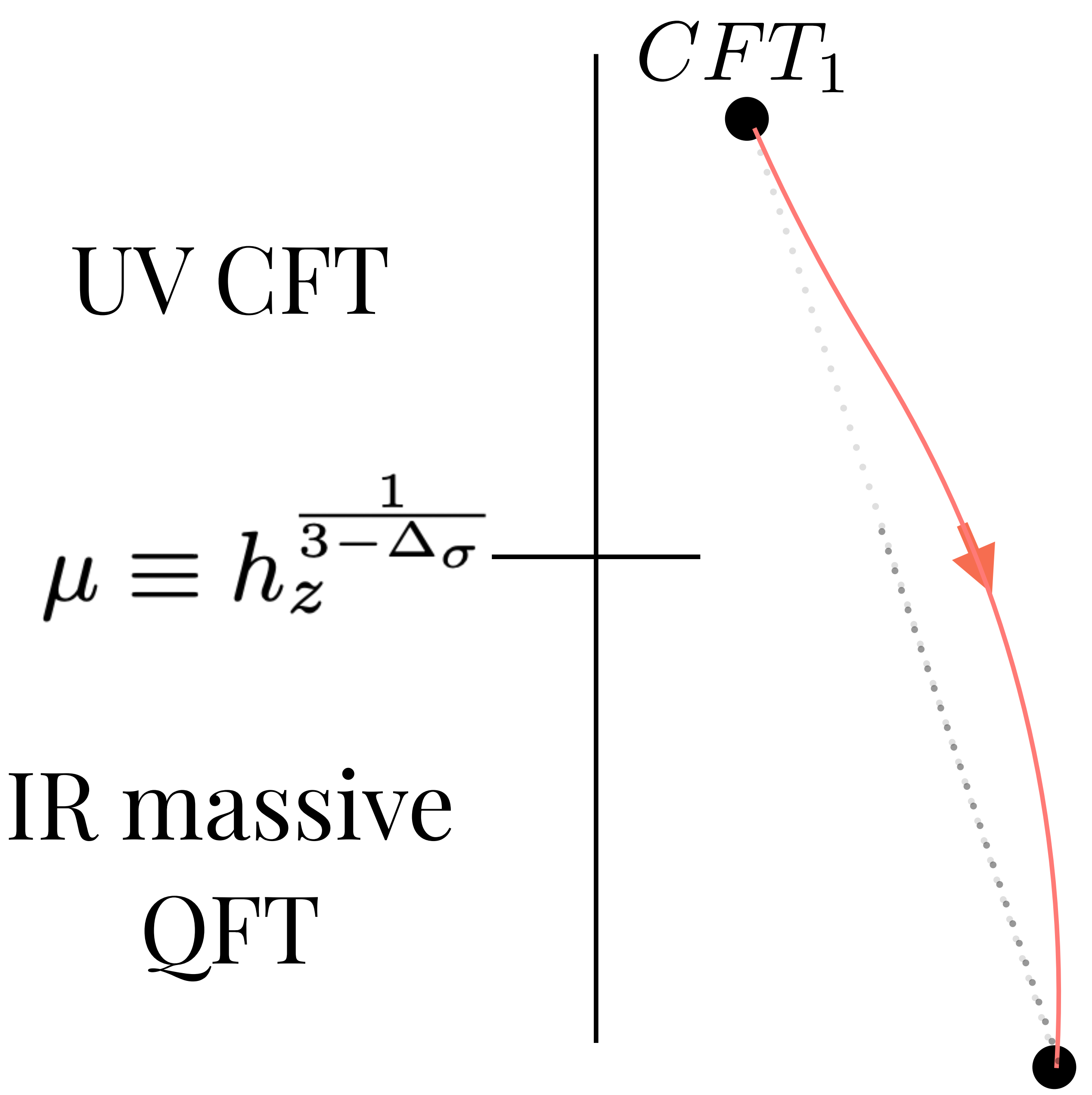
Large coupling and Dimensional Analysis

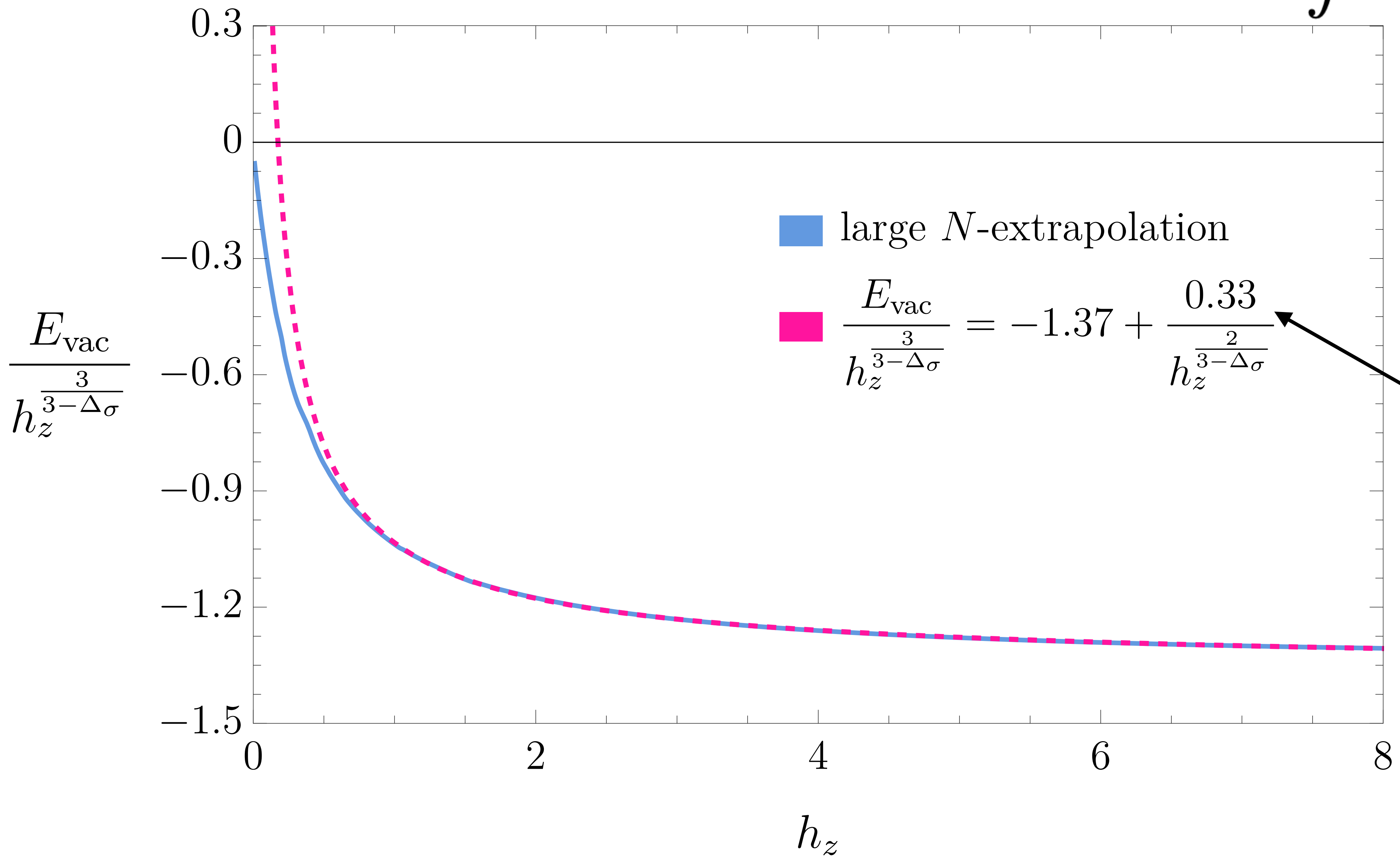
$$H = H_{\mathrm{FS}} + h \int d^2\Omega \psi^\dagger \sigma^z \psi$$

Fuzzy sphere at infinite N: $\mu R = h^{\frac{1}{3-\Delta_\sigma}} R$

One dimensionless ratio

At large μ : $S_{\mathrm{IFT}} = \int d^3x \left(\mathcal{E}_0 + \frac{c_1}{\mu^2} \mathcal{R} + \dots \right)$





curvature corrections ~ $1/R^2$ from sphere

$N \to \infty$

Large coupling h = large volume

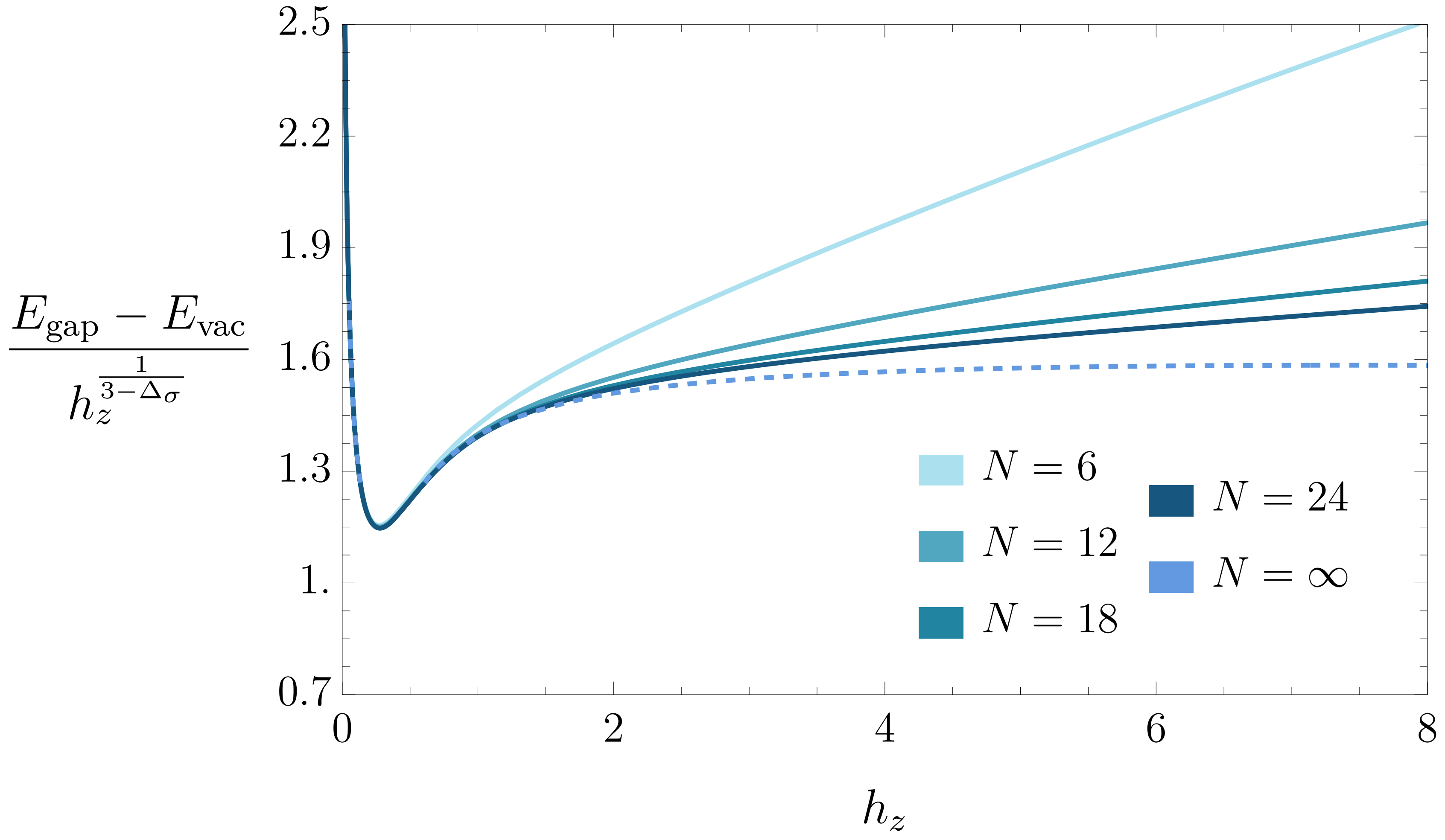
$$E_{\mathrm{vac}} = \mathcal{E}_{\mathrm{vac}} R^2 = \mu^3 R^2 f_0(\mu R) \xrightarrow{h \to \infty} A_0 \mu^3 R^2 \qquad A_0 \approx -1.37$$

Large coupling and Dimensional Analysis

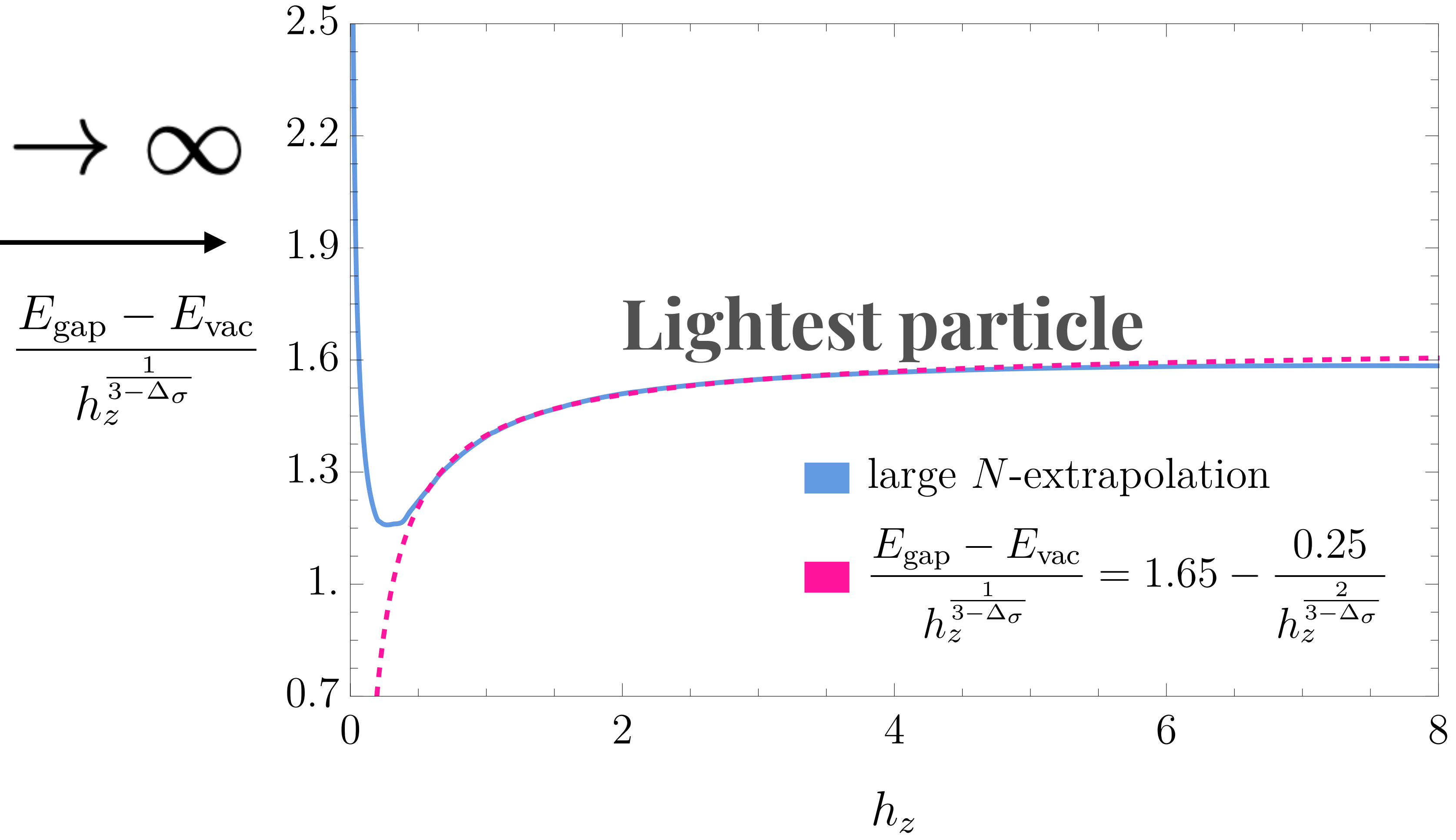
$$H = H_{3d \text{ Ising CFT}} + h_z \int d^2\Omega\, \sigma(\Omega)$$

