Black Holes, Random Variables and the Bootstrap

Eric Perlmutter, IPhT Saclay

Bootstrap, August 7 2026

Based on 2607.02233 and work to appear

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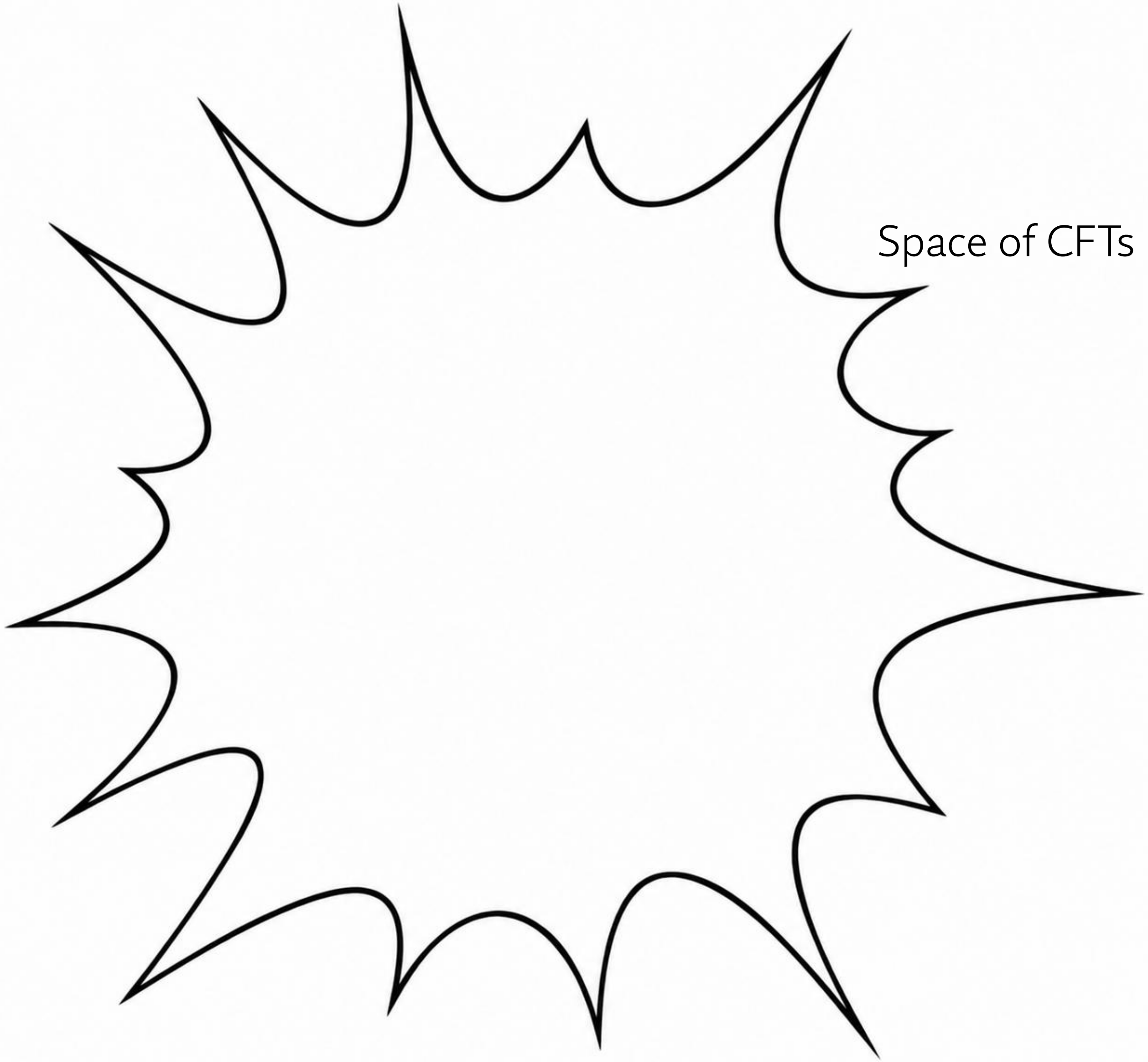
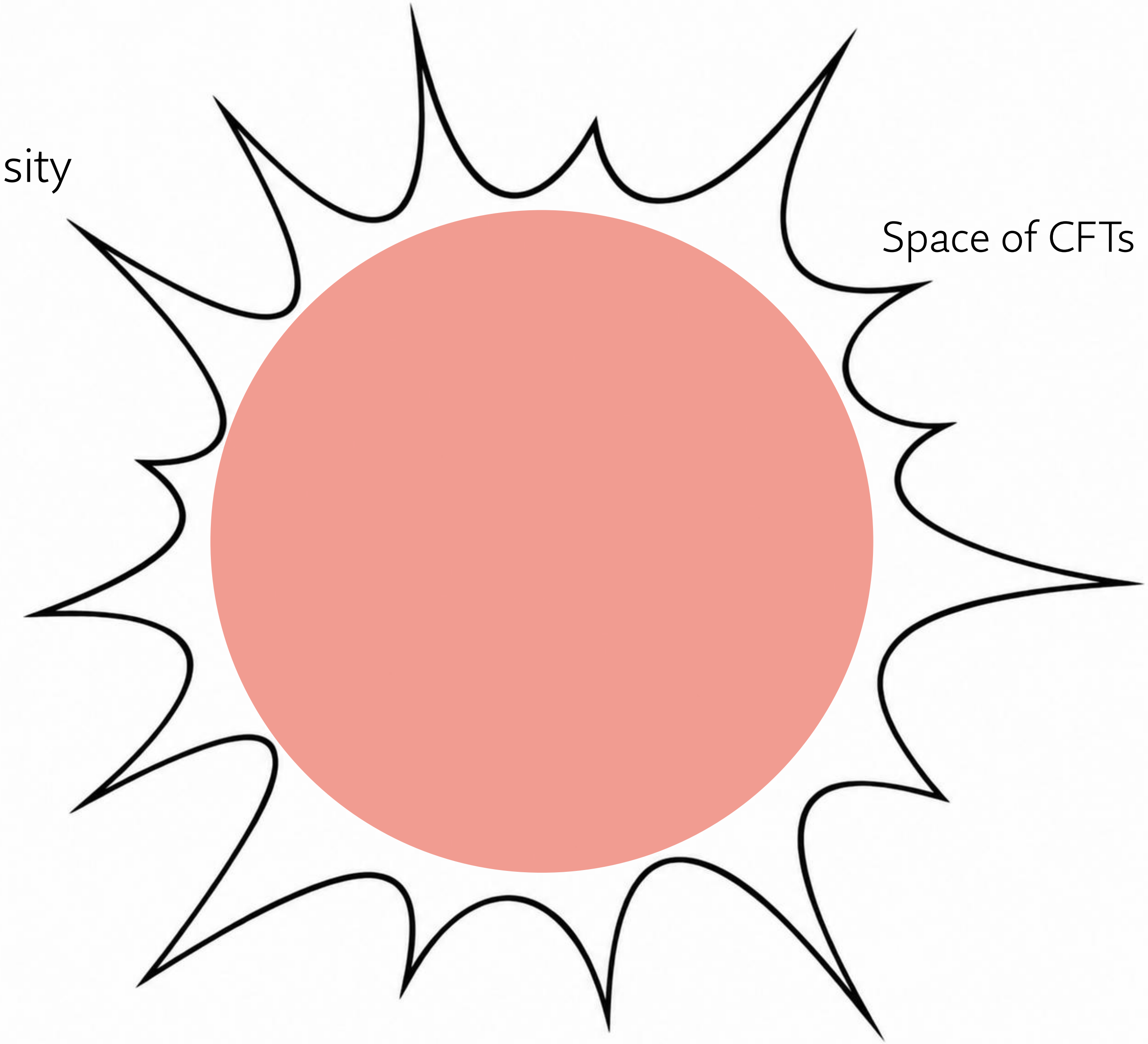


Space of CFTs

Proposal: stratify the space of CFTs *using spectral statistics*.

"Normal CFTs"

Linear statistics of the symmetry-resolved spectral density obey a **central limit theorem** at high energy.



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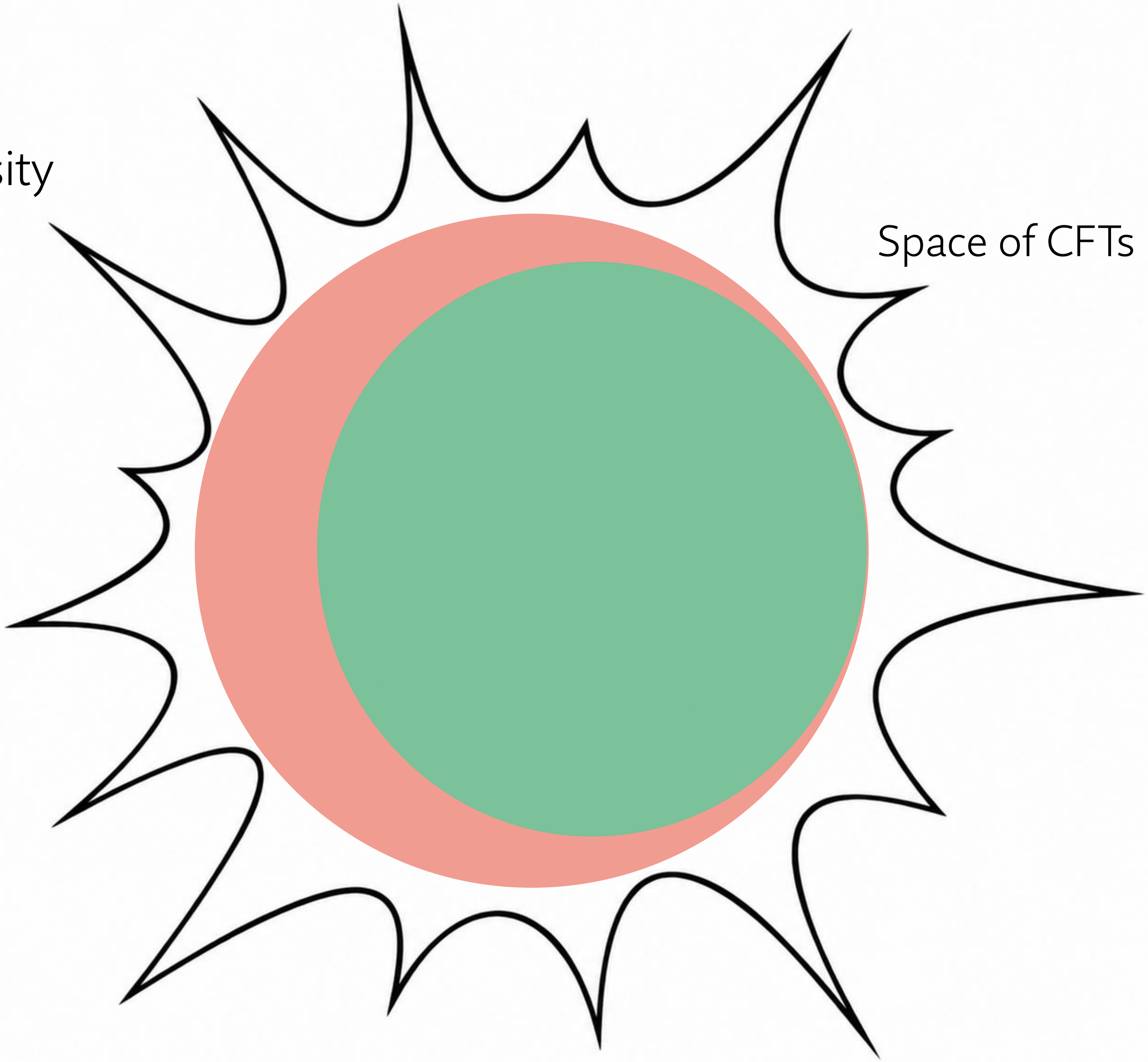

