Multi-particle S-matrix analyticity from Mellin amplitudes

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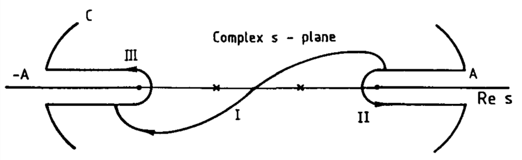
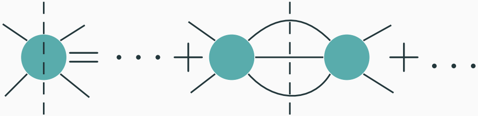
Motivation

Motivation: S-matrix bootstrap

S-**matrix bootstrap** requires analyticity



Multi-particle processes are inevitable



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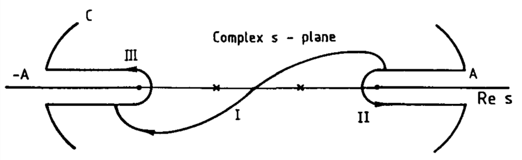

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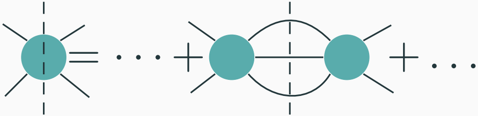

⚠ **Analyticity of multi-particle amplitudes is not well understood**

- hard to derive from basic axioms (microcausality & spectrum)
- crossing not proven [Bros'85]

Motivation: Mellin amplitudes

Mellin amplitudes are AdS analogues of S-matrix elements [Mack'09,...,FKRvR'11,...]

Simple analyticity properties: poles determined by CFT spectrum

Explicit prescription for $M \to S$ in flat space limit [PPTvRV'16,...,KPvRZ'19]

$\Longrightarrow$ What implications for the S-matrix?

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$\implies$ What implications for the S-matrix?

This talk:

- N-particle scattering of lightest scalar particle m_ϕ
- N-point function of lowest scalar ϕ in CFT$_d$ dual to QFT in AdS$_{d+1}$

Plan

1. Motivation

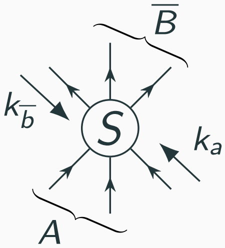
2. S-matrix analyticity

3. Mellin amplitudes

4. Flat space limit

S-matrix analyticity

2.1 N-particle kinematics

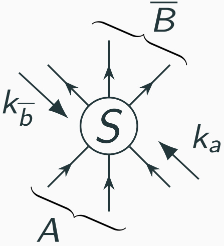


- $N = N_{in} + N_{out}$
- $k_i \in \mathbb{R}^{1,d}$, $k_i^2 = -m_\phi^2$, $\sum_i k_i = 0$
- Lorentz invariants:

$$s_I := -\left(\sum_{i \in I} k_i\right)^2, \quad I \subset \{1, \ldots, N\}$$

- $\langle out | in \rangle_{conn} = \delta(\sum_i k_i)\, \mathcal{T}(s_I)$

2.1 N-particle kinematics



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Physical region $\mathcal{M}_{phys}$

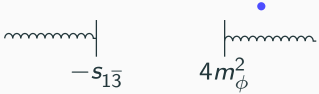
$N-3$ energies $s_A \geq |A|^2 m_\phi^2$, $s_{\overline{B}} \geq |\overline{B}|^2 m_\phi^2$

Remaining d.o.f.s = angles: $s_{A\sqcup\overline{B}} \leq \left(\sqrt{s_A} - \sqrt{s_{\overline{B}}}\right)^2$

$\dim \mathcal{M}_{phys} = \min(N(N-3)/2, Nd - \dim SO(1, d+1))$

$\boxed{s_{12},\ \text{fixed } s_{1\overline{4}}}$

Why is $\mathcal{T}(s_{12} + i\epsilon, s_{1\overline{4}} \pm i\epsilon)$ analytic?



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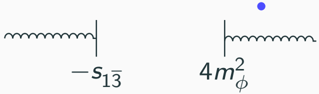
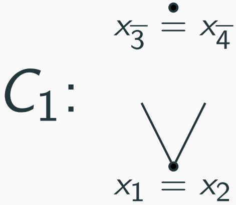


From microcausality and spectrum

- Support in x_i-space reduces to C_1
- $\mathcal{T}(s_{12}, s_{1\bar{4}})$ analytic for $(\mathrm{Im}\, k_i) \in \check{C}_1$



2.2 Analyticity near physical region: $N > 4$ [Bros et al'72,...,Iagolnitzer'94...,Caron-Huot et al'23]

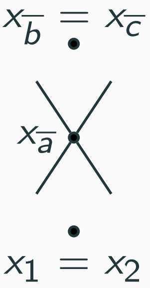
From microcausality and spectrum

- Support in x_i-space reduces to $\bigcup_{\beta=1}^{N_F} C_\beta$
- $\mathcal{T}(s_I) = \sum_{\beta=1}^{N_F} F_\beta(s_I)$
- F_β analytic near $s_I^0 \in \mathcal{M}_{phys}$ for $(\mathrm{Im}\, k_i) \in \check{C}_\beta$

Ex $N_F = 3, 9$ for $N = 5, 6$.