Fuzzy sphere excited states:

$$m_i = \mu f_i(\mu R) \overset{h\to\infty}{\longrightarrow} A_i \mu$$

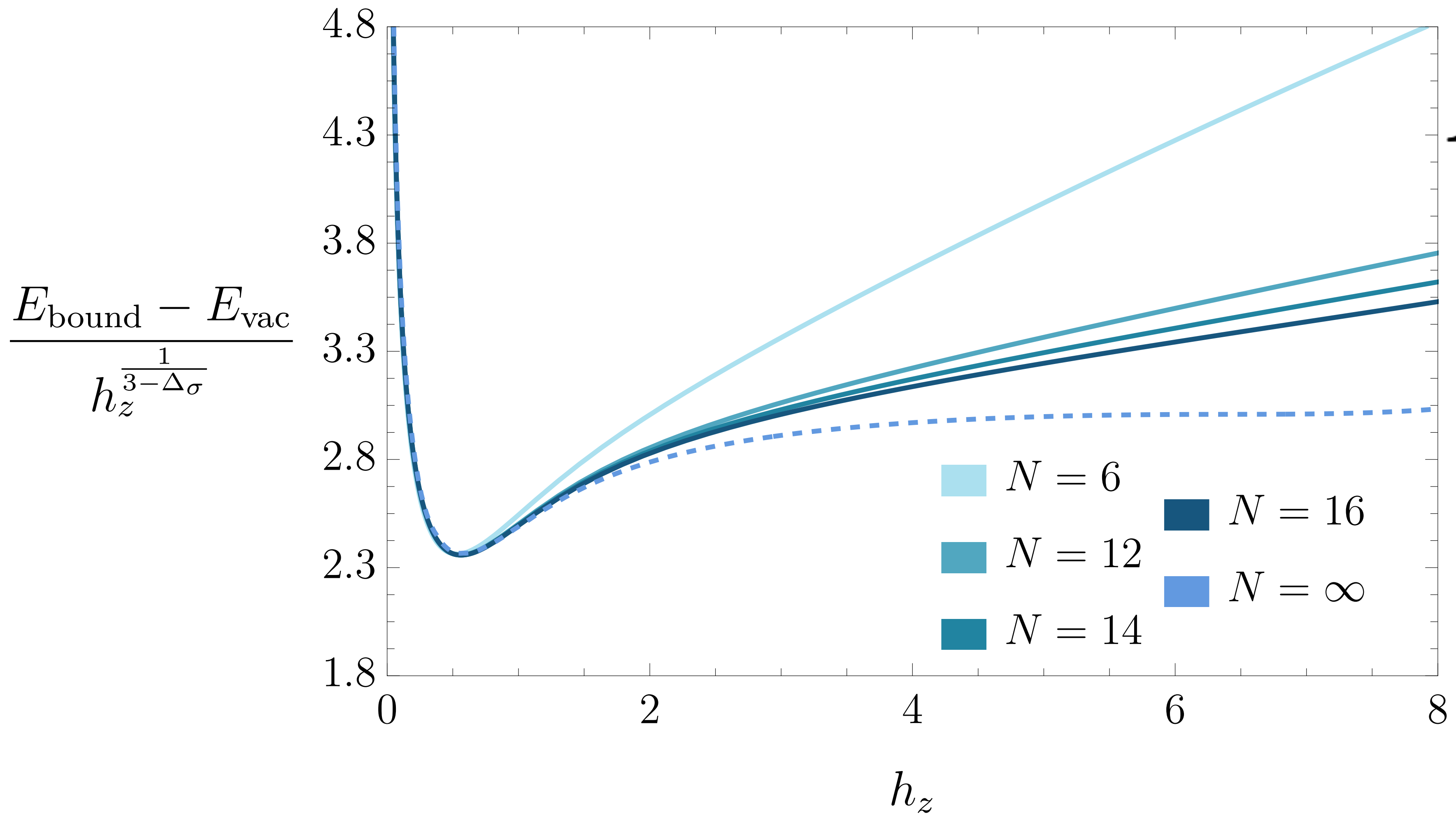


$N \to \infty$

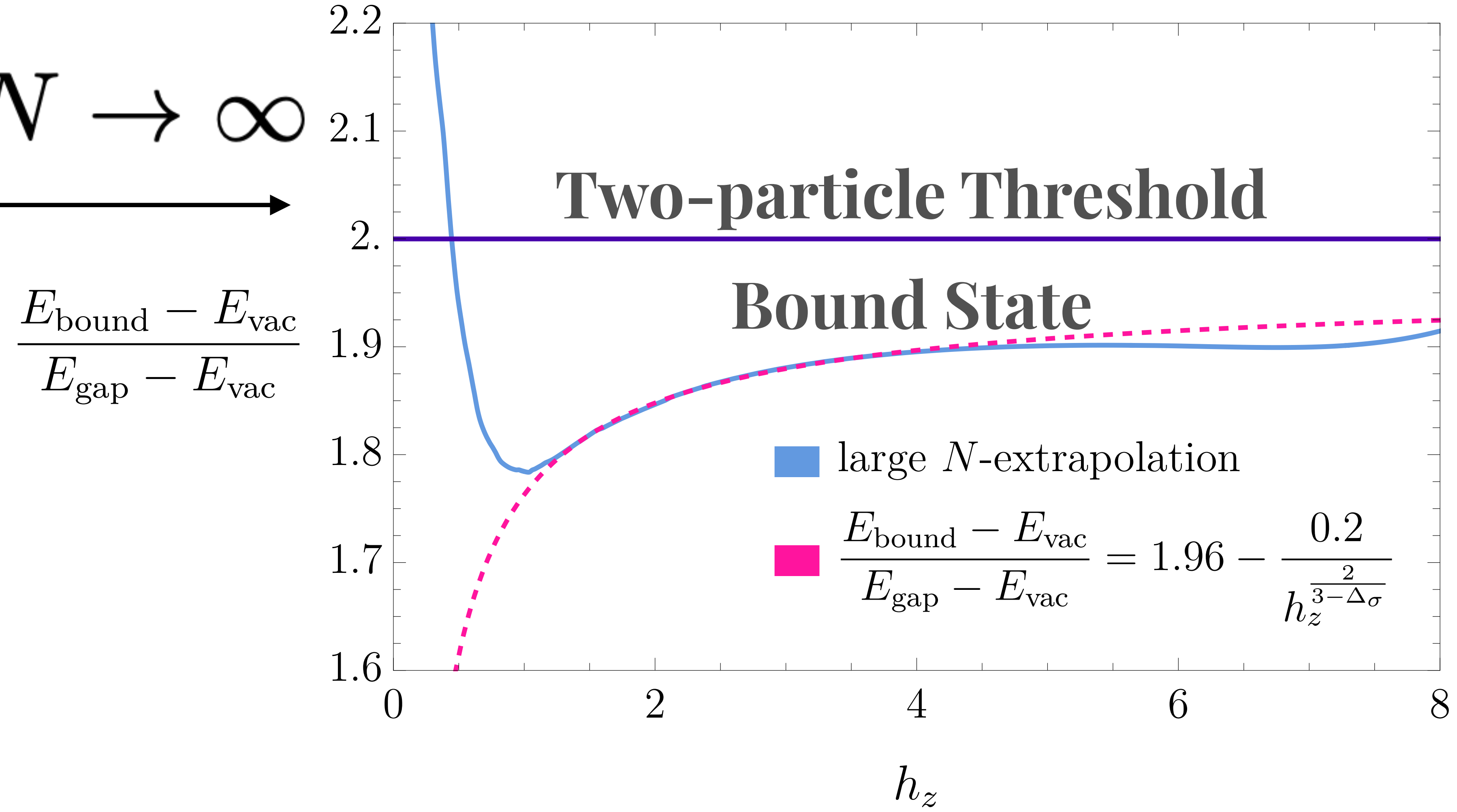


$A_1 \approx 1.65$ **Vacuum State**

Lightest one-particle state



$N \to \infty$

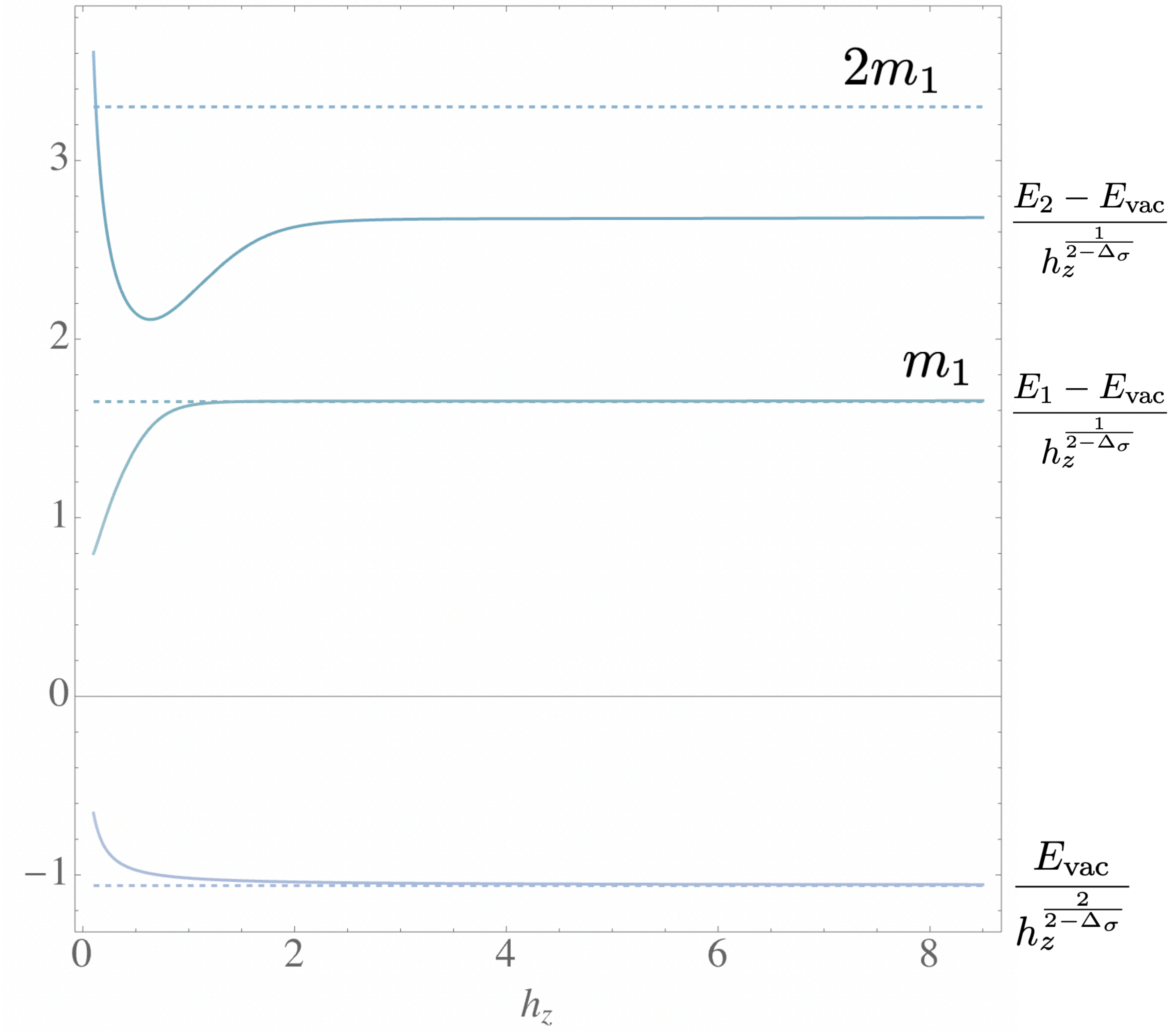


$$\frac{m_2}{m_1} \approx 1.96$$

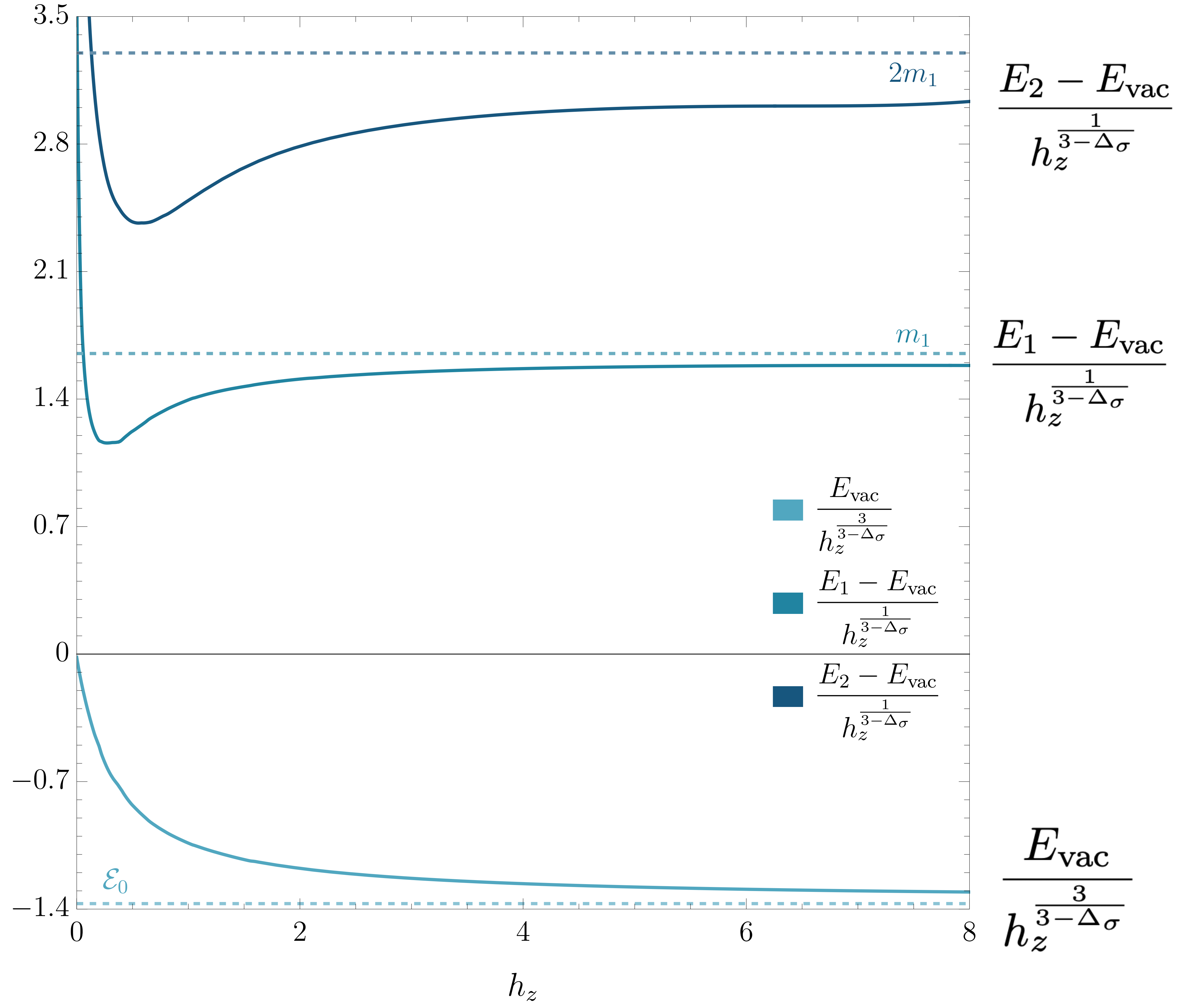
Weakly-bound state below 2-particle threshold