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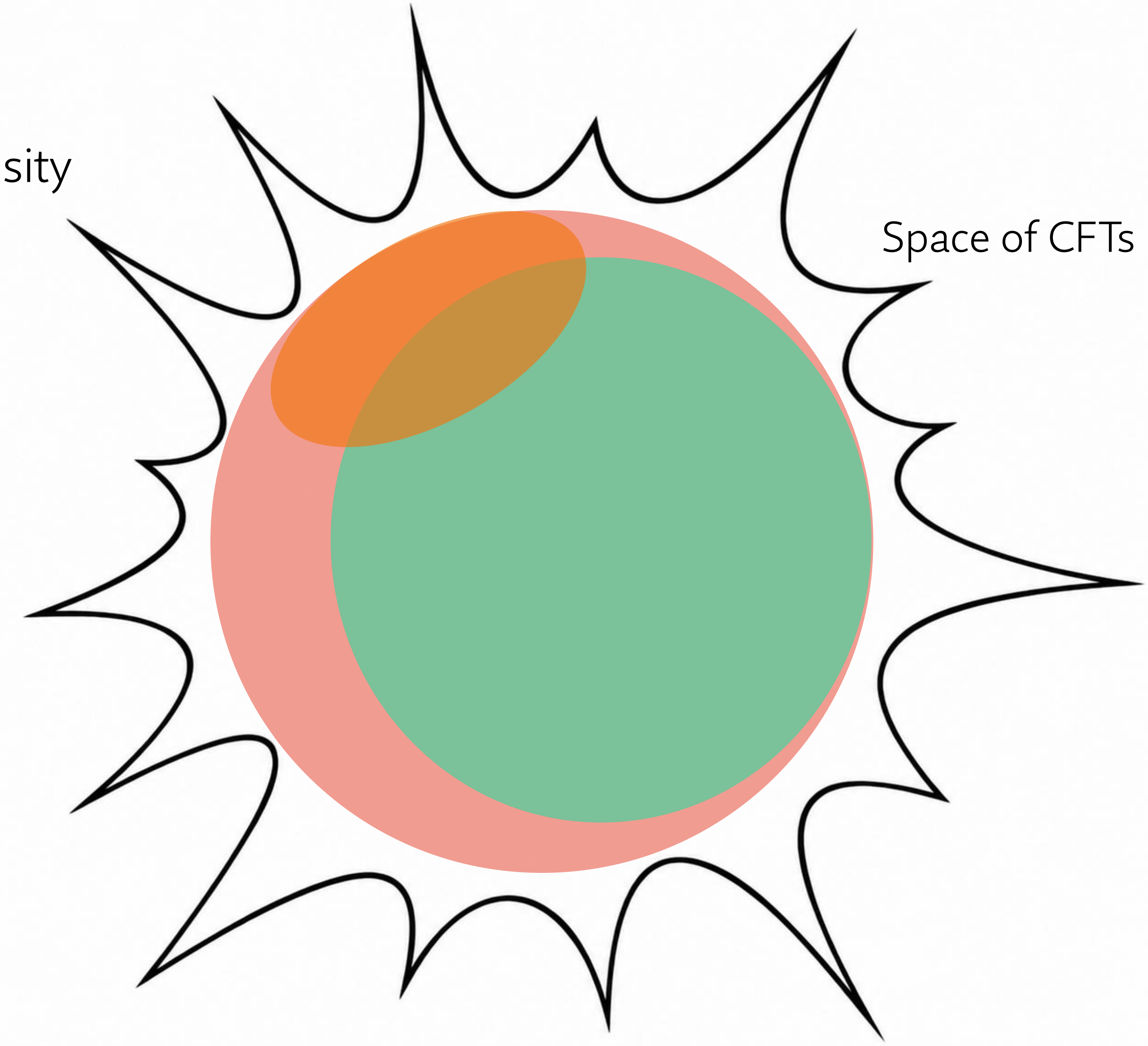

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- Hyperuniform statistics

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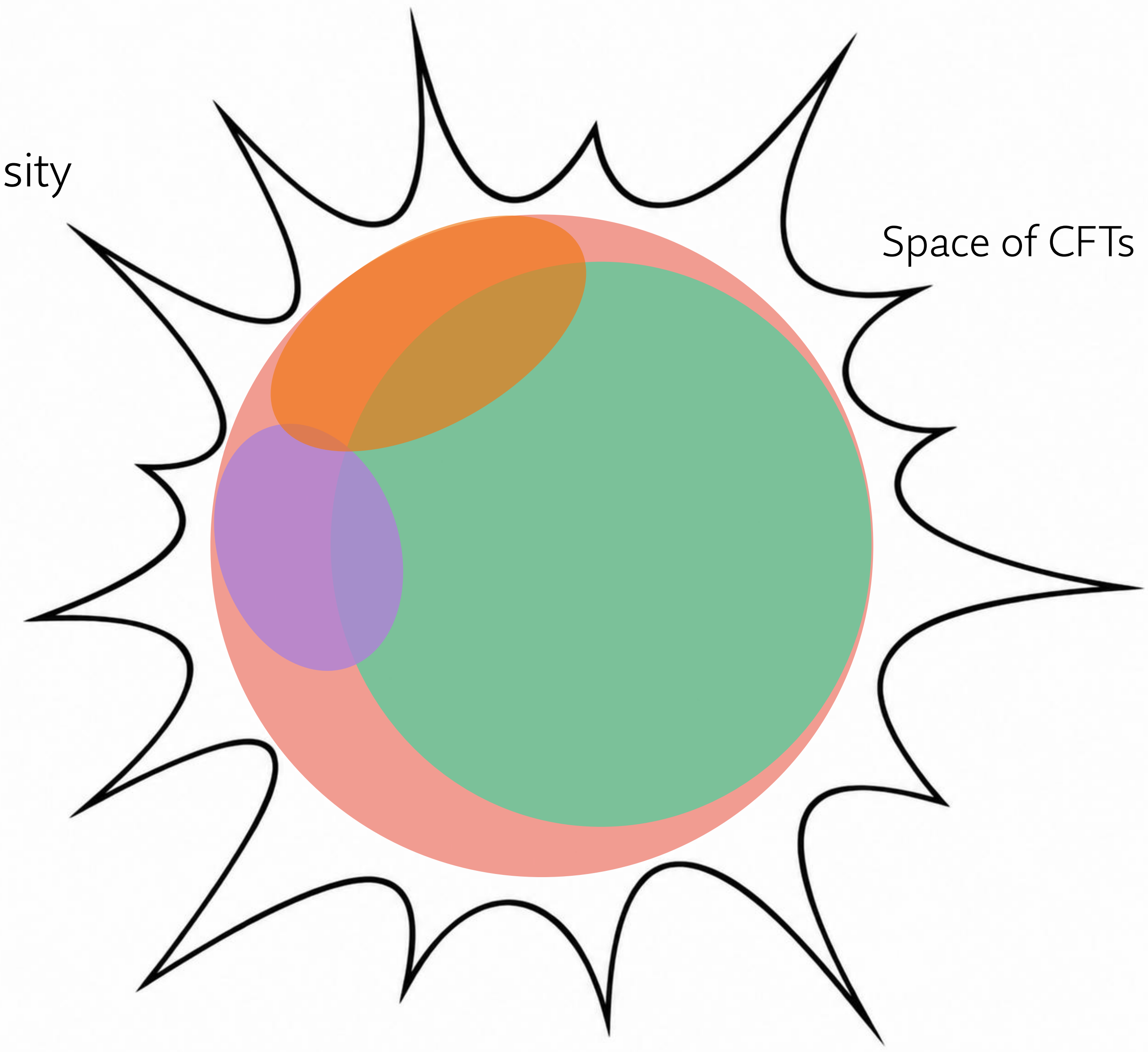

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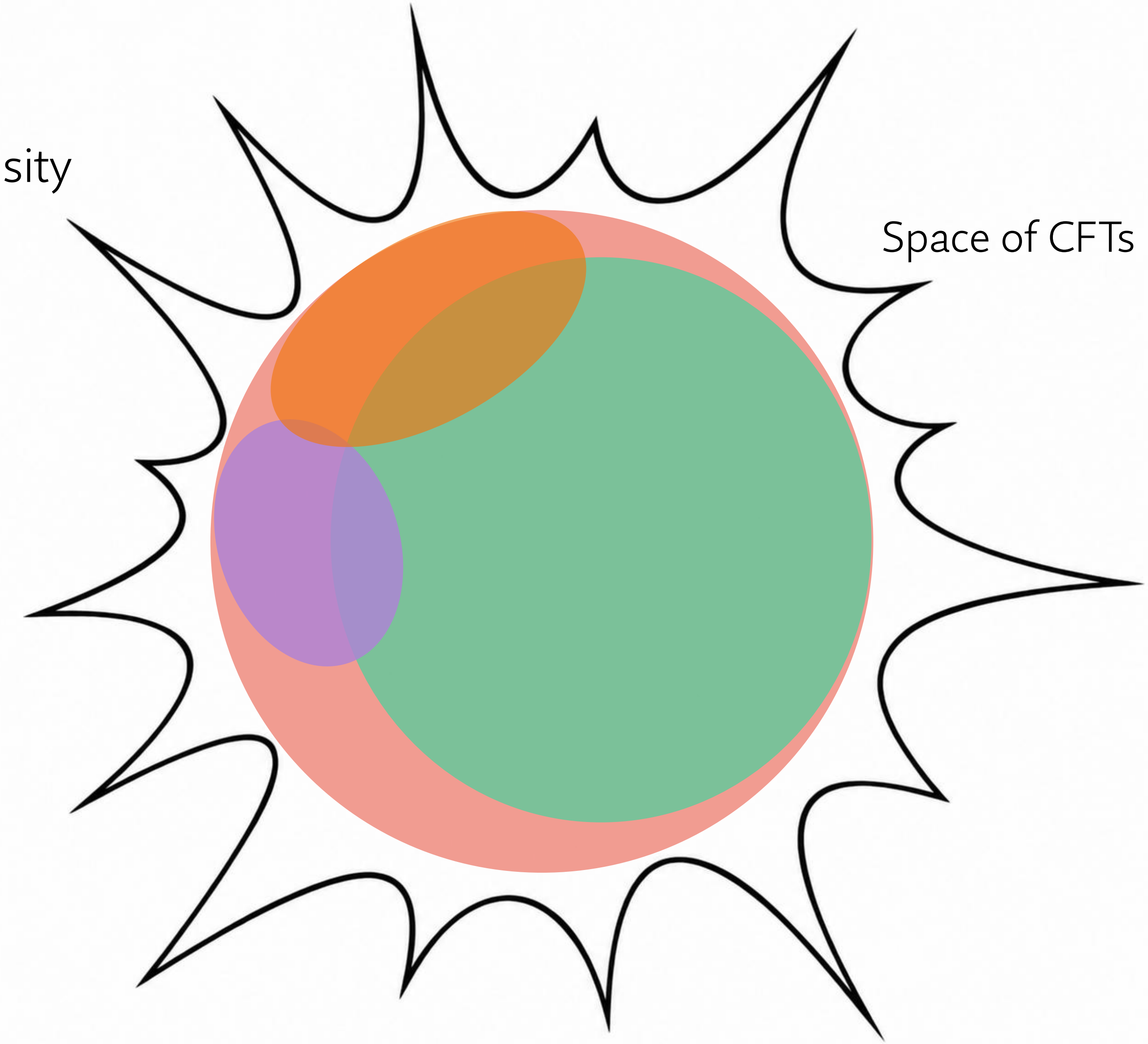
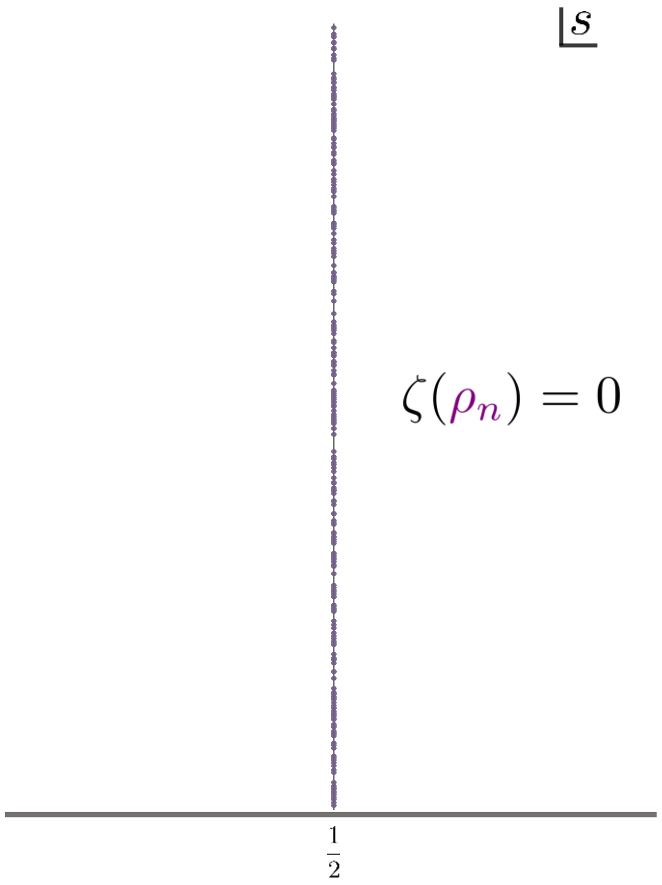


Today's talk is about a property of the random matrix subclass of normal CFTs.