⚠ $\bigcap_\beta \check{C}_\beta = \{0\}$ for many $s_I^0 \to$ S-matrix $\neq$ analytic?

C_β:



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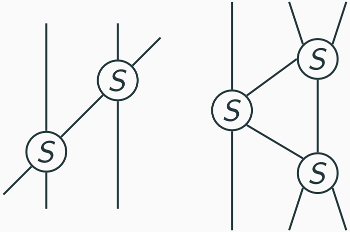


Example: factorized $3 \to 3$ **in** $d = 1$ [Iagolnitzer'78]

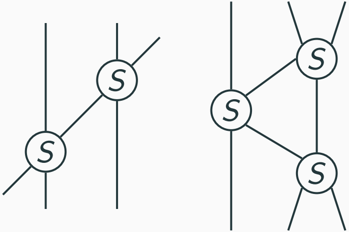
$$2\pi i \delta(z = k_1^1 + k_{\frac{1}{6}}) = \frac{1}{z + i\epsilon} - \frac{1}{z - i\epsilon}$$

2.3 Local analyticity: Landau diagrams

From perturbation theory, unitarity, macrocausality

- singularities $\stackrel{?}{=}$ classical scattering events
- codim≥ 1 surfaces $L \subset \mathcal{M}_{phys}$
- $\mathcal{T} \neq$ analytic only for $s_I^0 \in L_1 \cap L_2$ with opposite $i\epsilon$'s

$\Longrightarrow$ Avoidable for codim$(L_1 \cap L_2) \geq 2$



Improvement for $N = 5, 6$, $s_{tot} \leq (4m_\phi)^2$ from unitarity (asymptotic completeness) [Epstein-Glaser-Iagolnitzer'81]

Mellin amplitudes

3.1 The inverse Mellin transform

$$\langle \phi(x_1)\dots\phi(x_N)\rangle_{conn} = G(x_{ij}^2) = \int [d\gamma_{ij}]\, M(\gamma_{ij}) \prod_{1\le i<j\le N} \frac{\Gamma(\gamma_{ij})}{x_{ij}^{2\gamma_{ij}}}.$$

- $x_i \in$ principal Euclidean sheet
- Covariance under $x_{ij}^2 \to \lambda_i\lambda_j x_{ij}^2$
- $\gamma_{ij} \in$ complex contour
- Mellin constraints $\sum_{j\neq i}\gamma_{ij} = \Delta_\phi$

3.1 The Mellin transform

$$M(\gamma) = c_N \prod_{i<j} \int_0^\infty \frac{dx_{ij}^2}{\Gamma(\gamma_{ij})} x_{ij}^{2(\gamma_{ij}-1)} \delta^{(N)}\left(\sum_{j\neq i} \log x_{ij}^2 \right) G(x)$$

- Evaluate on $\sum_{j\neq i} \gamma_{ij} = \Delta_\phi$ surface
- For $N = 4$:

  $\gamma_{12} = \gamma_{34}$, $\gamma_{13} = \gamma_{24}$, $\gamma_{14} = \gamma_{23}$

- Gauge fix $x_{ij}^2 \to \lambda_i \lambda_j x_{ij}^2$
- For $N = 4$:

  $u = (x_{12}^2/x_{13}^2)^2$, $v = (x_{14}^2/x_{13}^2)^2$

3.2 Poles from lightcone OPE

$$M(\gamma) = \cdots \prod_{i<j\in I} \int_0^\infty dx_{ij}^2 \, x_{ij}^{2(\gamma_{ij}-1)} \ldots \langle \phi(x_1) \ldots \phi(x_N) \rangle_{conn}$$

codim-1 singularities = lightcone limits

generated by $e^{-t_E \tau}$, $\tau = \Delta - J \geq 0$

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$x_{ij}^2 \to 0$: lightcone OPE

$x_{ij}^2 \to \infty$: lightcone OPEs in overlapping channels

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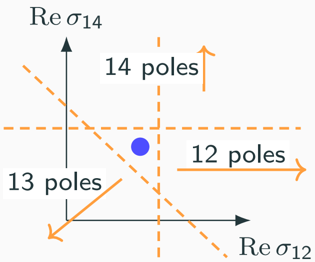
Maximal Mellin analyticity conjecture

$$\text{poles at } \sigma_I := |I|\Delta_\phi - 2 \sum_{i<j\in I} \gamma_{ij} = \tau_{\mathcal{O}}, \quad \mathcal{O} \in \phi \times \cdots \times \phi.$$