Summary: 2d vs 3d Ising Field Theory

2d



3d



2d: Finite volume corrections $\sim e^{-mR}$

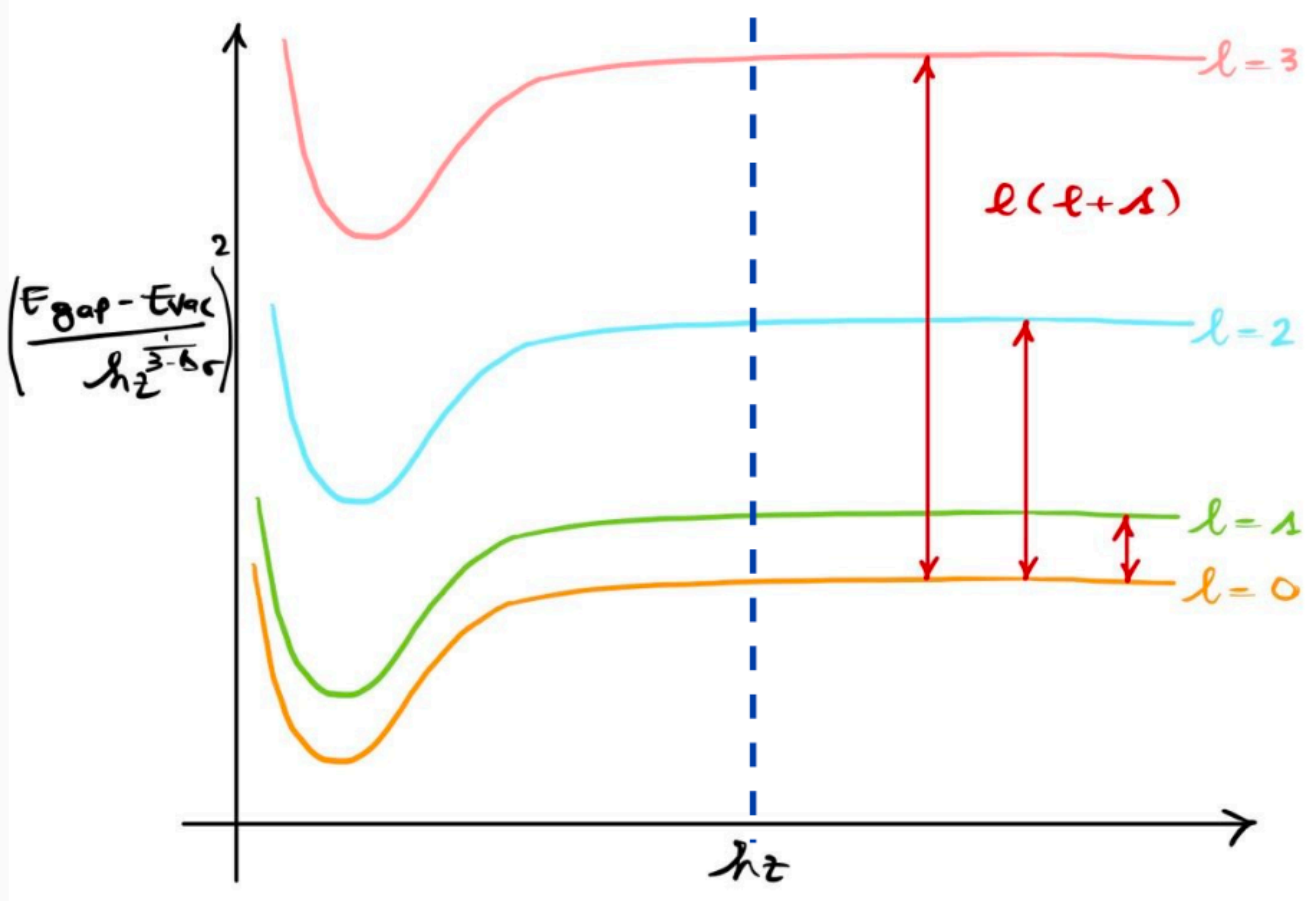
3d: Finite volume corrections $\sim \frac{1}{(mR)^2}$

Spinning Sectors

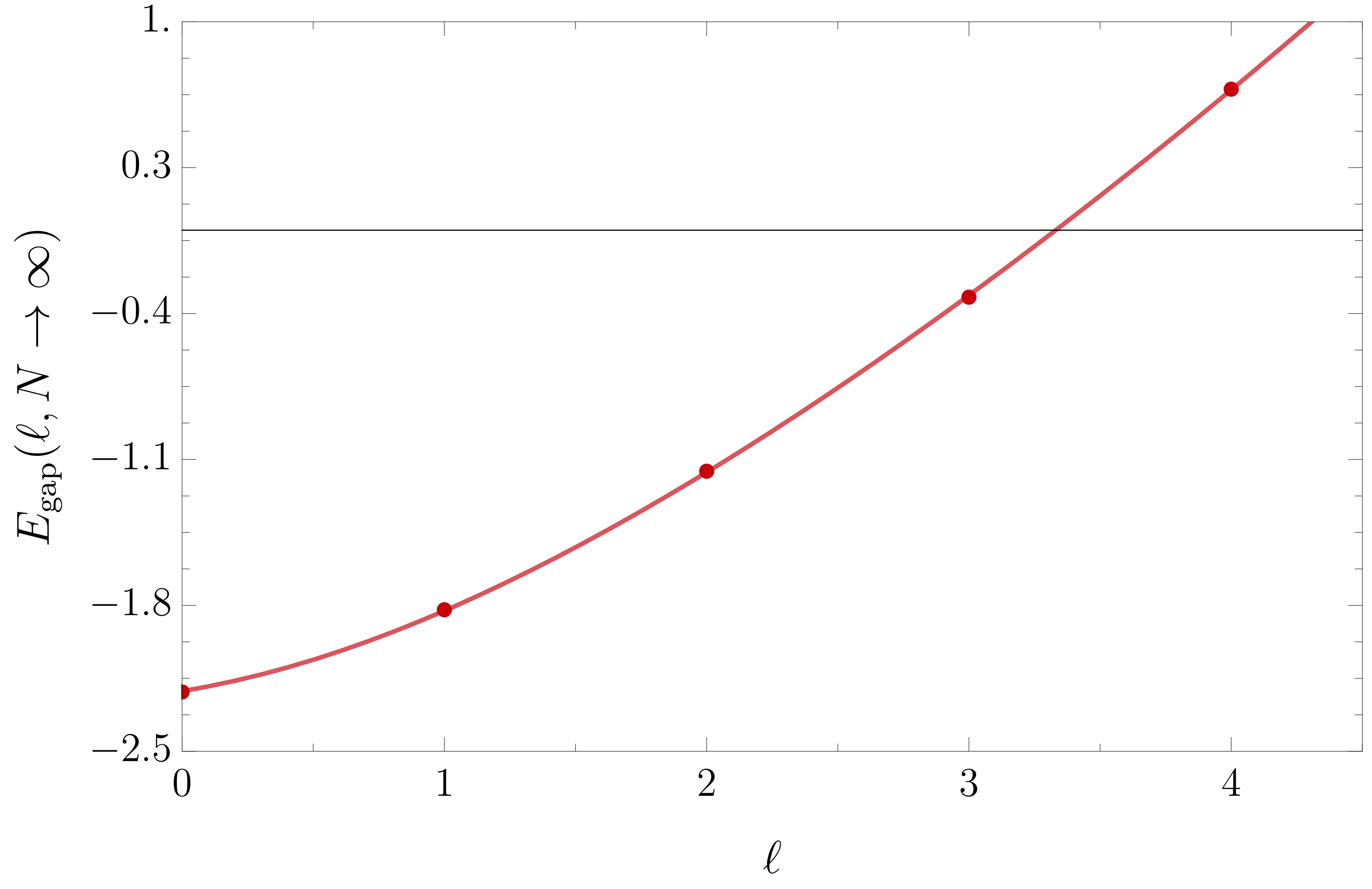
Spinning Sectors

We can also look at the lightest state in each spin ℓ>0 sector:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{c}{m^2}(\nabla^2\phi)^2 + \dots$$



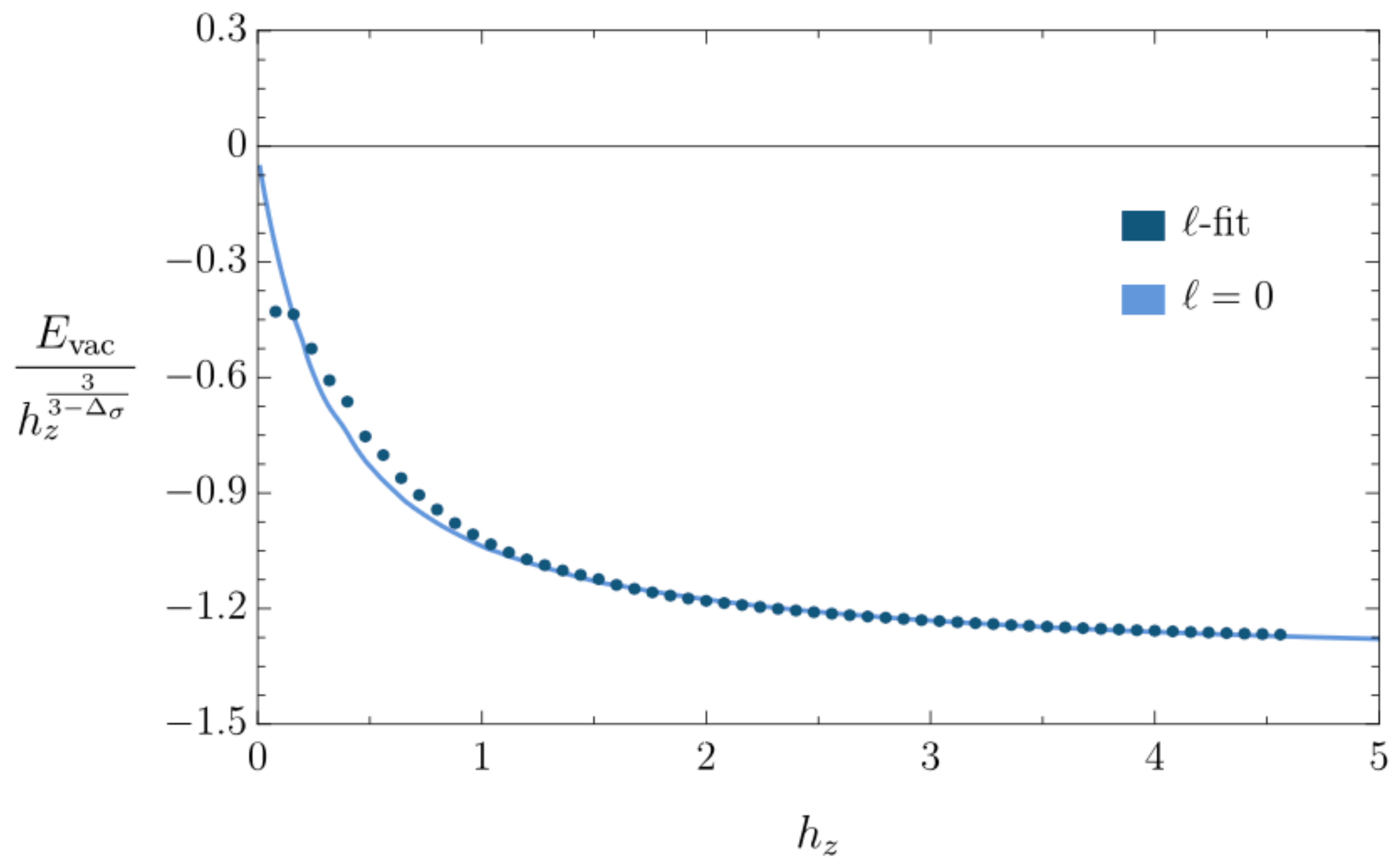
At h=3:

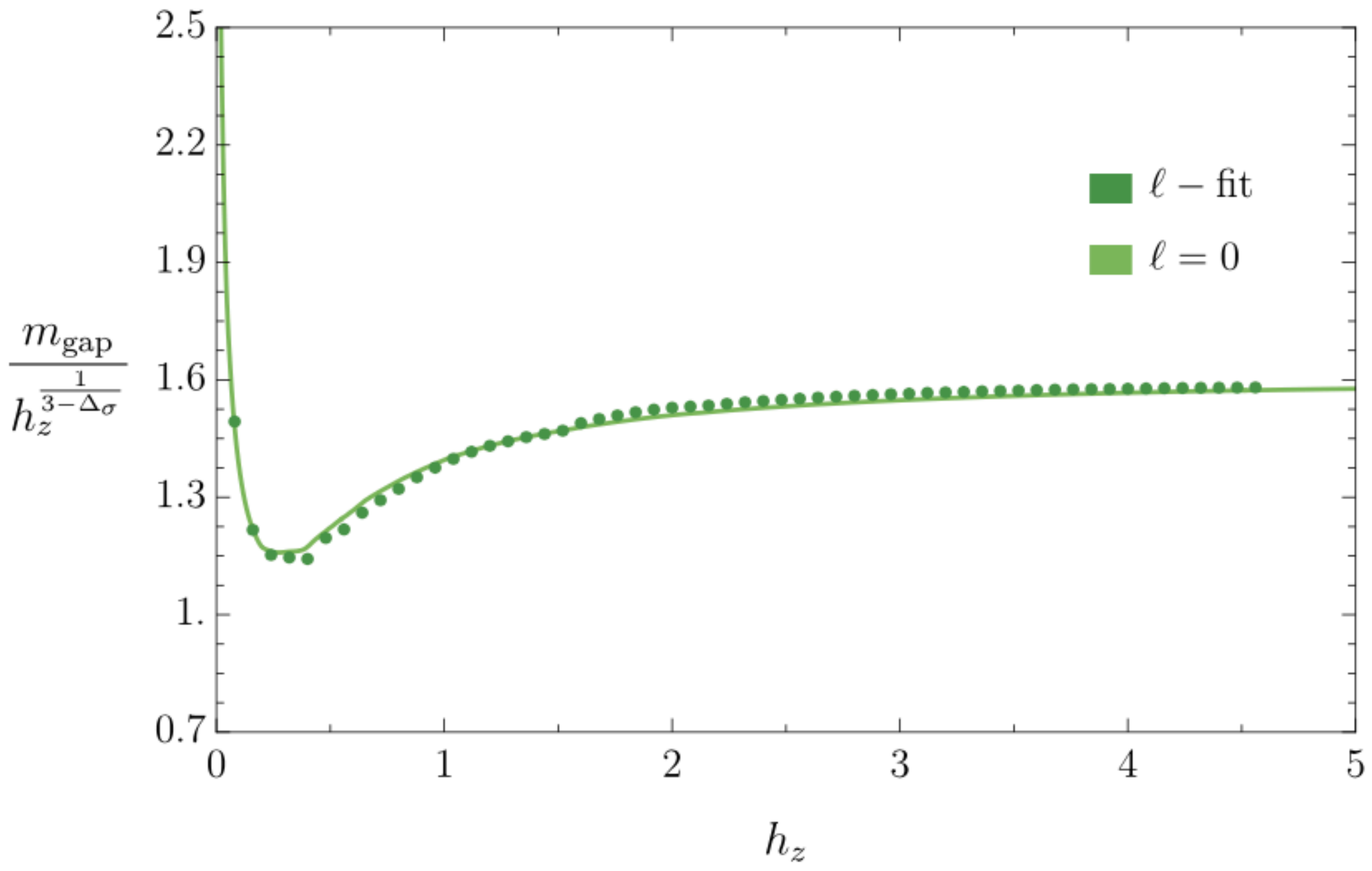


From Fit to ℓ>0 spectrum: $E_1(\ell) = -1.23\mu^3 + \sqrt{(1.56\mu)^2 + \ell(\ell+1) + 0.006\ell^2(\ell+1)^2}$

Spinning Sectors

Fitting $E_1(\ell) = E_{\text{vac}} + \sqrt{m_{\text{gap}}^2 + \ell(\ell+1) + c\ell^2(\ell+1)^2}$ to $\ell > 0$ sector gives us an independent measure of vacuum Energy E_{vac} and mass gap m_{gap} (fit doesn't use $\ell = 0$ sector)





$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{c}{m^2}(\nabla^2\phi)^2$$

Emergent Lorentz Invariance at large volume + spin!

Our Goal Today: 3d Ising Field Theory on R x S²

Next: redo it with TCSA

Fuzzy Sphere: Use **fuzzy sphere** microscopic description

$$H = H_{\mathrm{FS}} + h \int d^2\Omega \psi^\dagger \sigma^z \psi$$

N non-rel fermions in Lowest Landau Level

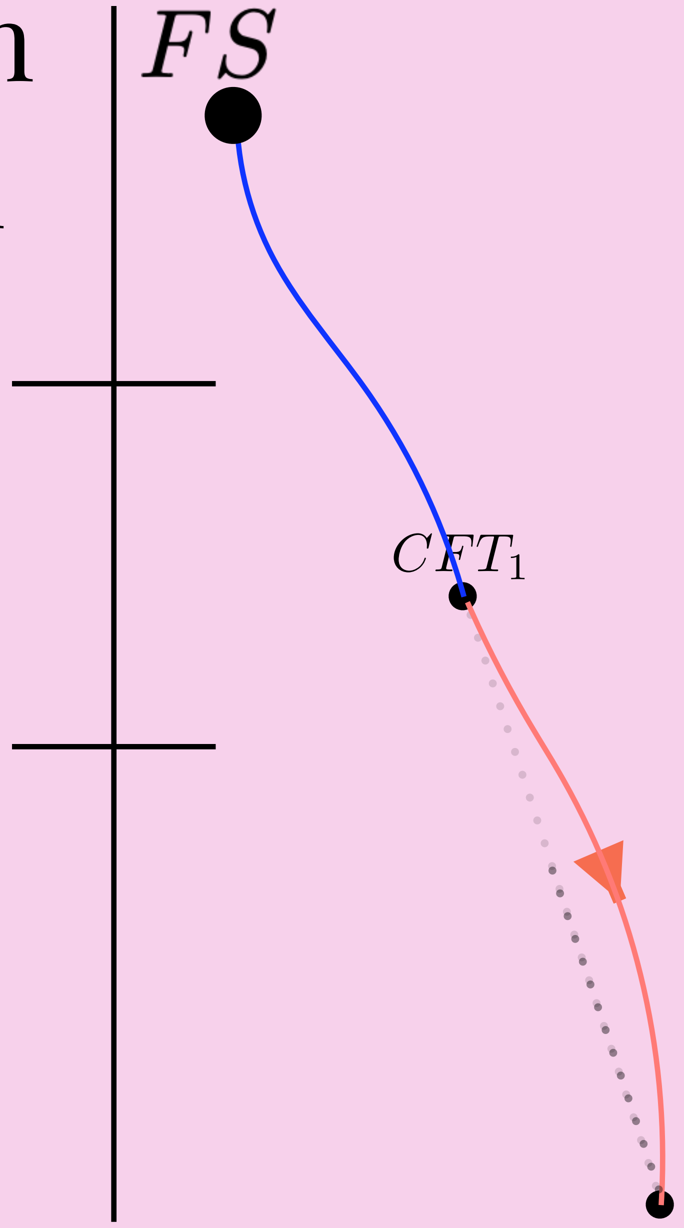
$\Lambda_{\mathrm{UV}} \sim \sqrt{B} \sim \sqrt{N}$

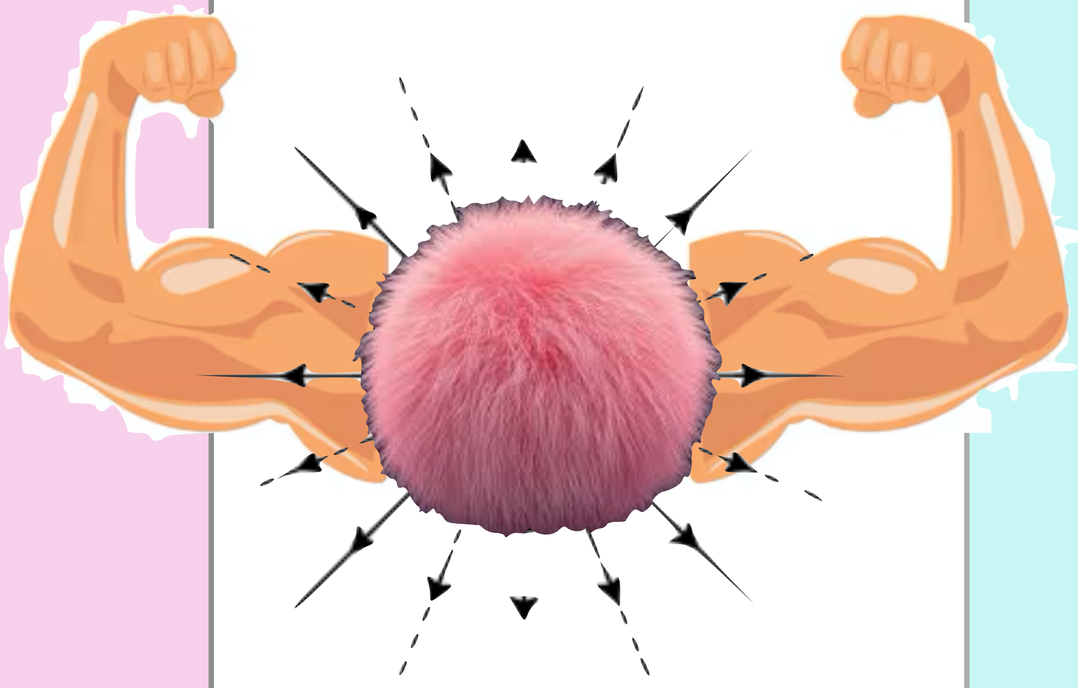
two scale transitions

UV CFT

$\mu \equiv h_z^{\frac{1}{3-\Delta_\sigma}}$

IR massive QFT





TCSA: Using only CFT Data

(get CFT data from **Fuzzy Sphere**)

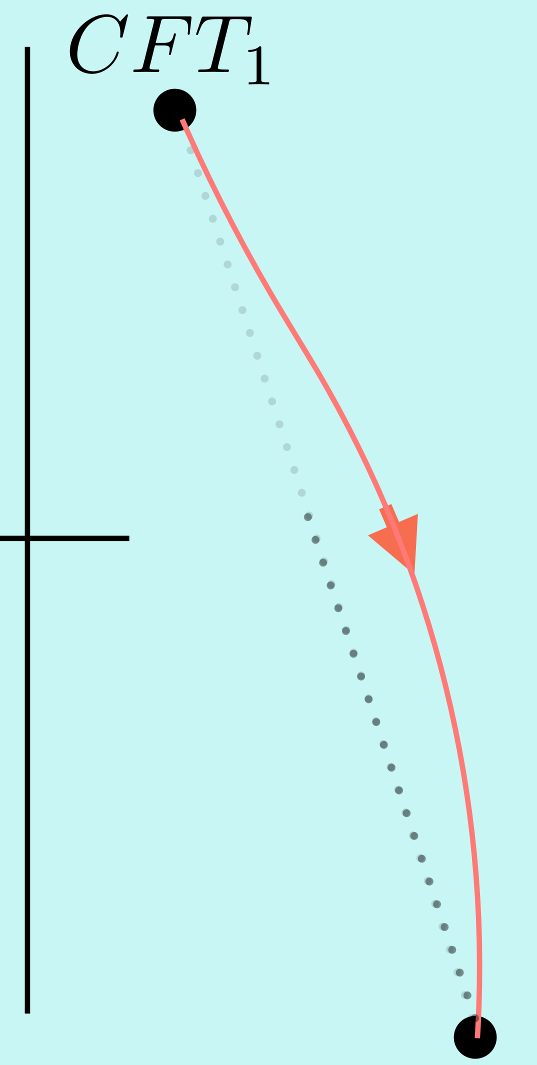
$$H = H_{3d\ \text{Ising CFT}} + h_z \int d^2\Omega\ \sigma(\Omega)$$

UV CFT

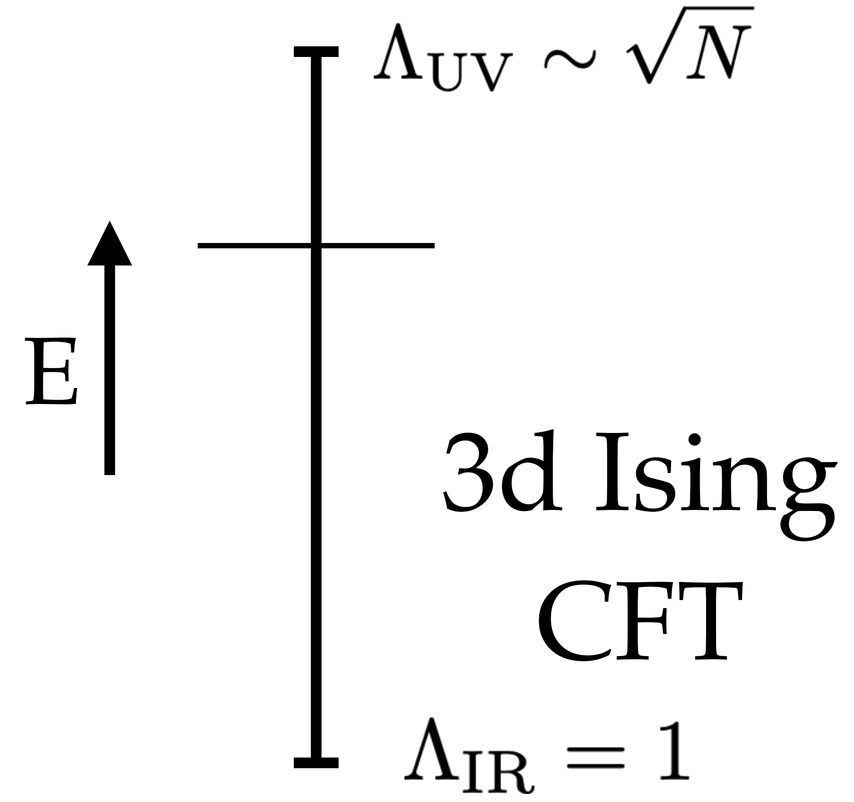
one scale transition

$\mu \equiv h_z^{\frac{1}{3-\Delta_\sigma}}$

IR massive QFT



Excavating High Dimension Primaries



Identify primaries vs descendants by using the fact that P_μ raises dimension by 1

$\Delta_{\partial_\mu \mathcal{O}} = \Delta_{\mathcal{O}} + 1$ etc.