Here is the Riemann Hypothesis:



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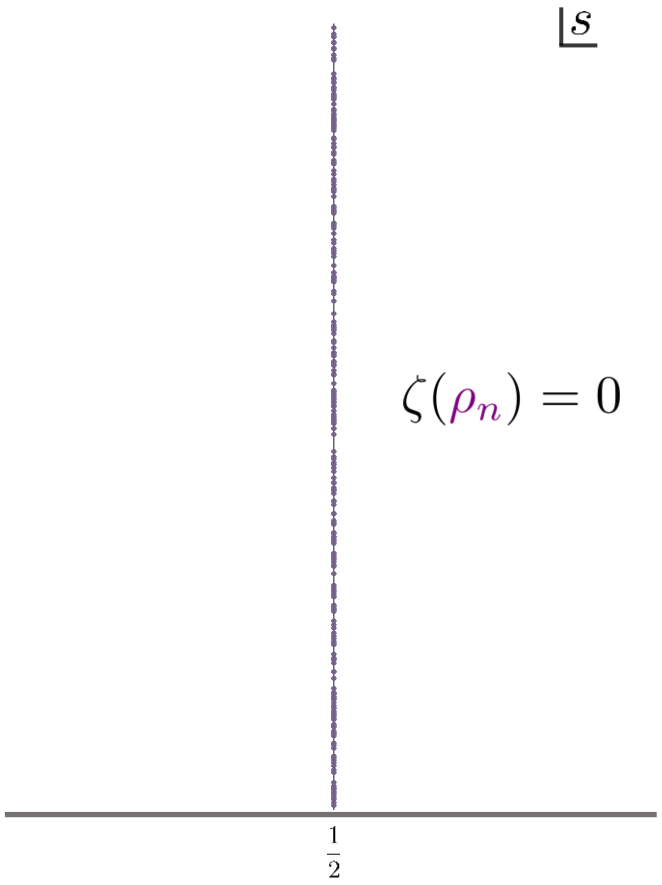

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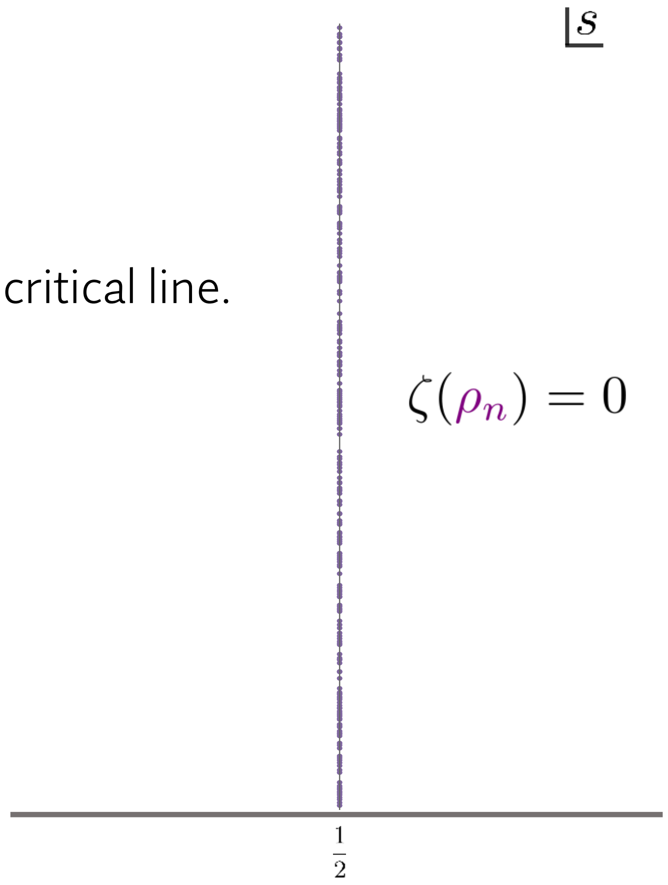

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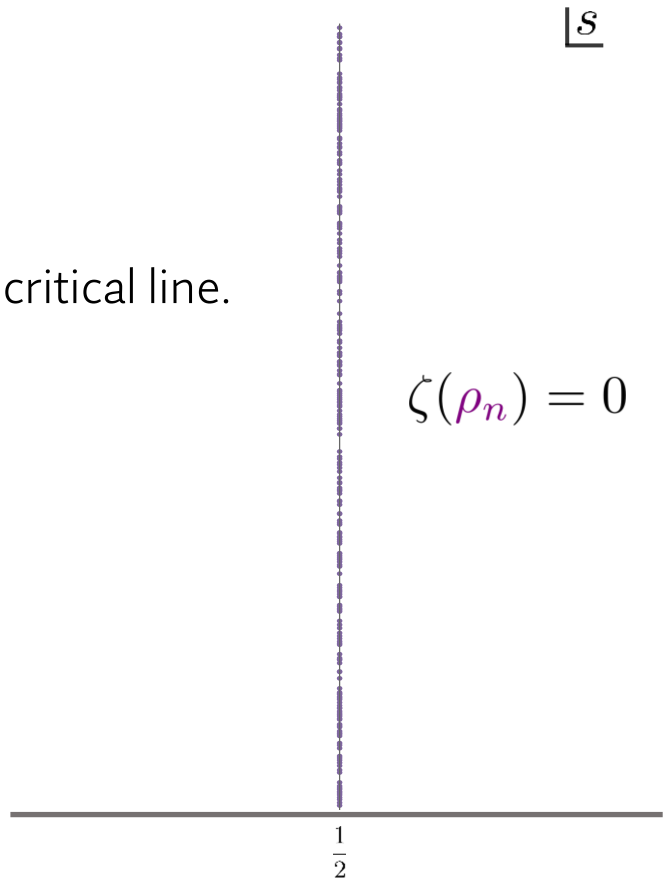
In the Riemann zeta case, one would be obviously foolish to dismiss it.



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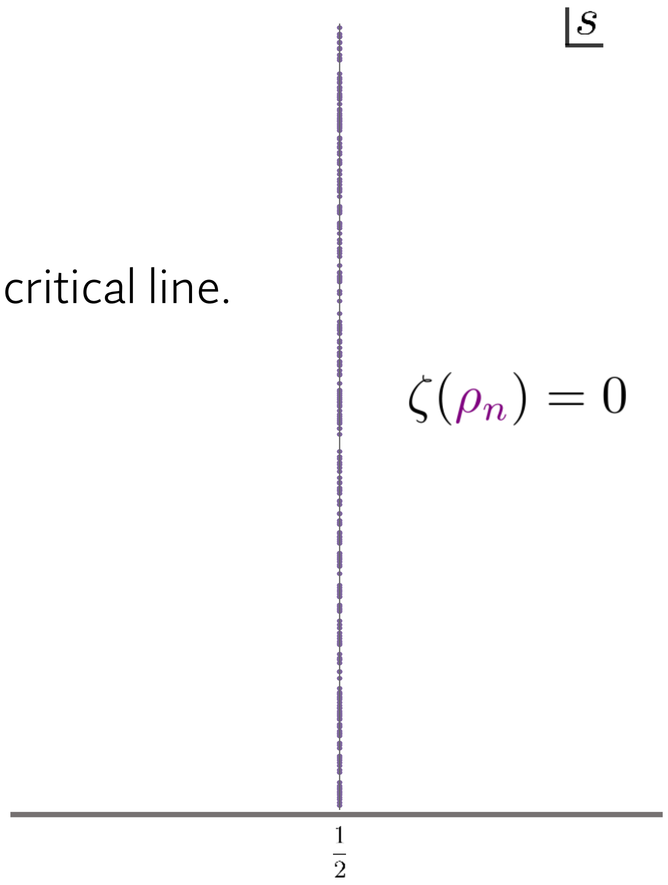

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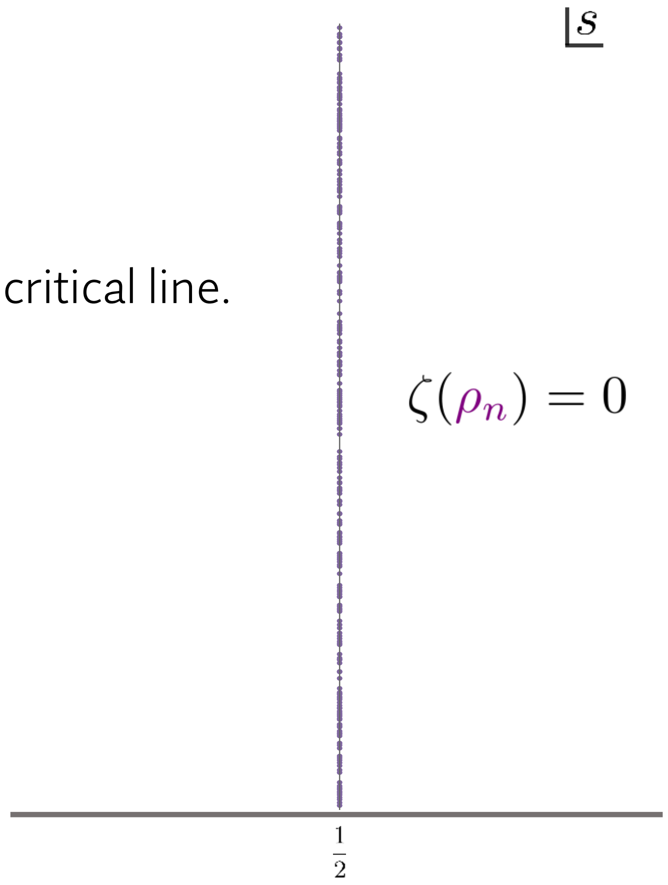
Why shouldn't the same be true of QFT?

(And would we win a Quantum Fields Medal?)



A different sort of retreat from RH involves studying statistical *correlations* among the critical zeros – an approach with remarkable results [Montgomery; Odlyzko]:

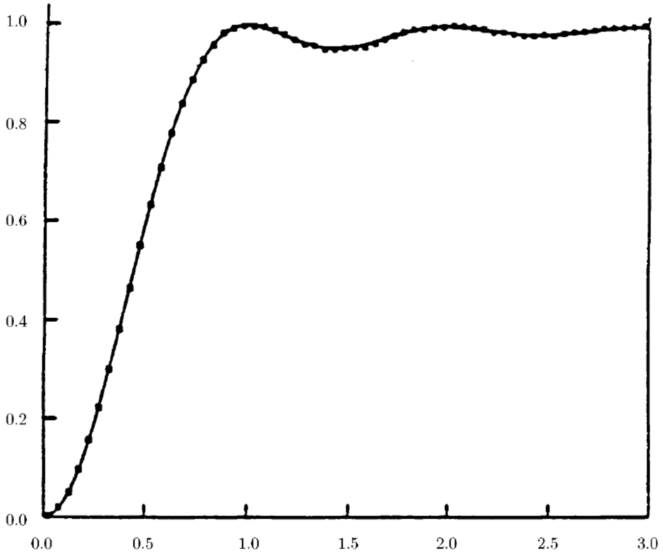
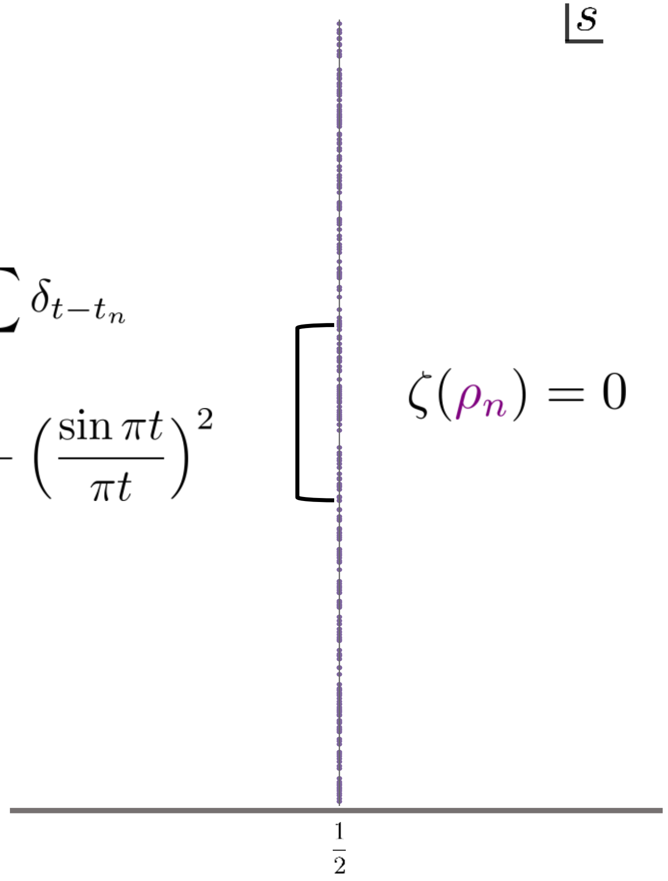


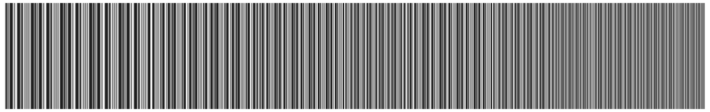
FIGURE 2. Pair correlation for zeros of zeta based on 8×10^6 zeros near the 10^{20}-th zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

$$X(t) := \sum_n \delta_{t-t_n}$$

$$\mathbb{E}[X(t)X(0)] = 1 - \left(\frac{\sin \pi t}{\pi t}\right)^2$$



The empirical adherence to random matrix theory predictions encourages an analogy between zeta zeros and quantum chaotic spectra – a refinement of Hilbert-Polya.



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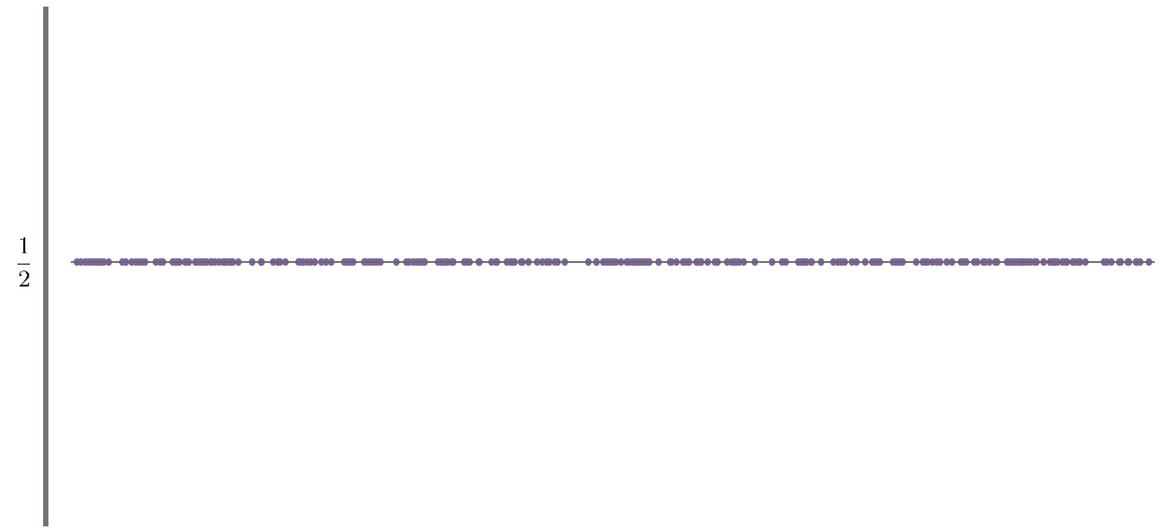

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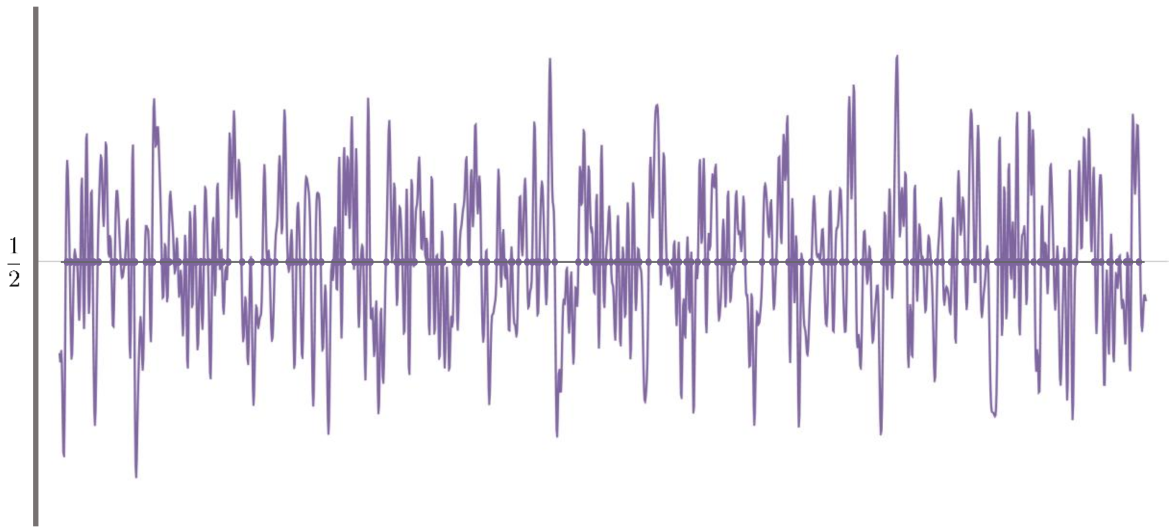

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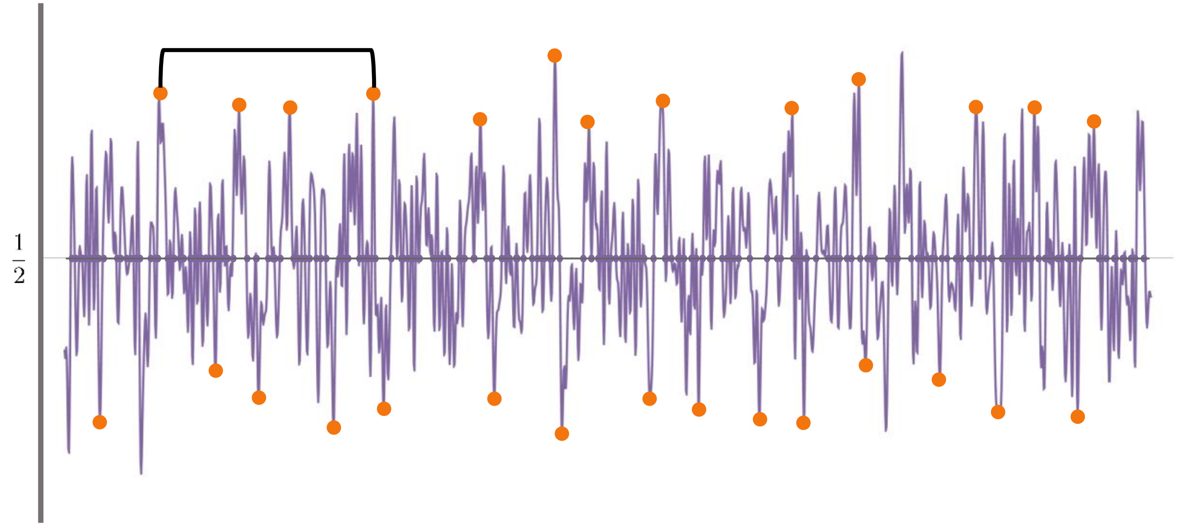

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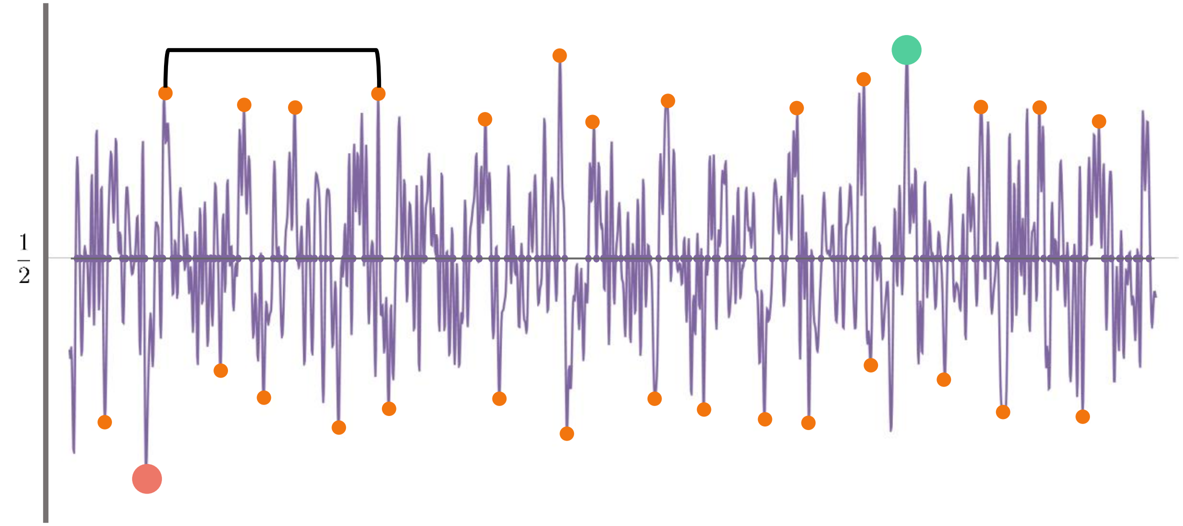

There is a very precise answer to a version of this question, high on the critical line:

$$\sup_{t\in[\tau,\tau+2\pi]} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| \approx \log\log T\,, \quad \tau \sim \mathrm{Unif}([T,2T])$$

(As we will see later, much more is known than the leading term.)

Unlike the observed random statistics of the critical zeros, this has actually been *proven*.

The previous result pertains to what are known as *extreme value statistics*.

The law for the supremum follows from the log-correlation structure of the critical Riemann zeta field.