But this gets harder at higher energies where spectrum is dense

Δ=0	ℓ=0	Id
Δ=1.40906	ℓ=0	ϵ
Δ=2.40153	ℓ=1	$\partial_\mu \epsilon$
Δ=3	ℓ=2	$T_{\mu\nu}$
Δ=3.41479	ℓ=2	$\partial_\mu \partial_\nu \epsilon$
Δ=3.46556	ℓ=0	$\partial^2 \epsilon$
Δ=3.81164	ℓ=0	ϵ'
Δ=4.03867	ℓ=3	$\epsilon^{\mu\nu\rho} \partial_\nu T_{\rho\sigma}$
Δ=4.0798	ℓ=2	
Δ=4.39303	ℓ=3	
Δ=4.55349	ℓ=1	
Δ=4.81803	ℓ=1	
Δ=4.97802	ℓ=4	
Δ=5.11837	ℓ=4	
Δ=5.16312	ℓ=3	
Δ=5.18702	ℓ=2	
Δ=5.26222	ℓ=4	
Δ=5.53438	ℓ=2	
Δ=5.5791	ℓ=2	

E.g. which of these is the primary "$C^{\mu\nu\rho\sigma}$" and which is $\partial_\rho \partial_\sigma T_{\mu\nu}$?

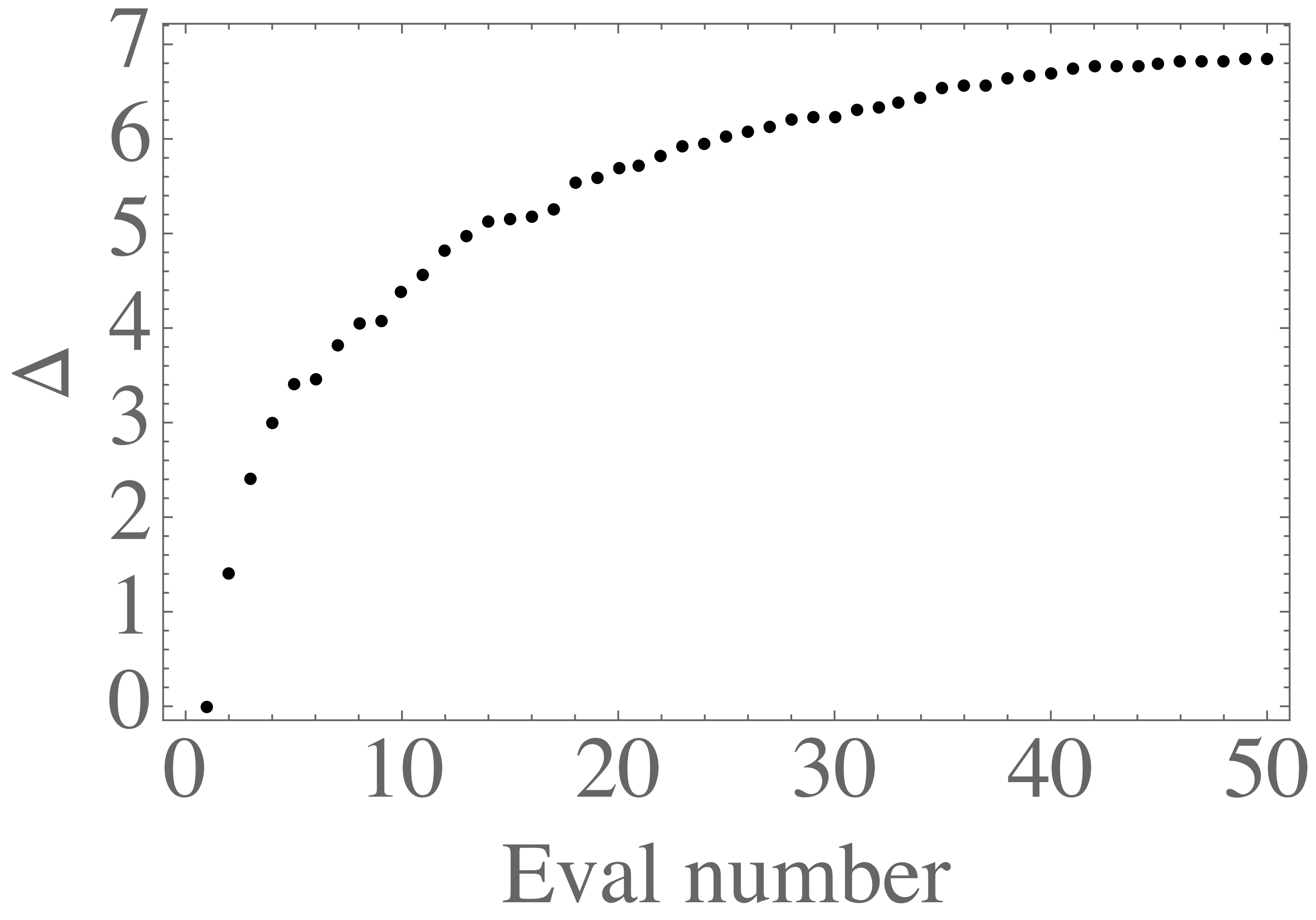
$\Delta_{C^{\mu\nu\rho\sigma}} = 5.023$

$\Delta_{\partial_\rho \partial_\sigma T_{\mu\nu}} = 5$

E.g. which of these is the primary T′ and which is $\partial^2 \partial_\mu \partial_\nu \epsilon$?

$\Delta_{T'} = 5.509$

$\Delta_{\partial^2 \partial_\mu \partial_\nu \epsilon} = 5.413$



We need to construct the special conformal generators K:

$K_a |\text{primary}\rangle = 0$

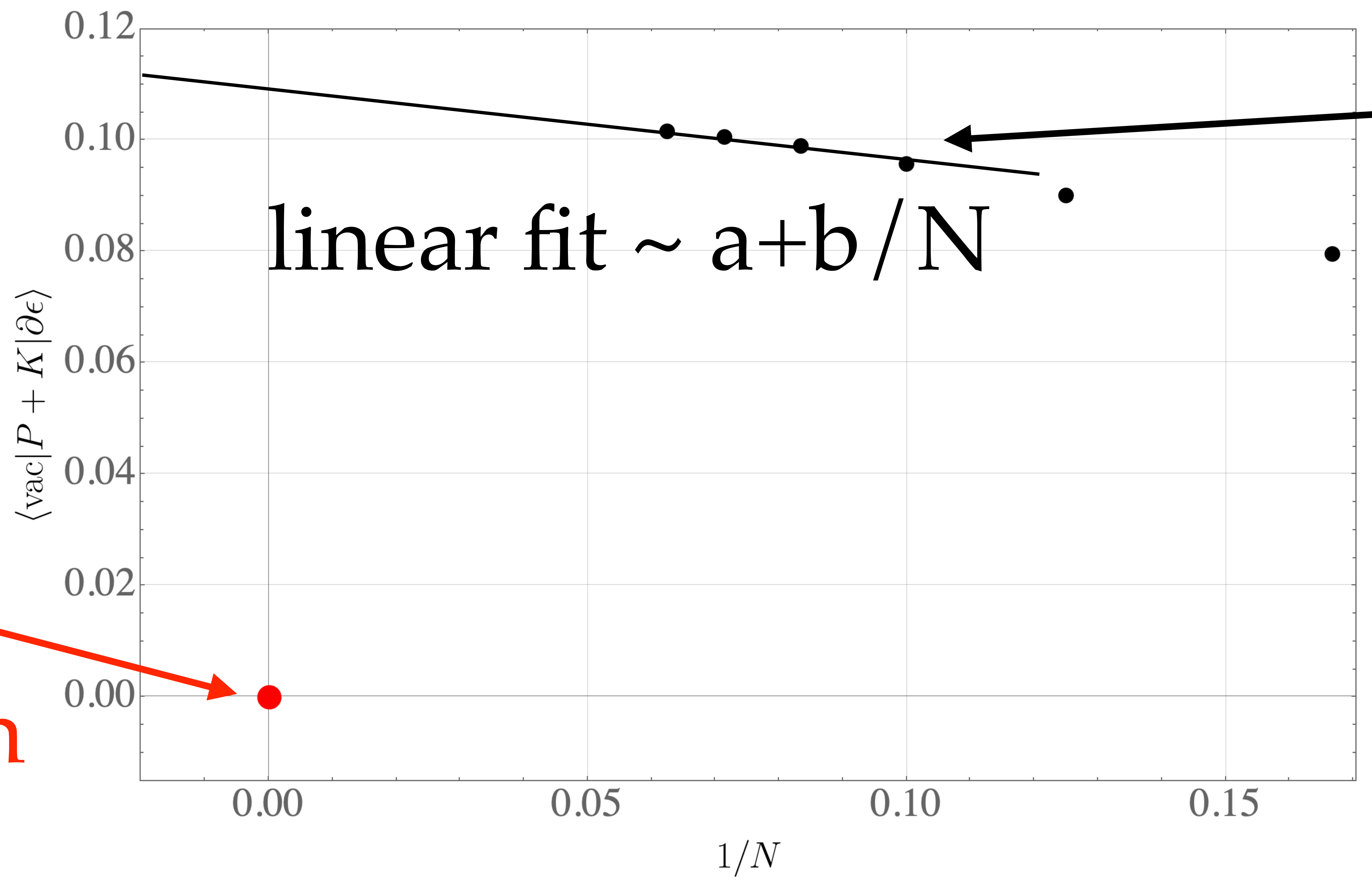
Constructing $P^i + K^i \approx 2\int d^2\Omega \hat{x}^i \mathcal{H}$

Δ=0	ℓ=0	$\mathrm{Id} = \|\mathrm{vac}\rangle$
Δ=1.40906	ℓ=0	ϵ
Δ=2.40153	ℓ=1	$\partial_\mu \epsilon$
Δ=3	ℓ=2	$T_{\mu\nu}$

Numeric test: conformal invariant vacuum implies

$P^i|\mathrm{vac}\rangle = 0 \quad K^i|\mathrm{vac}\rangle = 0$

So $\langle \mathrm{vac}|P^i + K^i|\partial\epsilon\rangle = 0$ in the CFT



CFT prediction

Fuzzy sphere results

Uh oh!

What's wrong?

Constructing $P^i + K^i \approx 2 \int d^2\Omega \hat{x}^i \mathcal{H}$

removed by tuning V_0, h

$$P^i + K^i \approx 2 \int d^2\Omega \hat{x}^i \mathcal{H} = 2 \int d^2\Omega \hat{x}^i (T^0_0 + \cancel{g_\epsilon} \epsilon + g_{\partial^2\epsilon} \partial^2 \epsilon + \dots)$$

$$\Delta_{\partial^2\epsilon} = 2 + \Delta_\epsilon = 3.413$$

Descendants contribute now!

We haven't tuned away contribution from $\partial^2 \epsilon$

Scales like $\Lambda_{\mathrm{UV}}^{3-\Delta_{\partial^2\epsilon}} = N^{\frac{3-3.413}{2}} = N^{-0.206}$

0.2 is a very small power! N is nowhere near large enough to suppress this term

Constructing Conformal Generators

We can easily fix this by removing the leading derivative terms from $\mathcal{H}$!

$$\mathcal{H} \to \mathcal{H} + a_1 \nabla^2 \psi^\dagger \sigma^x \psi \qquad \nabla^2 \psi^\dagger \sigma^x \psi \to \nabla^2 \epsilon + \ldots$$

Finally, how do we get P and K from P+K? Easy!

K_i is lowering operator

P_i is raising operator

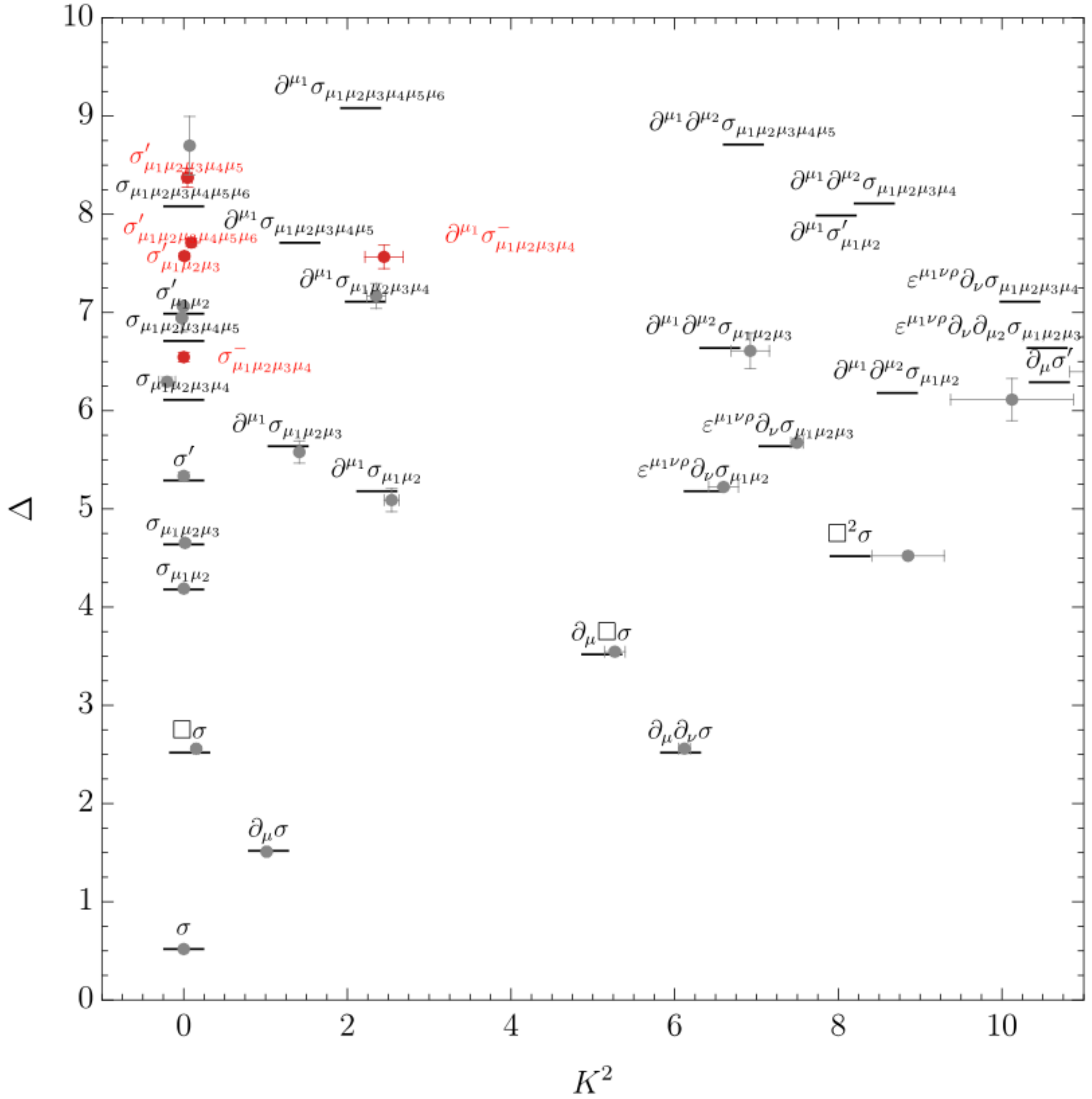
In the basis of eigenstates of H, P+K is just a big matrix

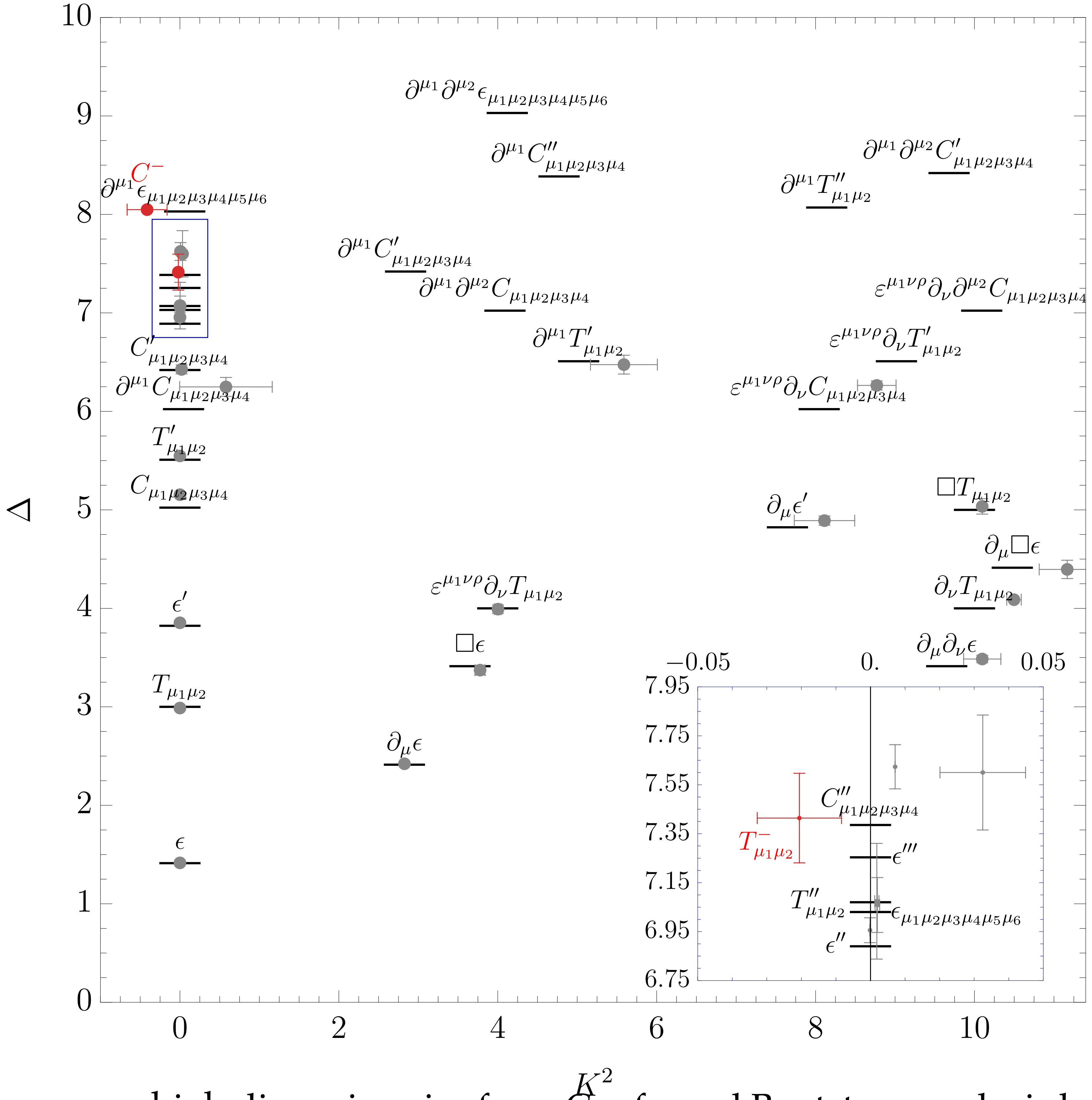
$$\begin{pmatrix} 0 & 0 & -0.374889 & 0 & 0 & 0 \\ 0 & 0 & -3.62057 & 0 & 0 & 0 \\ -0.374889 & -3.62057 & 0 & 0.3875 & 5.69698 & 2.39937 \\ 0 & 0 & 0.3875 & 0 & 0 & 0 \\ 0 & 0 & 5.69698 & 0 & 0 & 0 \\ 0 & 0 & 2.39937 & 0 & 0 & 0 \end{pmatrix}$$

Finding High Dimension Primaries with Fuzzy Sphere

K_a=Special conformal generators, $K^2 = K^\dagger \cdot K$

In 3d Ising CFT, descendants* have $\mathrm{K}^2 \geq 1$, primaries have K^2=0





New primaries

$\mathcal{O}$	Δ	ℓ	$\mathbb{Z}_2$	$\mathcal{P}$
$\sigma'_{\mu_1\mu_2\mu_3}$	7.53(6)	3	-	+
$\sigma^-_{\mu_1\mu_2\mu_3\mu_4}$	6.52(6)	4	-	-
$\sigma'_{\mu_1\mu_2\mu_3\mu_4\mu_5}$	8.37(10)	5	-	+
$\sigma'_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}$	7.73(5)	6	-	+
$T^-_{\mu_1\mu_2}$	7.41(18)	2	+	-
$C^-_{\mu_1\mu_2\mu_3\mu_4}$	8.04(2)	4	+	-

Fardelli, Fitzpatrick, Katz '26

to appear: high dim primaries from Conformal Bootstrap analysis by Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Matthew, Poland, Simmons-Duffin

*except for descendants of "almost-conserved" currents

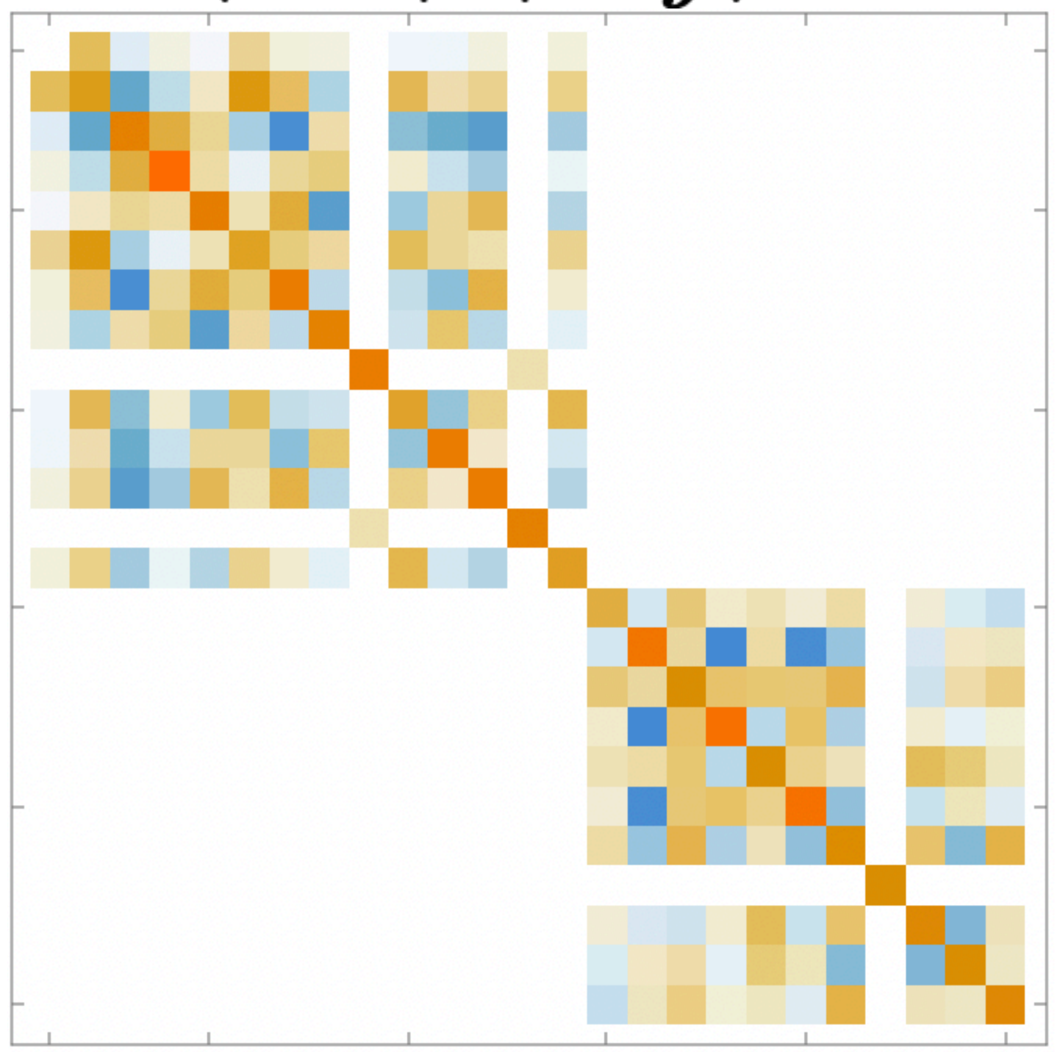
Next: redo it with TCSA

To do 3d Ising Field Theory $H = H_{\text{CFT}} + \int d^2x g_\epsilon \epsilon(x) + g_\sigma \sigma(x)$

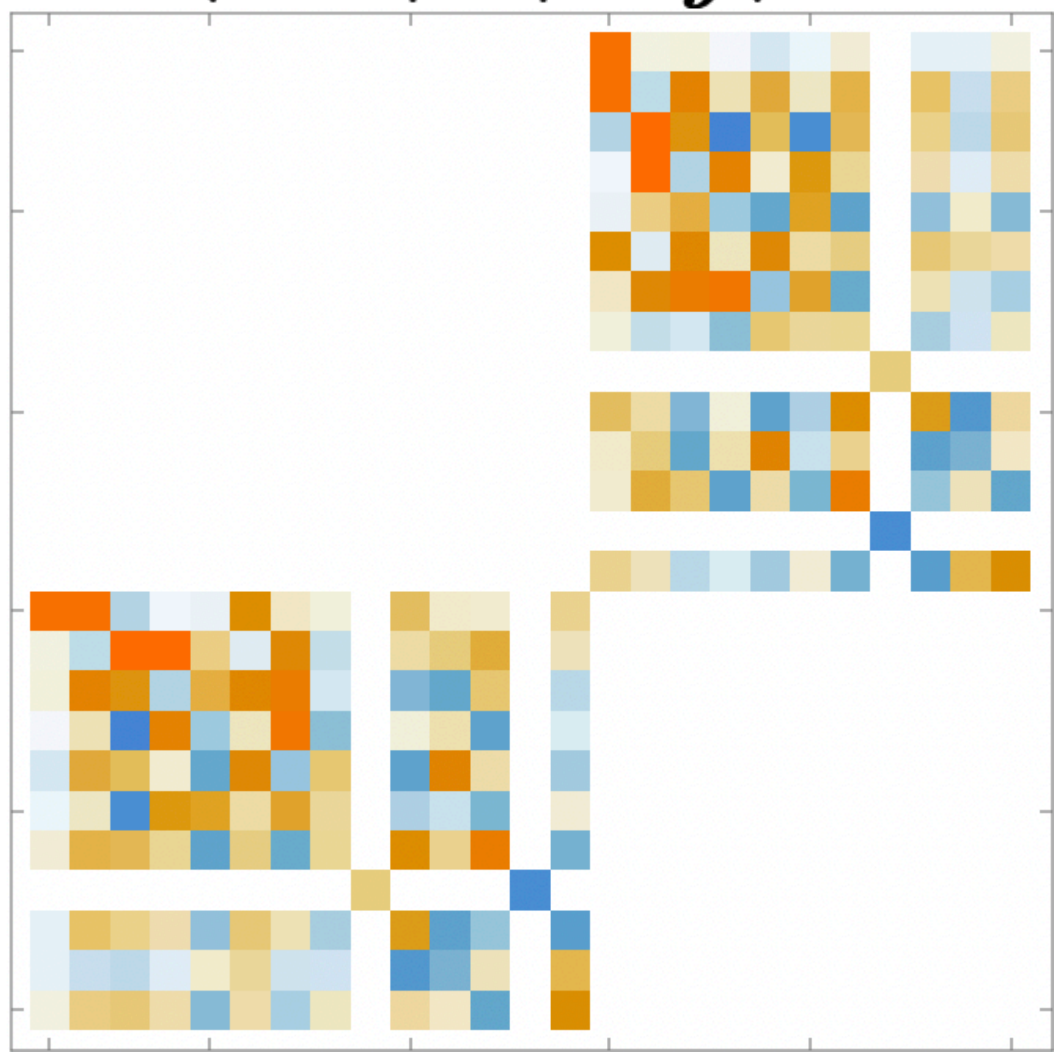
we need matrix elements of H: OPE coefficients $\langle \mathcal{O}_i | \epsilon | \mathcal{O}_j \rangle$ and $\langle \mathcal{O}_i | \sigma | \mathcal{O}_j \rangle$

Now we have them up to Δ ~ 9!
(from Fuzzy Sphere)