The same property holds for large *N* random matrices. In fact, the previous result should be viewed as

$$\sup_{t\in[\tau,\tau+2\pi]} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| \approx \sqrt{2\beta}\times\frac{1}{\beta}\log N\,, \quad \tau\sim \mathrm{Unif}([T,2T])$$

with $\beta = 2$ and $N \sim \log T$

In this powerful unification of number theory and random matrix theory – part of an original conjecture by **Fyodorov, Hiary and Keating** (FHK), since largely proven – *the Riemann zeta function is not just an analogy: it is an example.*

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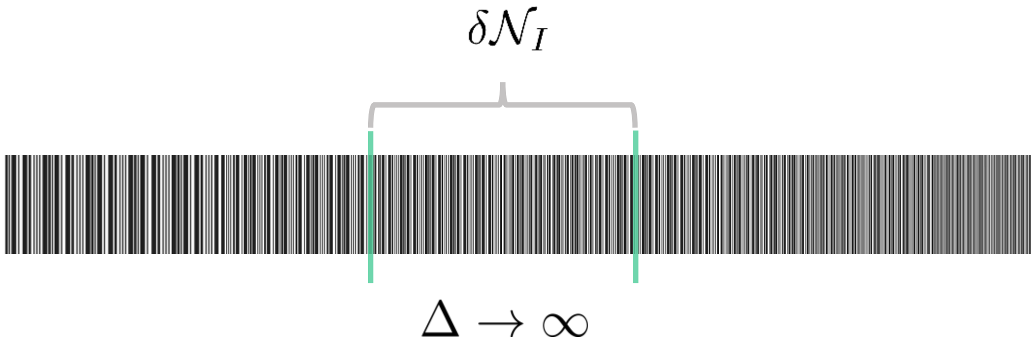
Today, we suggest a new perspective on high-energy spectra in normal CFTs with random matrix behavior.

These are the “nonperturbative CFTs” that the bootstrap is aiming for.

We propose to study extremal statistics of CFTs and AdS quantum gravity from an FHK point of view.

The Question

The quantity we will study – ***interval counts of high-dimension primary operators*** – is familiar to bootstrappers.



- <u>Bootstrap</u>: How well can asymptotic formulas count states over high-energy spectral intervals?
- <u>Quantum chaos</u>: How badly can spectral rigidity fail?
- <u>Semiclassical gravity (large *N* version)</u>: What is the maximal resolution of the gravitational path integral?

As we will review, this basic quantity is also constrained by (somewhat lesser-known) FHK extremal laws.

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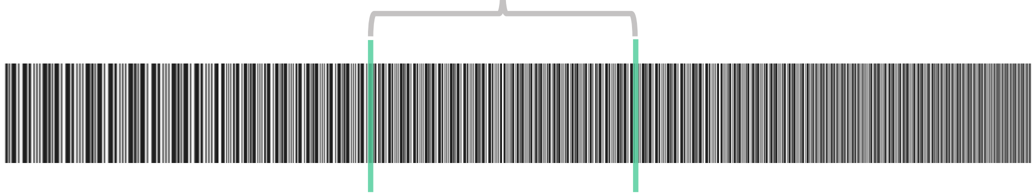
$$\sup_{I} \delta\mathcal{N}_I = ?$$



$$\Delta \to \infty$$

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The Plan

Goals:

- Introduce a vocabulary that is new to AdS/CFT/hep-th.
- Formulate an avatar of the FHK conjecture in the language of CFT and AdS quantum gravity.
- Analyze the bootstrap consequences = statistical sharp bounds.
- Make some brief remarks on consequences for semiclassical AdS gravity.

In doing so,

- Formulate new types of questions.
- Clarify the role of spectral statistics in CFT.

Part I. The FHK Conjecture

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See [Bailey, Keating] for nice review.

General shape

The FHK conjecture and variants thereof take the following form.

Consider a centered (zero-mean) spectral field *X*, on a domain $\mathcal{D}$ containing *O(N)* microscopic degrees of freedom in a statistical universality class **U**.

As $N \to \infty$,

$$\boxed{\sup_{z\in\mathcal{D}} X_{N,\mathsf{U}}(z) = C_{\mathsf{U}}\Big(\log N - \frac{3}{4}\log\log N + Y_{N,\mathsf{U}}\Big)}$$

where $Y_{N,\mathsf{U}}$ is a sequence of $O_{\mathbb{P}}(1)$ random variables, converging in law to a limiting variable $Y_{\infty,\mathsf{U}}$ with exponentially small tail probability as $y \to \infty$:

$$\mathbb{P}(Y_{\infty,\mathsf{U}} > y) \sim e^{-2y+O(\log y)}$$

The source of randomness depends on context:

a) Ensemble draw (e.g. RMT)

b) Statistical sampling of a deterministic spectrum

FHK I. Random Matrices

The first conjecture pertains to spectral determinants of $N \times N$ random matrices M_N:

$$P_N(z) := \det(z - M_N) = \prod_{j=1}^{N}(z - \lambda_j)$$

In the so-called circular unitary ensemble (CUE), eigenvalues live on the unit circle. At large N,

FHK: $$\sup_{|z|=1} \log|P_N(z)| = \log N - \frac{3}{4}\log\log N + Y_N$$

where $\{Y_N\}$ is a sequence of $O_{\mathbb{P}}(1)$ random variables with a specific limiting law at large N:

$$p(y) = 4e^{-2y}K_0(2e^{-y}) \qquad\qquad \mathbb{P}(Y_{\infty,\mathrm{U}} > y) \approx 2ye^{-2y}$$

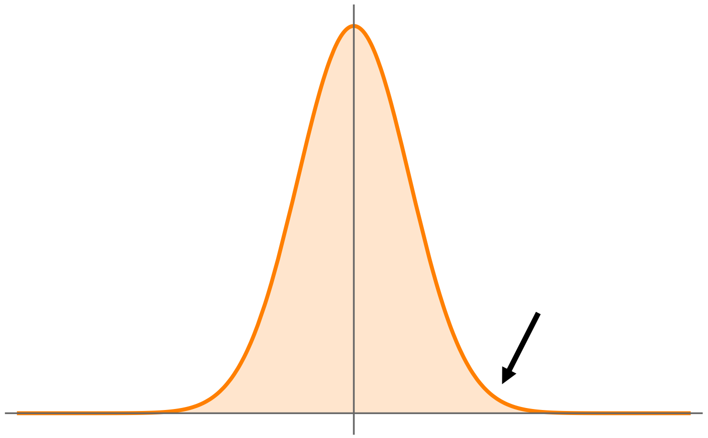
A slight generalization to other RM ensembles was made soon thereafter:

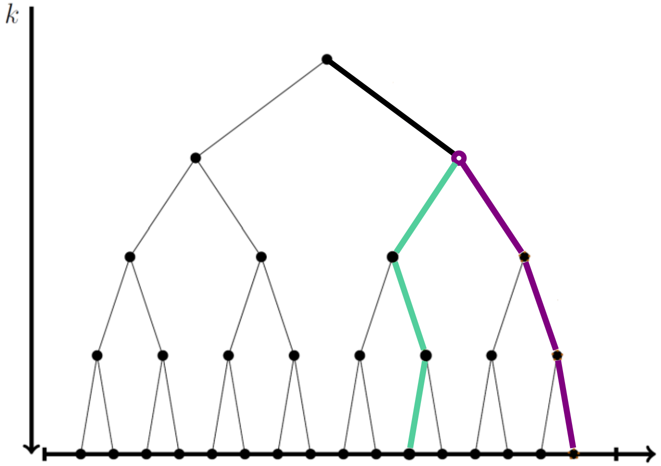
$$\boxed{\sup_{z\in\mathcal{I}}\left(\log|P_N(z)| - \mathbb{E}[\log|P_N(z)|]\right) = \sqrt{\frac{2}{\beta}}\left(\log N - \frac{3}{4}\log\log N + Y_{N,\beta}\right)}$$

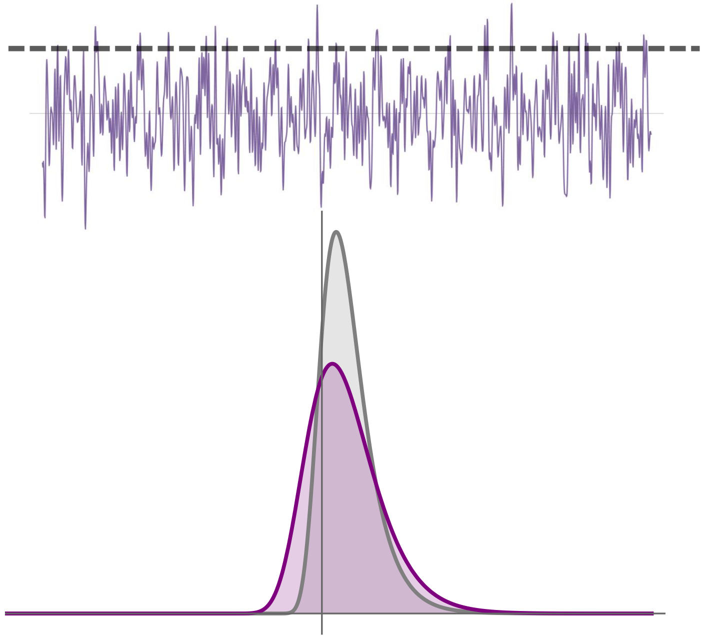
$\beta = 1 : \mathrm{GOE}$
$\beta = 2 : \mathrm{GUE}$
$\beta = 4 : \mathrm{GSE}$

where Y_∞ obeys the same ye^{-2y} tail asymptotic (depends otherwise on β, size of $\mathcal{I}$).

Each layer of the conjecture is governed by a different paradigm:







$$\log N - \frac{3}{4}\log\log N + Y_{N,\beta}$$

Large deviation

Log-correlation

Random fluctuation

Leading order: Large deviation

Consider the leading order term. (This will already be significant from the bootstrap point of view.)

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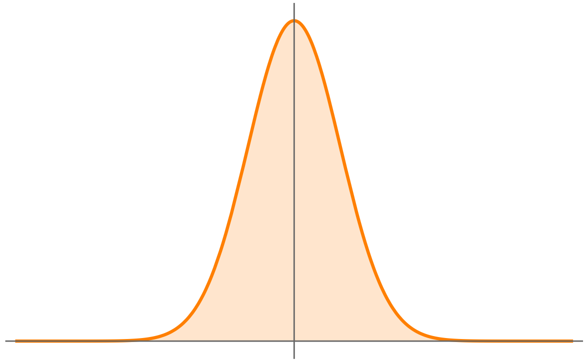
This is easy to motivate heuristically for any X_N that obeys a CLT with log variance – a proven property of the RMT log-determinant [Hughes, Keating, O'Connell; Diaconis, Evans; Soshnikov] – as follows.



$$X_N \Longrightarrow \mathcal{N}(0, \beta^{-1} \log N)$$

Treating the N levels as i.i.d., solve $$N\,\mathbb{P}(X_N(z) > y_*) = 1$$

Extrapolating the CLT out to large scales yields

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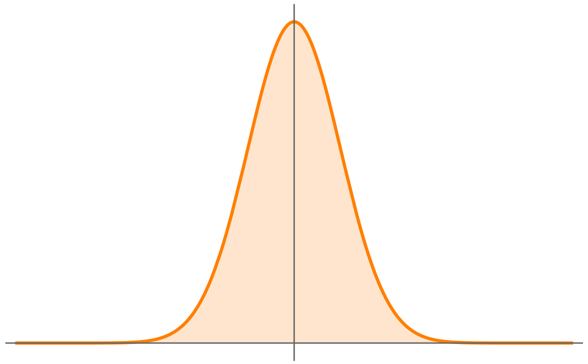

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Now, if we keep the determinant too,

$$y_* \approx \sqrt{\frac{2}{\beta}} \left(\beta V_N - \frac{1}{4} \log V_N \right)$$

Why is this wrong? The field is log-correlated!

Subleading order: Log-correlation

As we will justify momentarily, X_N is well-approximated by a Gaussian random model in frequency space.

At large N,

$$X_N(z) := \log|P_N(z)| - \mathbb{E}[\log|P_N(z)|] = \sqrt{\frac{2}{\beta}}\Re\left[\sum_{k\leq N} g_k \frac{e^{2\pi ikz}}{\sqrt{k}}\right]$$

with g_k asymptotically i.i.d. complex Gaussian. From this, one derives the large N covariance

$$\mathbb{E}[X_N(z)X_N(z')] \approx \frac{1}{\beta}\log\min\left(N, \frac{1}{|z-z'|}\right)$$

For every e-fold of separation beyond the mean level spacing $1/N$, the correlation decreases by one unit.

This hierarchical structure can be explained by splitting the frequency sum into weakly-interacting Gaussian blocks of O(1) variance.

$$X_N(z) = \sum_{r=1}^{\log N} X_r(z)$$

e-folds with $r >> -\log|z-z'|$ are effectively decorrelated.

Subleading order: Log-correlation

This hierarchical structure, similar to that of a branching random walk, suppresses large excursions.

This is intuitive: local *k*-space correlations make it “harder” to generate a large value, because the field needs to climb a landscape of clustered peaks along the way.

Tweaking our earlier heuristic computation (“barrier probability” [Bramson]) yields

$$y_* \approx \sqrt{\frac{2}{\beta}}\left(\beta V_N - \frac{3}{4}\log V_N\right)$$

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Where did the Gaussian random model come from?

Gaussian random models from CLT

Given a linear statistic obeying a CLT, the variance is determined by the autocorrelation of the test function, weighted by the spectral form factor:

$$X(u) = \int_{\mathbb{R}} dv \, f_u(v) \delta\rho(v)$$

$$\mathbb{E}[X(u)X(u')] = \int_{\mathbb{R}} dk \, m_\rho(k) \hat{f}_u(k) \overline{\hat{f}_{u'}(k)}$$

SFF (microcanonical)

Diagonalizing into Gaussian noise,

$$X(u) = \int_{\mathbb{R}} dW(k) \sqrt{m_\rho(k)} \hat{f}_u(k)$$

The $k \ll 1$ scaling of SFF, or the number variance scaling, is an order parameter for statistical universality:

$$m_\rho(k) \sim k^\alpha$$

Bulk RMT: $\alpha = 1$ (linear ramp) $\Sigma^2(\ell) \approx \frac{2}{\beta\pi^2} \log \ell$

Poisson: $\alpha = 0$

Hyperuniform: $\alpha \in (0,1)$ [e.g. Torquato review]

For log-determinants or counting statistics, $\hat{f}_k(u) \sim 1/k$. So *random matrix scaling* $\Leftrightarrow$ *log correlation*.

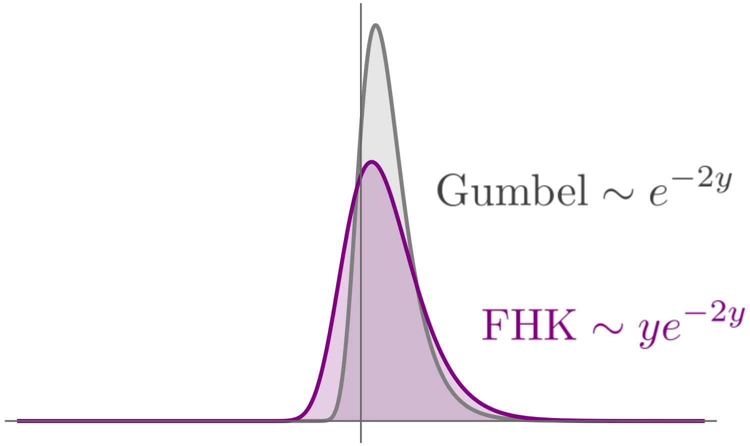
$O_{\mathbb{P}}(1)$ random fluctuation

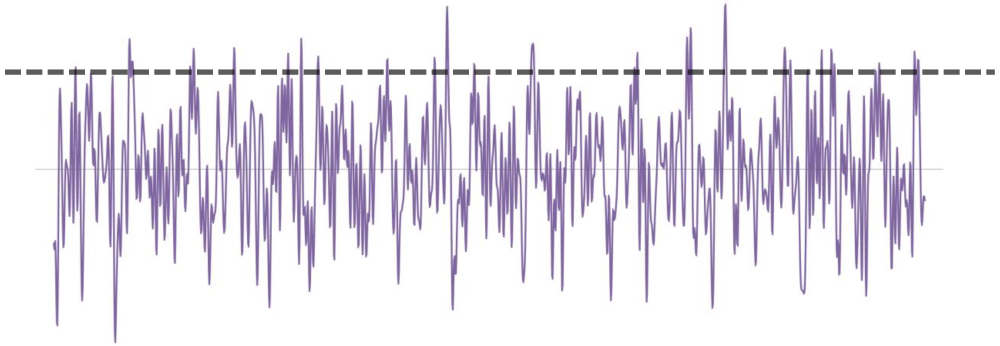
FHK is a statement of large *N* convergence in law to a random variable:

$$C_{\mathsf{U}}^{-1} \sup_{z \in \mathcal{D}} X_{N,\mathsf{U}}(z) - \left(\log N - \frac{3}{4} \log \log N \right) \Longrightarrow Y_{\infty,\mathsf{U}}$$

Y_∞ is a *randomly-shifted Gumbel*. (i.e. Y = *Gumbel* + Z where Z is itself a random variable.)

[Paquette, Zeitouni; Sabag, Zeitouni]





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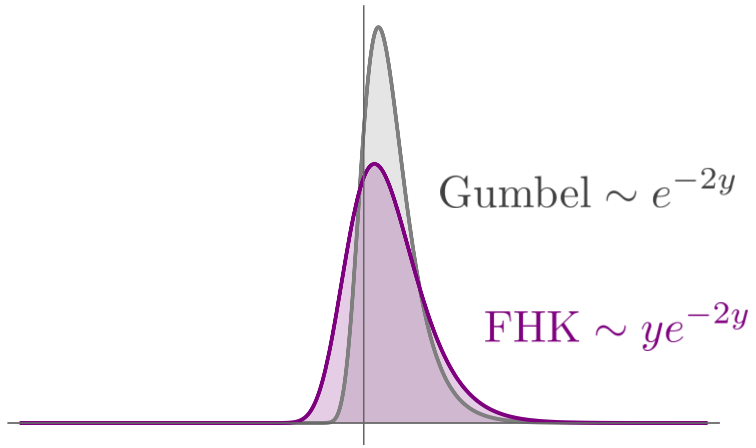

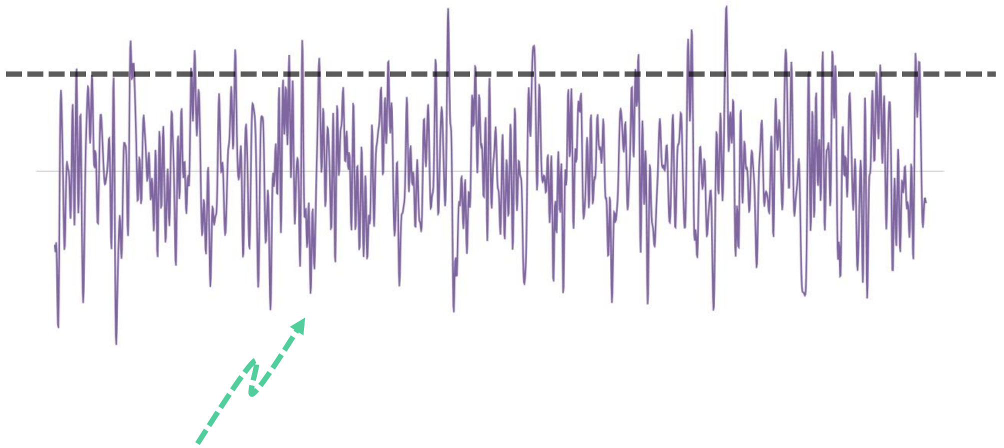

FHK is about the local extrema – a top slice of a whole "*extremal landscape*" (= marked point process charting the relative locations and local neighborhoods of these extrema).

Again, the randomness is in the draw (ensemble) or the basepoint of the sampling interval (single system).

FHK II. Riemann Zeta

The second part of the original FHK conjecture pertained to extrema of Riemann zeta on the critical line.

Over uniformly sampled "short intervals" $I := [\tau, \tau + 2\pi]\,, \quad \tau \sim \mathrm{Unif}([T, 2T])$

$$\sup_{t \in I} \log \left| \zeta\left(\frac{1}{2} + it\right) \right| = \log\log T - \frac{3}{4}\log\log\log T + Y_T$$

as $T \to \infty$, with the same log-correlated tail asymptotic for the limiting law. $\log\left|\zeta\left(\frac{1}{2} + iT\right)\right| \Longrightarrow \mathcal{N}\left(0, \frac{1}{2}\log\log T\right)$

On the one hand, various of the above ingredients had been established for Riemann zeta – e.g. CLT [Selberg], log-correlated covariance [Bourgade; Fyodorov, Keating] – which directly motivate the conjecture.

On the other, its main motivation came from years of increasingly refined observations that Riemann zeta, high on the critical line, behaves statistically like an *N x N* unitary random matrix upon identifying

$$N := \log \frac{T}{2\pi}$$

[Montgomery; Odlyzko; Keating, Berry; Keating, Snaith; Hughes, Keating, O'Connell]

FHK extends this correspondence to the extreme statistics, over short intervals.

FHK II. Riemann Zeta

So, there are two – logically distinct – ways to understand the FHK conjecture for Riemann zeta:

1. As an instantiation of the expected universality of extremal statistics of log-correlated fields.
2. Granting the FHK conjecture for RMT, as a consequence of convergence to random matrix statistics.

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Indeed, this lets us segue to an

Important point:

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RMT:

$$\sup_{z\in\mathcal{I}}\left(\log|P_N(z)| - \mathbb{E}[\log|P_N(z)|]\right) = \sqrt{\frac{2}{\beta}}\left(\log N - \frac{3}{4}\log\log N + Y_{N,\beta}\right)$$

[Holcomb, Paquette]
[Paquette, Zeitouni] x2
[Chhaibi, Maudale, Najnudel]
[Bourgade, Lopatto, Zeitouni]

Degree of proof depends on the ensemble (e.g. CUE v. GUE) and interval size (e.g. mesoscopic v. macroscopic)

Riemann zeta:

$$\sup_{t\in I}\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log\log T - \frac{3}{4}\log\log\log T + Y_T$$

[Arguin, Belius, Bourgade]
[Arguin, Belius, Bourgade, Radziwill, Soundararajan]
[Harper]
[Arguin, Bourgade, Radziwill] x2

Not yet proven:

- FHK prediction for RMT limiting law
- For RMT, ye^{-2y} tail
- For macroscopic intervals in β-ensembles, the subleading term

(See 2607.02233 for more detailed literature review.)

Part II. “Imaginary FHK” and Interval Counts

Extremal laws for interval counts

[Arguin, Belius, Bourgade;
Chhaibi, Maudale, Najnudel;
Paquette, Zeitouni;
Bourgade, Lopatto, Zeitouni]

Before passing to the AdS/CFT setting, we need a little bit more preparation.