$\langle \mathcal{O}_i | \epsilon | \mathcal{O}_j \rangle$

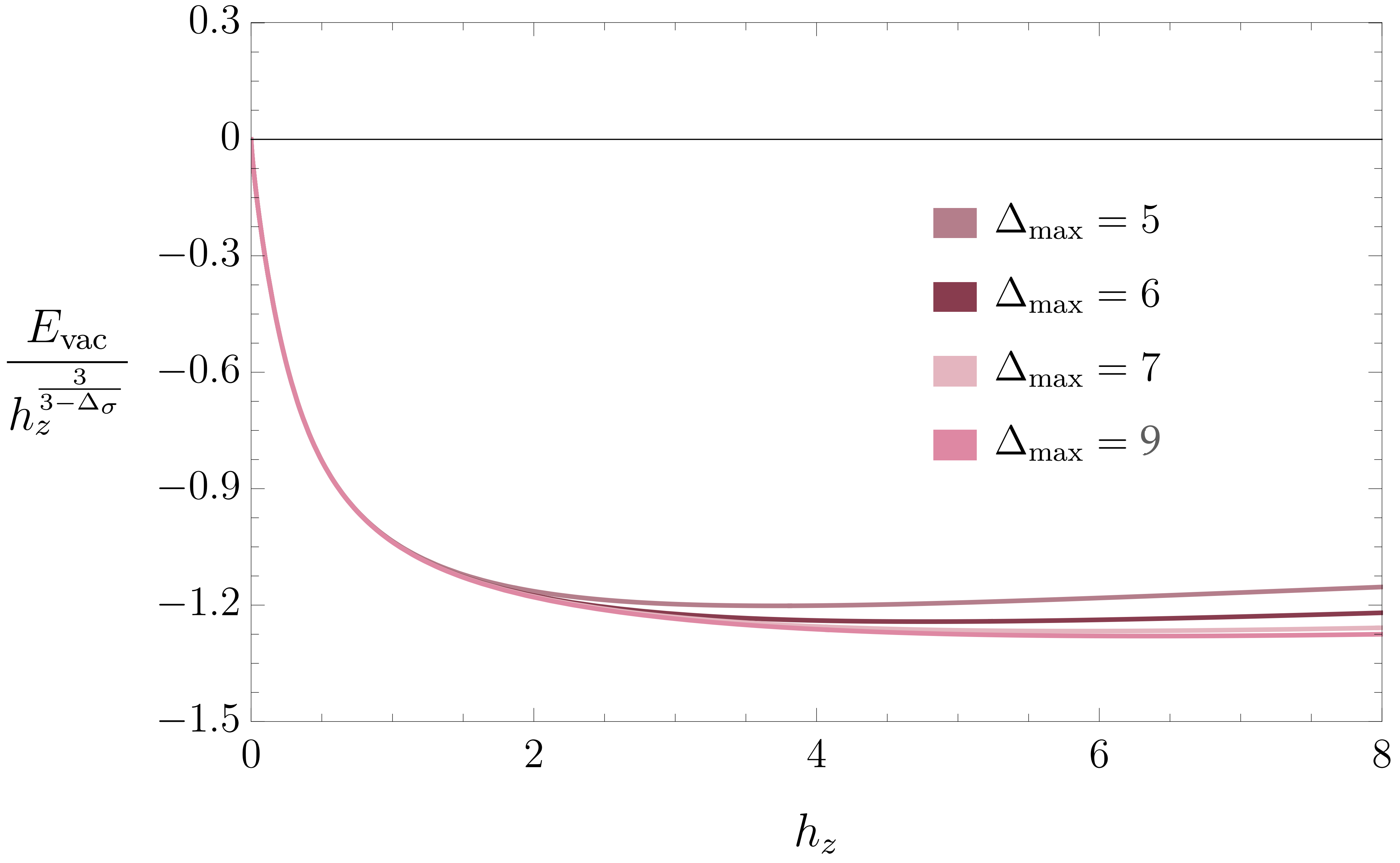


$\langle \mathcal{O}_i | \sigma | \mathcal{O}_j \rangle$



TCSA vs Fuzzy Sphere Comparison Vacuum Energy:

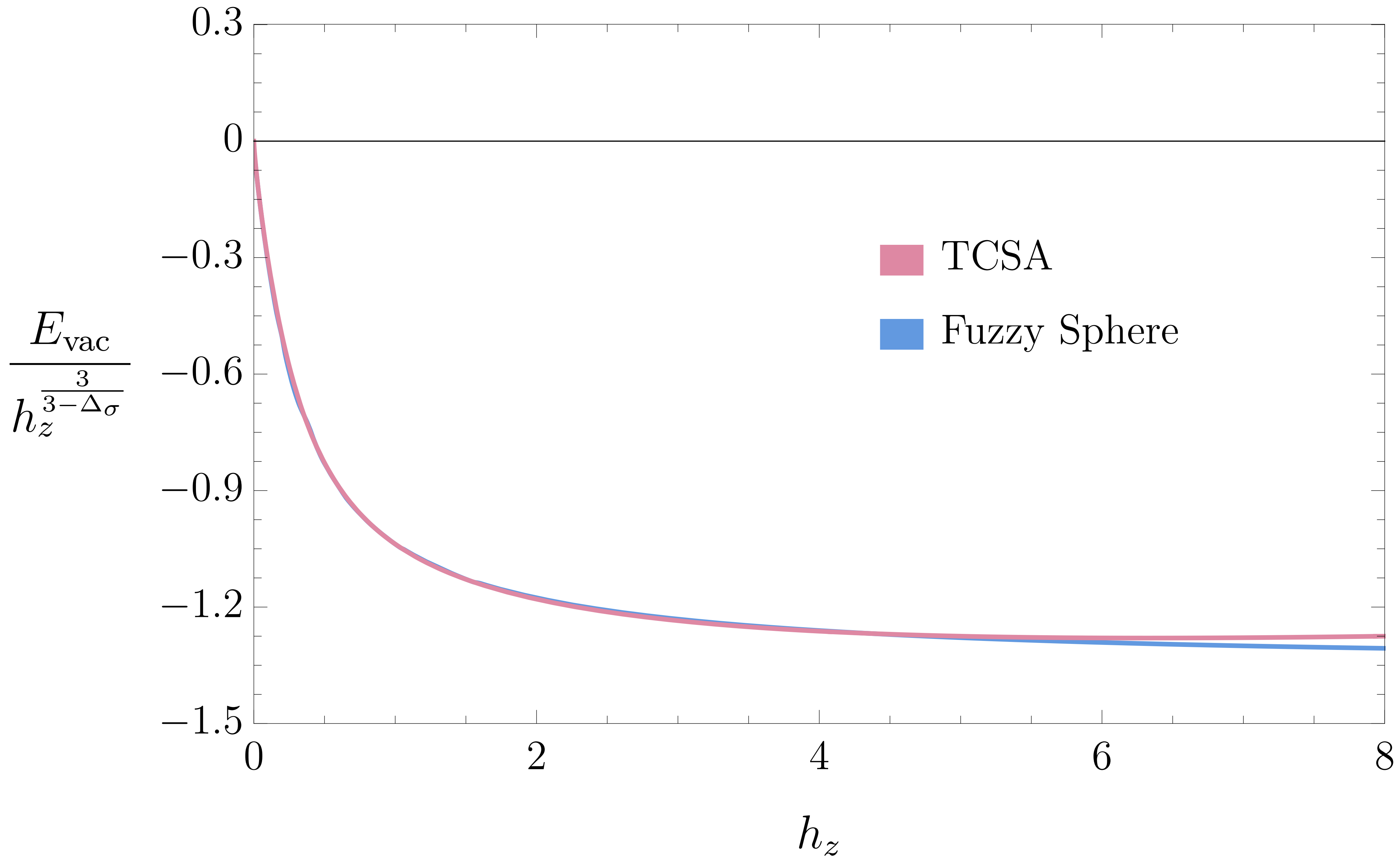
TCSA at finite truncation:



TCSA extrapolation

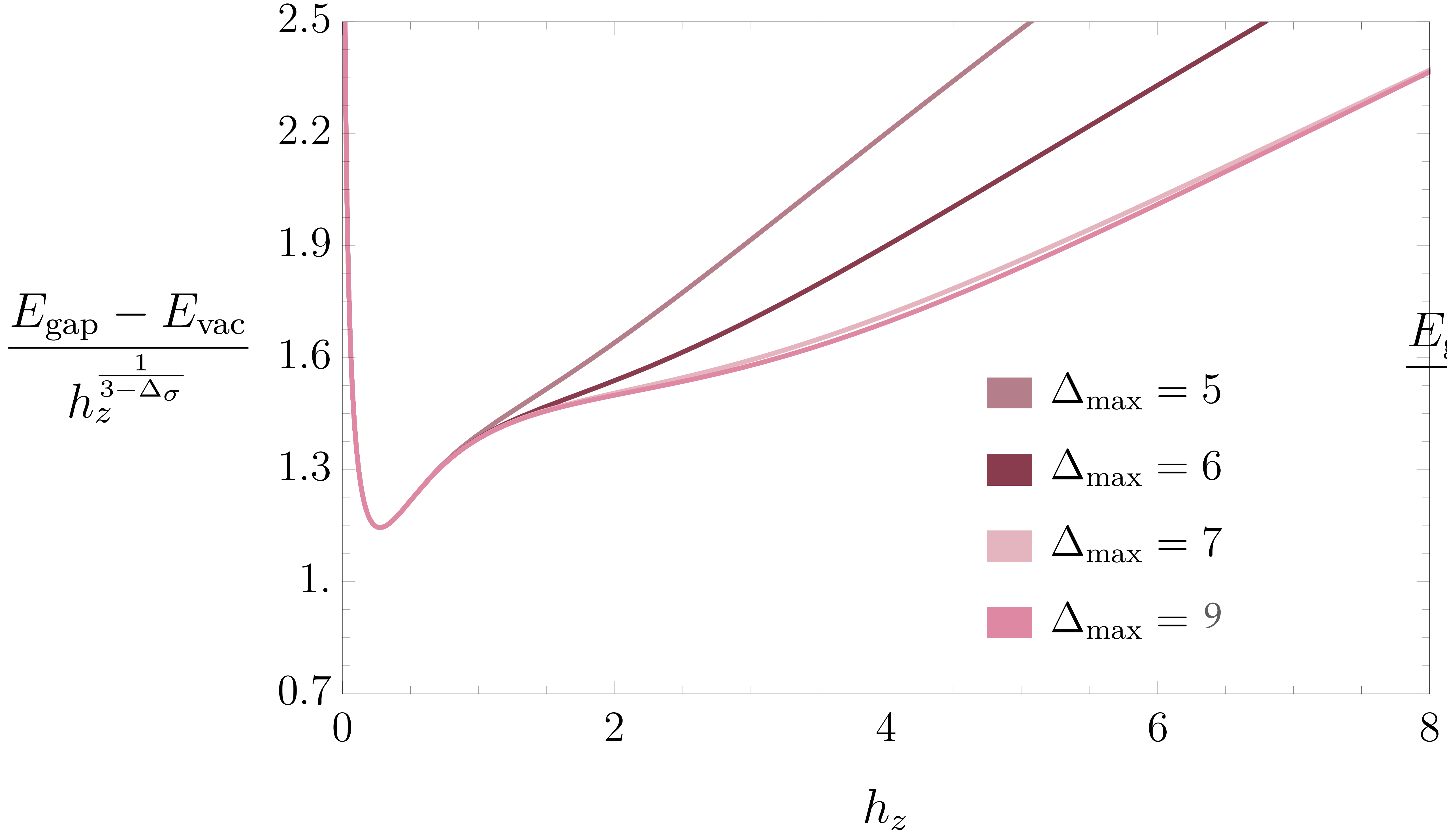
vs

Fuzzy sphere extrapolation

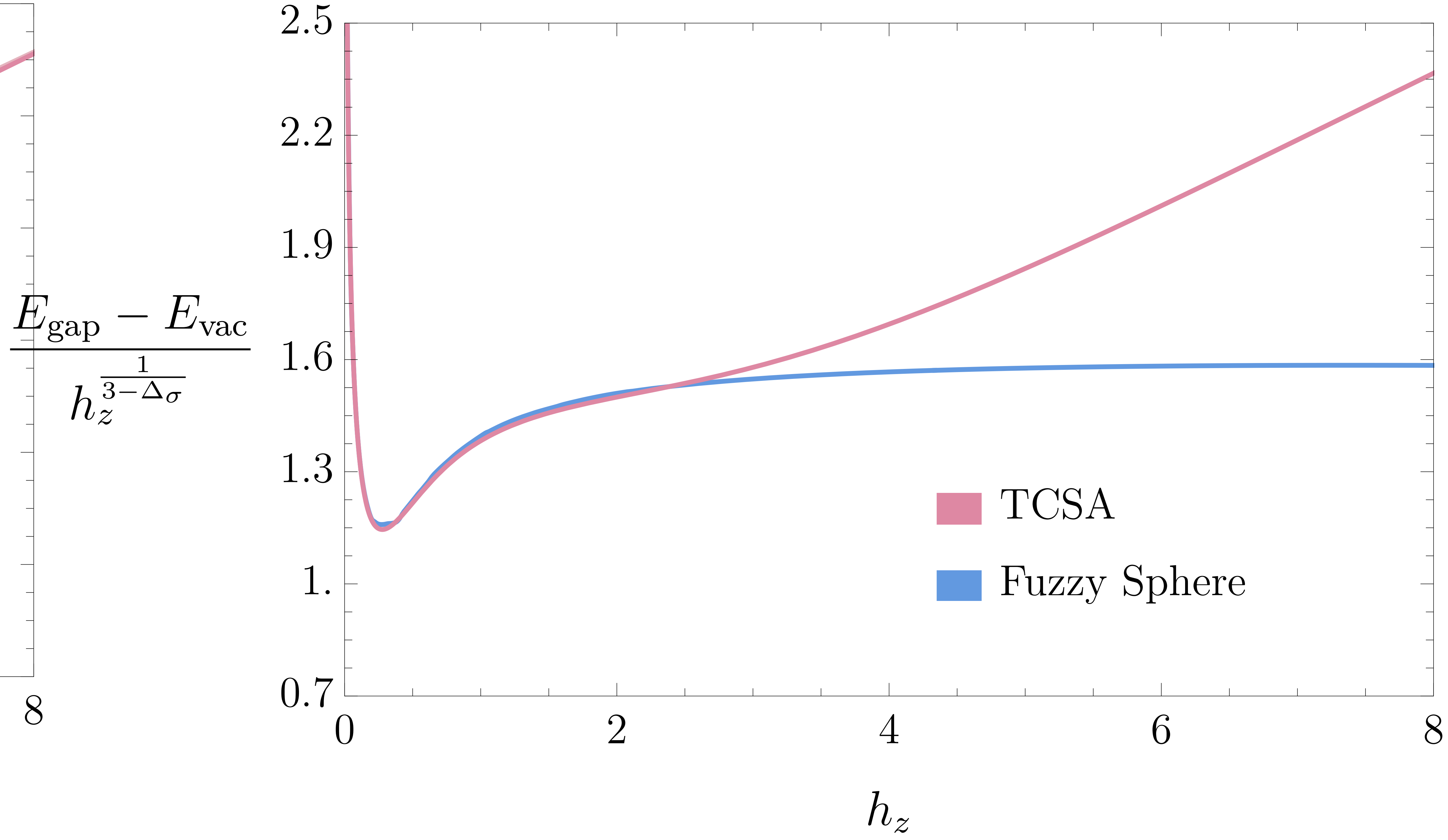


TCSA vs Fuzzy Sphere Comparison
Mass Gap:

TCSA at finite truncation:

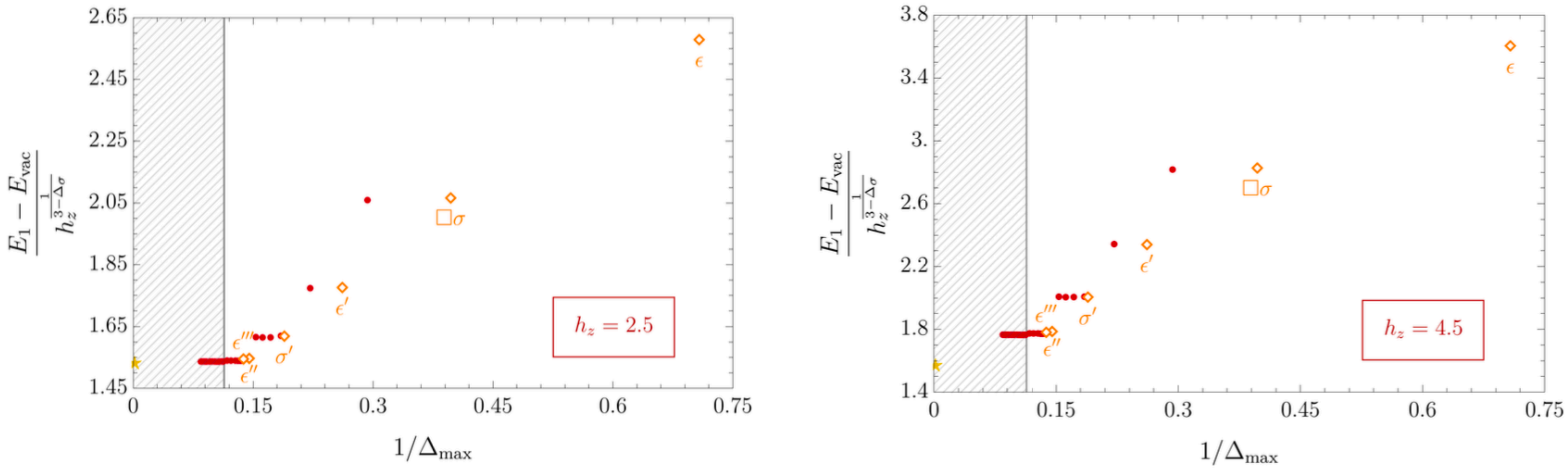


TCSA extrapolation vs Fuzzy sphere extrapolation



TCSA vs Fuzzy Sphere Comparison
Mass Gap:

Why does truncation diverge from FS around h ~ 4?



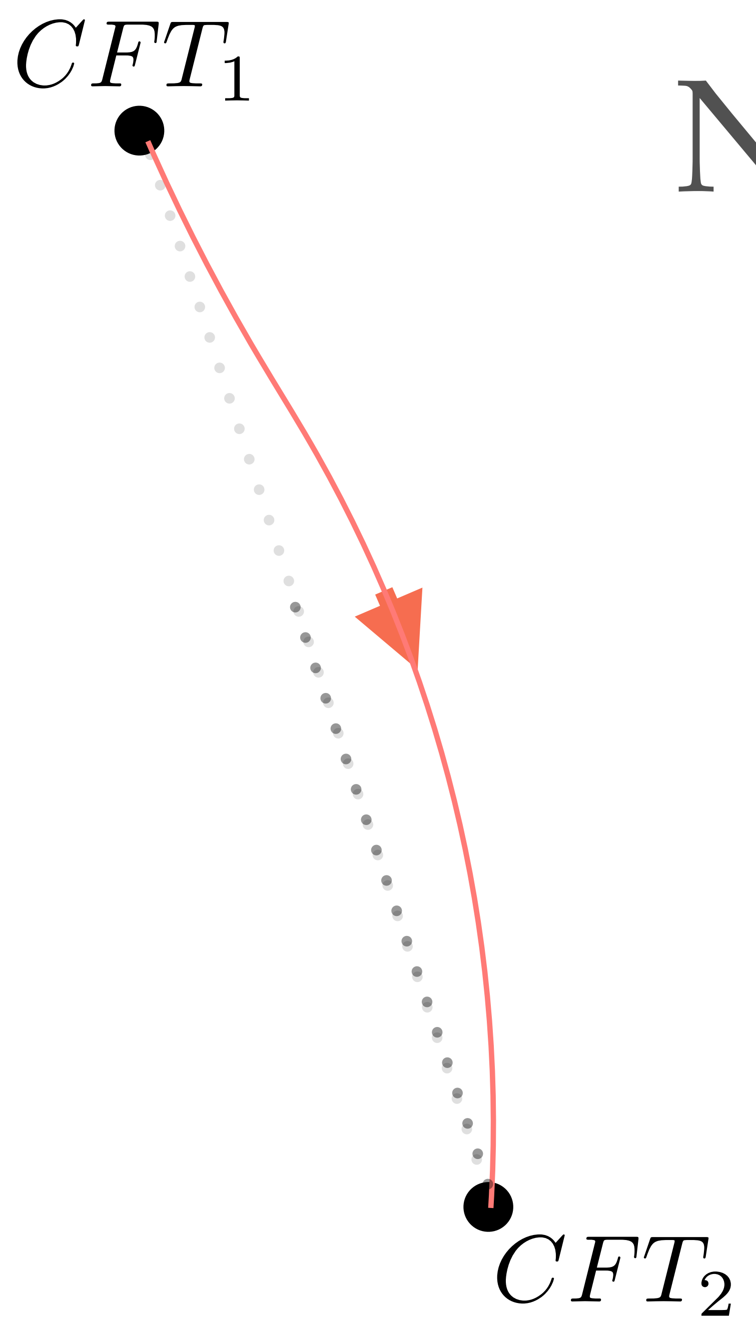
Shaded region is above our cutoff - we are missing Δ~ 9 primaries there

seen in ϵ-expansion analysis by
Henriksson, Kousvos, Nepveu '25

Goal: Solve QFT in 2 Steps

Step 1: Solve CFT

Step 2: Solve RG flow



Not a joke!

What does it mean to solve a CFT when the theory is chaotic?

Step 1: Solve CFT

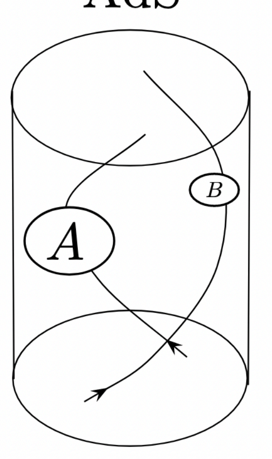
"Chaotic" means there will never be an efficient algorithm for all CFT data

If we want to "solve" 3d Ising, say, what should our goal be?

1) Some limits *will* still simplify and admit approximately integrable solutions/Effective descriptions - "scar states"

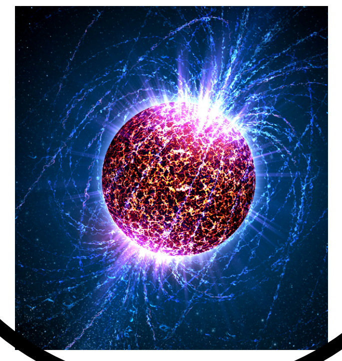
long-distance AdS physics

AdS



Large spin

CFT

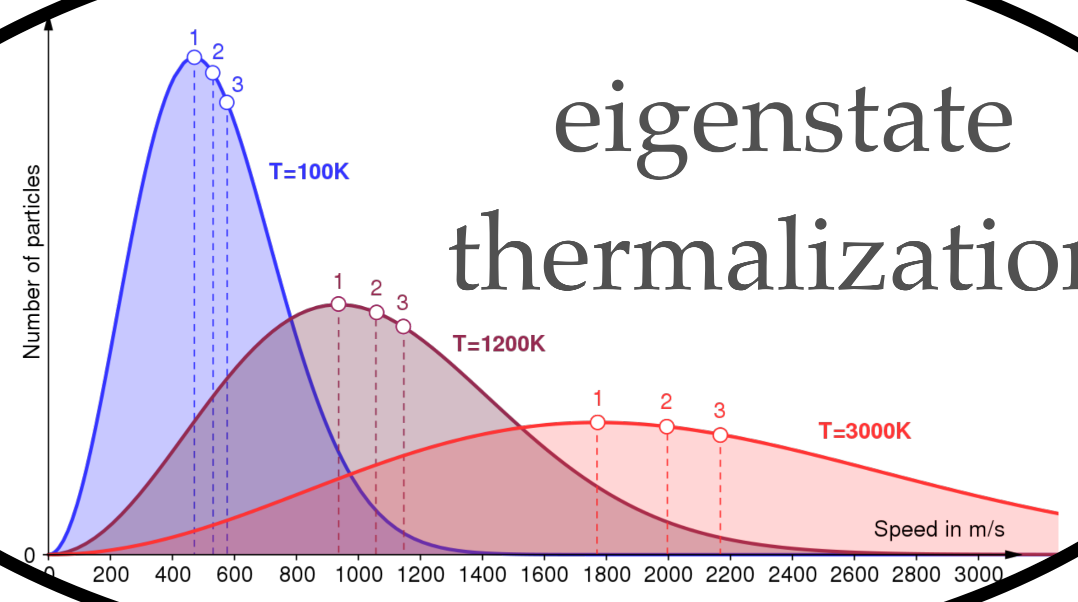


Large charge

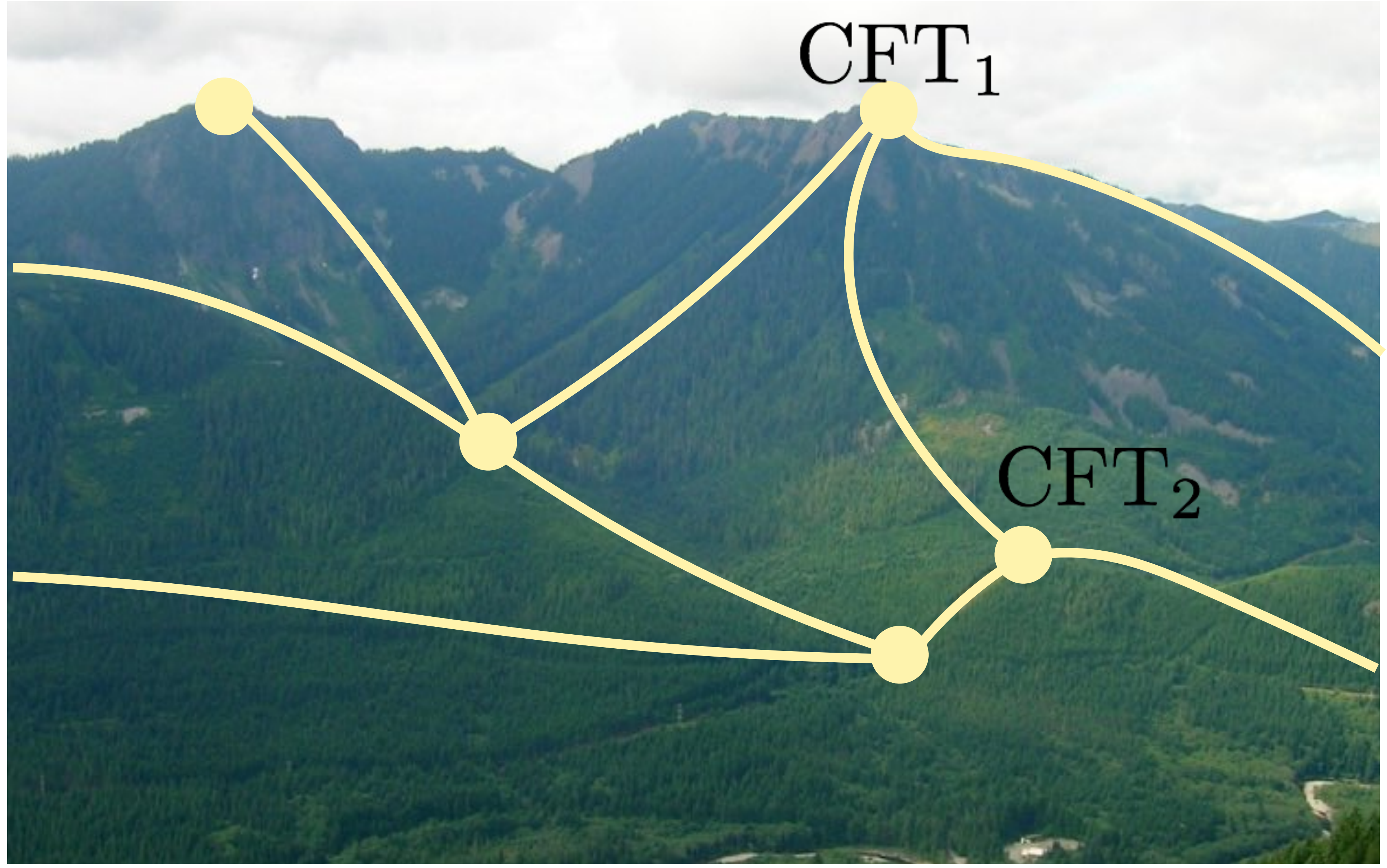
2) Chaotic regime will be described probabilistically

Generic high energy

eigenstate thermalization



The End



The wonder of the world
The beauty and the power,
The shapes of things,
Their colours, lights and shades,
These I saw.
Look ye also while life lasts.

— **Denys James Watkins-Pitchford,**
Alexander Morton Memorial