The spectral determinant is a complex field, so one can also consider extremal laws for

$$\Phi(z) := \Im \log P_N(z) - \mathbb{E}[\Im \log P_N(z)]$$

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Simple manipulations relate this field to centered interval counts:

Coarse-grained spectral density

$$\Phi(z) - \Phi(z') = \pi \left(\underbrace{\mathcal{N}_{[z,z']} - \overline{\mathcal{N}}_{[z,z']}}_{\delta\mathcal{N}_{[z,z']}} \right) \quad \text{where} \quad \overline{\mathcal{N}}_{[z,z']} := \int_z^{z'} dx\, \rho_0(x)$$

Indeed, there are FHK laws for sup/inf Φ

So, *Range law for* Φ $\Rightarrow$ *FHK law for* $|\delta\mathcal{N}_I|$

$$\boxed{\sup_{I \subseteq I_\ell} |\delta\mathcal{N}_I| = \frac{1}{\pi}\sqrt{\frac{2}{\beta}} \left[2\left(\log \ell - \frac{3}{4} \log\log \ell \right) + \mathcal{Y}_{\ell,\beta} \right]}$$

Determined by joint law of sup/inf

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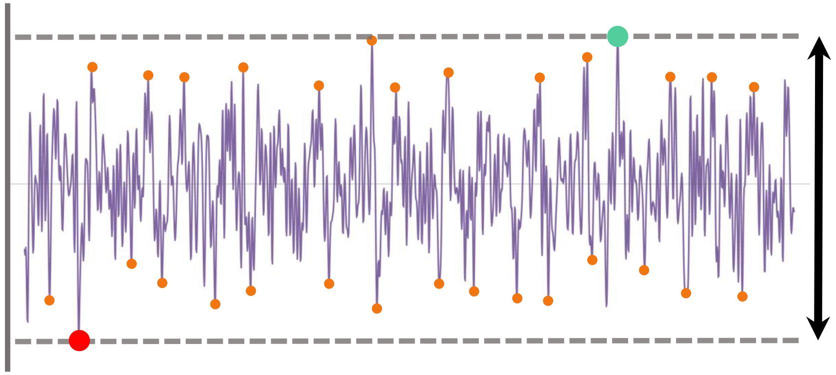

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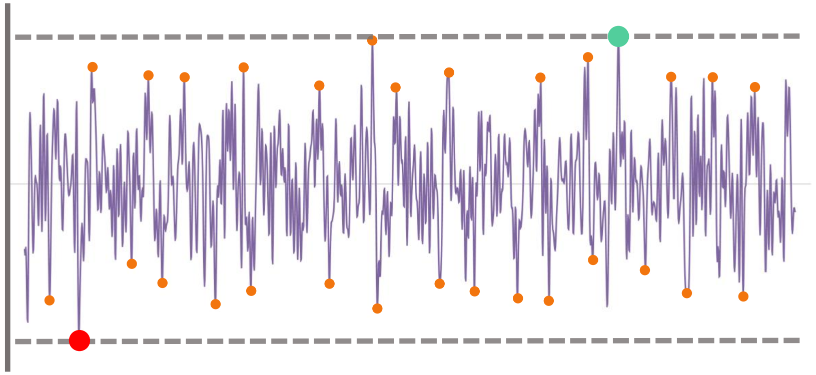

Similar laws for sup and inf independently (drawup/drawdown)

Extremal laws for interval counts

Interval counts are, of course, natural objects from the spectral/bootstrap point of view.

This is true especially when $N_{\mathrm{dof}} = \infty$ (as in QFT): whereas the relevant spectral determinants would require regularization, interval counts do not, as the ratio of determinants is finite.

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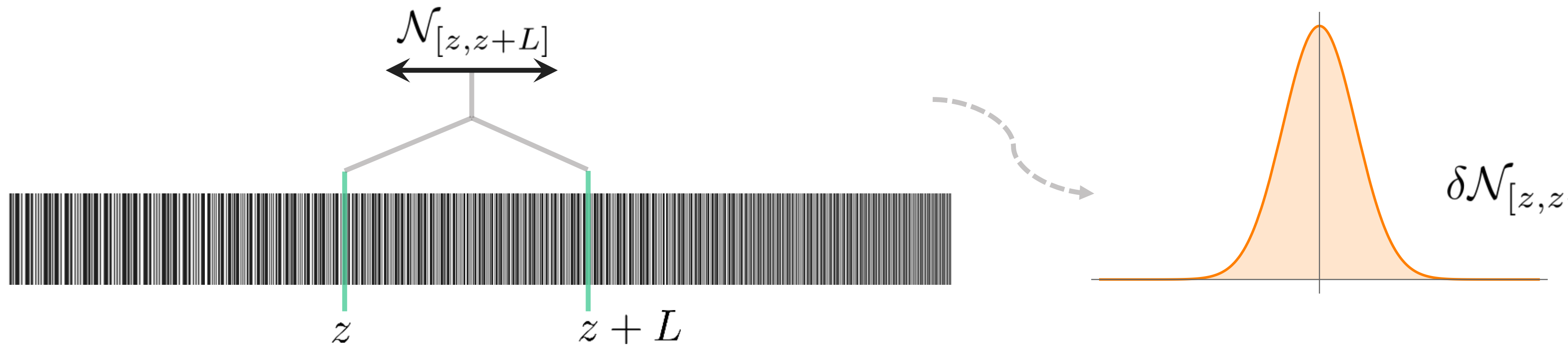
In particular, spectral sampling induces a probability distribution.



$$\delta\mathcal{N}_{[z,z+L]} \Longrightarrow \mathcal{N}(0, V_N)$$

This is what it means for, say, the Riemann zeta function, or a high-energy sector of a CFT, to obey a CLT.

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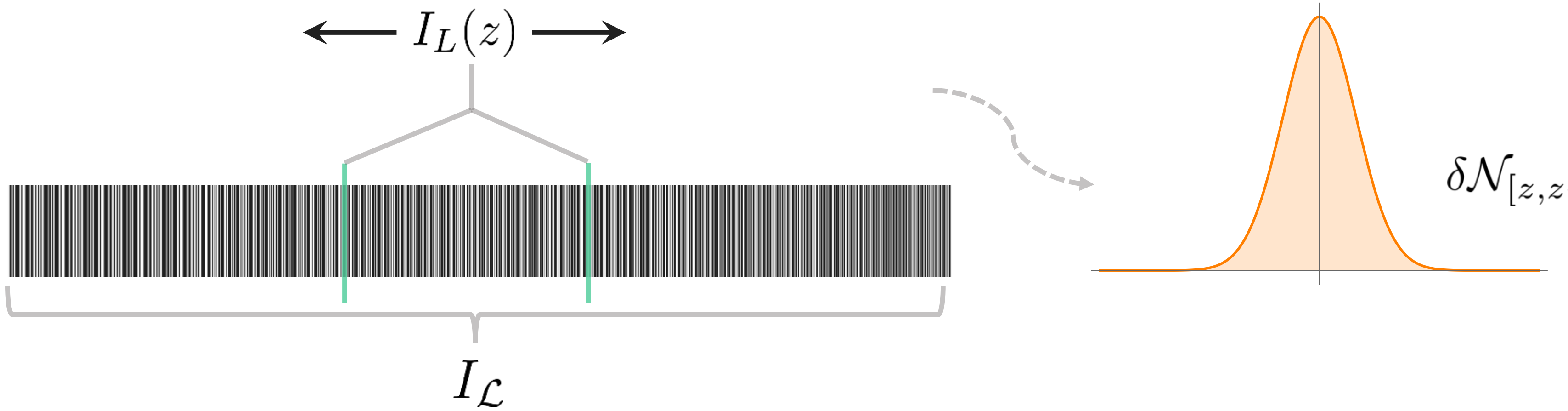

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Part III. FHK for AdS/CFT

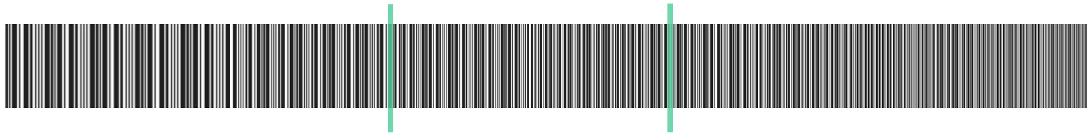
AdS/CFT avatar

We focus on interval counts (i.e. integrated density of states).

Consider a CFT in *d* spacetime dimensions.

In order to properly diagnose statistical universality in a quantum system, it is well-understood from the study of quantum chaos that one must do so within symmetry-resolved sectors.

Thus, we analyze spectra of local primary operators in fixed irreps of *SO(d)* ("fixed spin") at high dimension.



$\mathcal{N}_I^{(\boldsymbol{J})}$

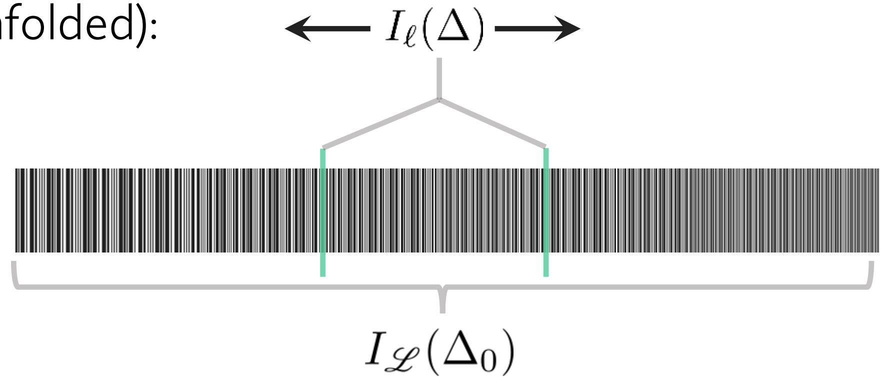
We define number fluctuations wrt to an asymptotic reference density $\rho_0^{(\boldsymbol{J})}(\Delta)$

(~ "d-dimensional Cardy formula")

[Cardy; Bhattacharya, Lahiri, Loganayagam, Minwalla; Shaghoulian; Benjamin, Lee, Ooguri, Simmons-Duffin]

FHK for AdS/CFT

Kinematics (unfolded):



$$I_\ell(\Delta) := [u, u+\ell]\,, \qquad \ell := \rho_0^{(\boldsymbol{J})}(\Delta) L$$

$$I_\ell(\Delta) \subset \mathcal{I}_{\mathscr{L}}(\Delta_0) := [u_0, u_0 + \mathscr{L}]\,, \quad \mathscr{L} := \rho_0^{(\boldsymbol{J})}(\Delta_0)\mathcal{L}$$

Scales: $\quad 1 \ll \ell \leq \mathscr{L} \ll \mathscr{L}_*(\Delta) \sim \Delta^{1/d}$

Define the spin-J counting fluctuation field on spectral intervals I:

$$\delta\mathcal{N}_I^{(\boldsymbol{J})} = \mathcal{N}_I^{(\boldsymbol{J})} - \overline{\mathcal{N}_I^{(\boldsymbol{J})}}\,, \quad \text{where} \;\; \overline{\mathcal{N}_I^{(\boldsymbol{J})}} := \int_I d\Delta\, \rho_0^{(\boldsymbol{J})}(\Delta)$$

is the unfolded length of I. Then as $\Delta_0 \to \infty$, for $u \sim \mathrm{Unif}(\mathcal{I}_{\mathscr{L}}(\Delta_0))$ and for every J,

$$\sup_{I \subseteq I_\ell(\Delta)} \left(\sigma \delta\mathcal{N}_I^{(\boldsymbol{J})}\right) = \frac{1}{\pi}\sqrt{\frac{2}{\beta}}\left[2\left(\log \ell - \frac{3}{4}\log\log \ell\right) + \mathcal{Y}_\ell^{(\sigma|\boldsymbol{J})}\right] \qquad (\sigma = \pm)$$

$O_{\mathbb{P}}(1)$ random variables, with $y \to \infty$ tail $\mathbb{P}(\mathcal{Y}_\infty^{(\sigma|\boldsymbol{J})} > y) \sim y^p e^{-2y}$

Comments

1. $\boldsymbol{J}$ fixes β. The random variable might depend on $\boldsymbol{J}$, or only on β.
2. It's an equality, not a bound! ("Optimal rigidity")
3. A milder, preliminary postulate: interval counts in normal CFT admit Gaussian random models at high energy.

$$\delta\mathcal{N}_I^{(\boldsymbol{J})} = \frac{1}{\pi\sqrt{\beta}} \sum_{1\leq k\leq \mathscr{L}} \frac{a_k^{(\boldsymbol{J})} \sin\left(\frac{2\pi k(u'+\ell_I)}{\mathscr{L}}\right) + b_k^{(\boldsymbol{J})} \cos\left(\frac{2\pi k(u'+\ell_I)}{\mathscr{L}}\right)}{\sqrt{k}} - (u' \to u' - \ell_I)$$

$$\mathbb{E}[(a_k^{(\boldsymbol{J})})^2] = \mathbb{E}[(b_k^{(\boldsymbol{J})})^2] = 1\,,\ \ \mathbb{E}[a_k^{(\boldsymbol{J})} b_k^{(\boldsymbol{J})}] = 0$$

4. *p=3* is a natural baseline expectation for the tail (ask me)

$$\sup_{I\subseteq I_\ell(\Delta)} \left(\sigma\delta\mathcal{N}_I^{(\boldsymbol{J})}\right) = \frac{1}{\pi}\sqrt{\frac{2}{\beta}}\left[2\Big(\log\ell - \frac{3}{4}\log\log\ell\Big) + \mathcal{Y}_\ell^{(\sigma|\boldsymbol{J})}\right] \qquad (\sigma = \pm)$$

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   That is, it applies when unfolded spin-J spectral statistics at high energy converge to those of Gaussian random matrix theory on mesoscopic scales.

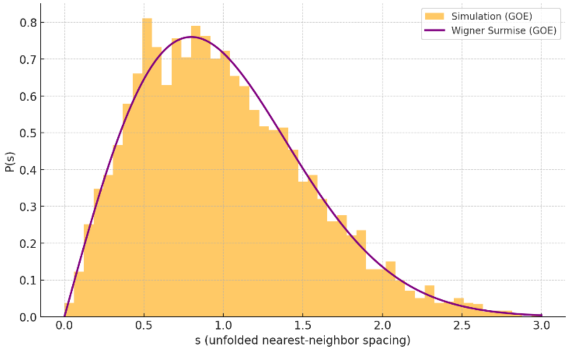
Riemann zeta is special because things can be proven – actually, certain extremal features of $\zeta(\frac{1}{2}+it)$ were proven *before* those of random matrix characteristic polynomials!

But as an instance of 2, Riemann zeta is not an analogy: it is an example.

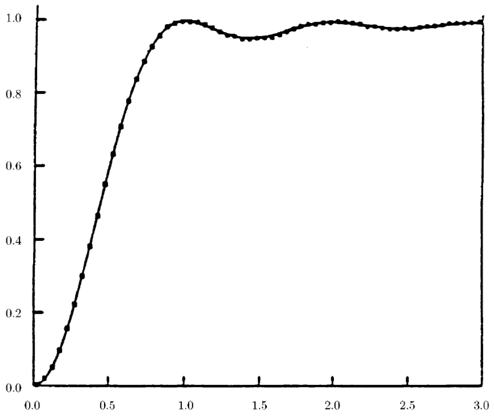
Interlude: Brief remark on spectral statistics and universality

RM statistics

- Nearest-neighbor: Wigner's surmise
  (k-point statistics in disguise)

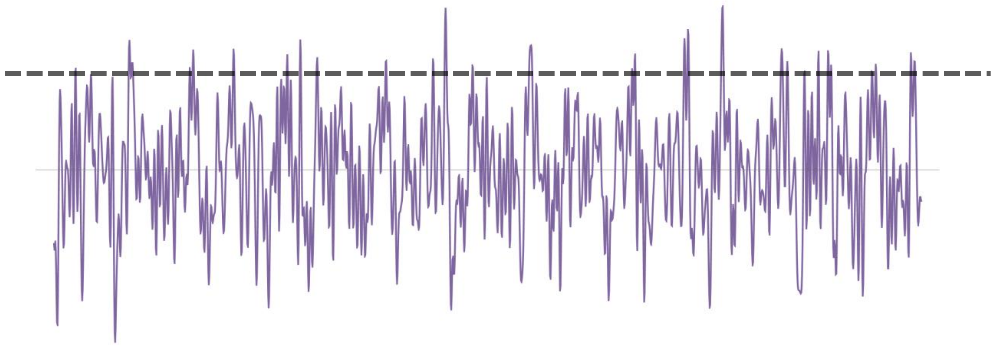


- 2-point: sine kernel



$$\Sigma^2(\ell) \sim \log \ell$$

- 1-point extremal: FHK



RM statistics

The essential idea of random matrix universality is that, *barring a structural reason not to*, any quantum system enjoys semiclassical spectral statistics of large N random matrices.

However, the *sense/degree* to which the universality holds – precisely which RMT statistics are obeyed – varies.

In this sense, the FHK conjecture for Riemann zeta and its AdS/CFT avatar above are statements of random matrix universality for 1-point extremal statistics.

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In this sense, the FHK conjecture for Riemann zeta and its AdS/CFT avatar above are statements of random matrix universality for 1-point extremal statistics.

Like other bootstrap inputs (e.g. gap assumptions), ***spectral statistics are a statement about a single CFT***.

They are not some auxiliary crutch, and there is no ensemble.

1. They allow us to select the types of CFTs we are trying to understand most: deeply ***nonperturbative CFTs***.

   (These are also the theories relevant for black hole physics via AdS/CFT.)

2. They actually help to bootstrap: one can use these criteria to learn new things – either deductively (2607.02233) or constructively (to appear).

Part IV. Bootstrap Application + Questions

Bootstrap

FHK gives a certain statistical version of a Tauberian formula.

Given a spectral statistical universality class **U**, what it means to bound a spectral field X_U in the sense of the bootstrap is ***to constrain large excursions of X_U subject to the probabilistic laws of U***.

This is exactly the domain of extreme value theory.

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This is exactly the domain of extreme value theory.

The leading-order extremal result is:

$$\boxed{\sup_{I \subseteq I_\ell(\Delta)} \left| \delta \mathcal{N}_I^{(\boldsymbol{J})} \right| \xrightarrow{\mathbb{P}} \frac{2}{\pi} \sqrt{\frac{2}{\beta}} \log \ell}$$

[Holcomb, Paquette; Bourgade, Lopatto, Zeitouni]

The extremal probabilistic excursion is determined once a spectral universality class is specified.

This is a very stringent statistical bootstrap property!

Bootstrap

What do we know about fixed-spin interval counts in unitary CFT$_d$?

[Pappadopulo, Rychkov, Espin, Rattazzi; Rychkov, Yvernay; Qiao, Rychkov; van Rees; Pal, Qiao, van Rees]

In d>2, Tauberian results for OPE densities.

In d=2, purely spectral, fixed-spin results are available:

[Mukhametzhanov, Zhiboedov; **Mukhametzhanov, Pal**]

$$\frac{1}{2} - |I|^{-1} \leq \liminf_{\Delta\to\infty} \frac{\mathcal{N}_I^{(J)}}{\ell_I^{(J)}} \leq \limsup_{\Delta\to\infty} \frac{\mathcal{N}_I^{(J)}}{\ell_I^{(J)}} \leq \frac{1}{2} + |I|^{-1}$$

$$\ell_I^{(J)} := \rho_0^{(J)}(\Delta)|I|$$

(The ½ is an artifact of including chiral, or holomorphically-factorized, CFTs – a modular obstruction.)

A folk belief is that a "chaotic CFT" should obey a bound *squeezed to Cardy* – something like

$$\liminf_{\Delta\to\infty} \frac{\mathcal{N}_I^{(J)}}{\ell_I^{(J)}} \approx \limsup_{\Delta\to\infty} \frac{\mathcal{N}_I^{(J)}}{\ell_I^{(J)}} \approx 1 \qquad \text{or} \qquad \sup_{I\subseteq I_\ell(\Delta)} \left|\delta\mathcal{N}_I^{(J)}\right| = o(\ell)$$

This is precisely the form of our previous statistical relation!

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So for Virasoro CFTs with spectral statistics converging to those of RMT in the sense articulated earlier, this result holds statistically.

$$\boxed{\sup_{I\subseteq I_\ell(\Delta)} \left|\delta\mathcal{N}_I^{(\boldsymbol{J})}\right| \xrightarrow{\mathbb{P}} \frac{2}{\pi}\sqrt{\frac{2}{\beta}}\log\ell}$$

Bootstrap

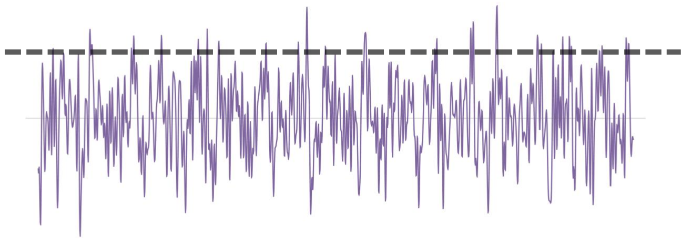
It would clearly be interesting to prove a deterministic counterpart of the above result.

And, conversely, to statisticalize Tauberian approaches.

Semiclassical gravity

(See the paper for gravity and large N stuff...)

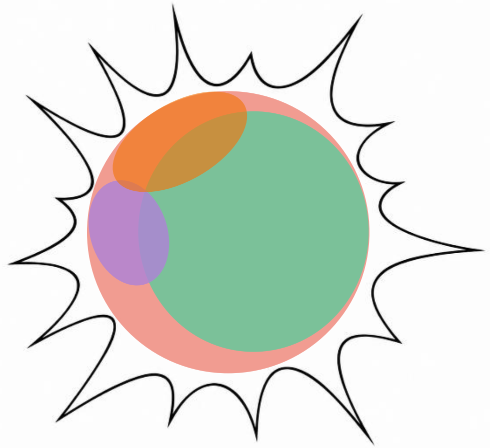
Future directions



Log-correlated phenomenology in CFT spectra.

Extremal statistics of CFT spectra.

- How do other bootstrap constraints shape the extremal landscape? e.g. modularity v. limiting laws for Y?
- Statistical Tauberian methods?



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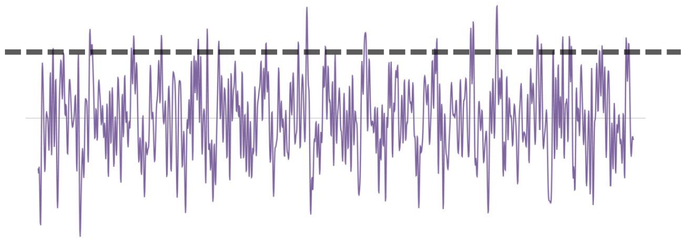

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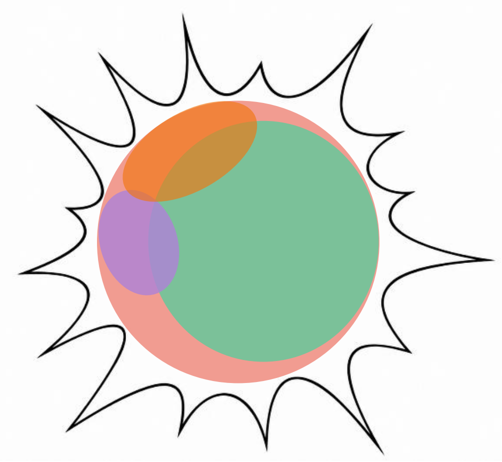
Better starting points:

- First principles search for log-correlatedness in CFT. (e.g. dense intervals cluster)
- Find log correlations numerically in the 3D Ising model.



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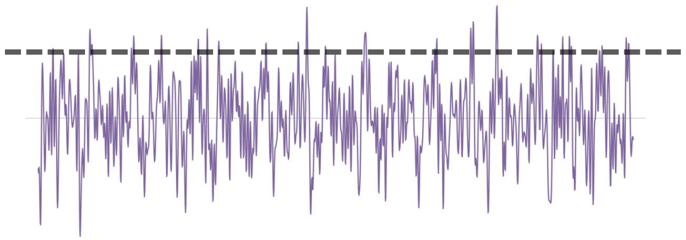

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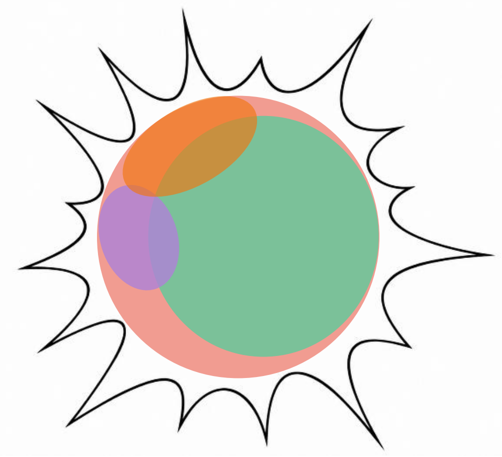
Can CFTs be hyperuniform? Not aware of previous appearance in CFT (nor number theory…).



Stratifying the space of CFTs with spectral statistics.

Bootstrapping **normal CFTs**.