3.3 Existence and cell decomposition



- Best convergence at crossing-symmetric point
- Singularity-free if $\sigma_I^{CS} = \frac{N-|I|}{N-1}|I|\Delta_\phi \leq \tau_{\mathcal{O}_I}$

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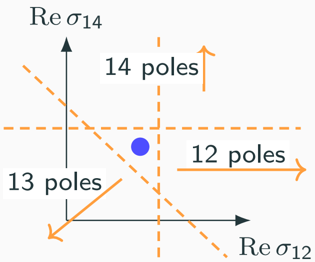

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Alternative: Decompose $M = \sum_\beta K_\beta$ **[Penedones-Silva-Zhiboedov'19]**

K_β analytic for $\mathrm{Re}\,\gamma \in \check{C}_\beta$ $\longleftrightarrow$ Restrict Supp G to $\log x_{ij}^2 \in C_\beta$

$\nexists$ 13 poles $\quad$ $x_{12}^2, x_{14}^2 \leq x_{13}^2$, $12 \cap 14$ converge

OPE convergence $\Rightarrow$ extend K_β meromorphically

3.3 Cell decomposition for general N [Behrend'13]

Goal: Find $\bigsqcup_\beta C_\beta = \mathbb{R}^{N(N-3)/2}$ that maximizes $\check{C}_\beta \sim \{\sigma < \tau_{min}\}$

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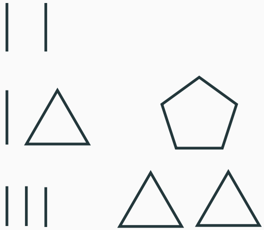
Example: $\check{C}_\beta = \{\sigma_e < \tau_{min} \,|\, e \notin E_\beta\}$,

$E_\beta =$ partition of $\{1, \ldots, N\}$ into disjoint edges and odd cycles

$N = 4$

$N = 5$

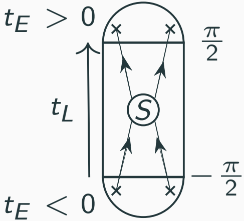
$N = 6$



- E_β-OPE limit $\notin \check{C}_\beta \iff \nexists\, E_\beta$-poles in C_β
- Extend K_β to common domain?

Flat space limit

$$\mathcal{T}(s_I) \stackrel{?}{=} \lim_{R\to\infty} \frac{\langle \phi(\tau_1, \hat{n}_1)\ldots\phi_N(\tau_N, \hat{n}_N)\rangle_{conn}}{D_{contact}(\tau_i, \hat{n}_i)}$$



- $k^0 = m_\phi \tanh\tau,\ \vec{k} \propto \hat{n}$
- $\tau = t_E + i\,\mathrm{sgn}(t_E)(\pi/2 - \epsilon)$

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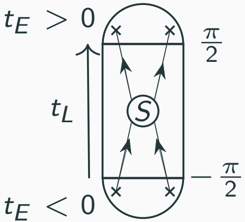


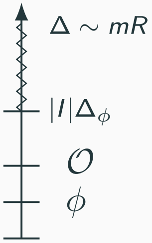
- $k^0 = m_\phi \tanh\tau,\ \vec{k} \propto \hat{n}$
- $\tau = t_E + i\,\mathrm{sgn}(t_E)(\pi/2 - \epsilon)$
- Saddle point of $\mathrm{Mellin}^{-1}[M]$ at $\gamma_{ij} \propto \Omega_{S^d}(x_i)\Omega_{S^d}(x_j)x_{ij}^2$.

  Avoid AdS Landau diagrams at small/large enough s

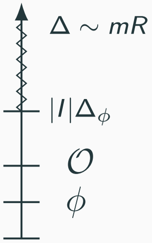
$\Longrightarrow$ Prescription for analytic $\mathcal{T}(s_I)$ near $\mathcal{M}_{phys}$

Maximal Mellin analyticity $\cap$ Flat space limit:

$$s_I^* = m_\phi(N m_{\mathcal{O}} - |I|(N - |I|)m_\phi)$$



4.2 Implications for analyticity



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Near physical region

- $s_I^* = (|I|m_\phi)^2$ at $m_\mathcal{O} = |I|m_\phi$
- only crossed channel poles $s^*_{A\sqcup\overline{B}} \leq (\sqrt{s_A} - \sqrt{s_{\overline{B}}})^2$ interfere

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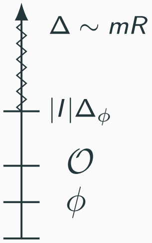

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Near crossing-symmetric point

- $s_I^{CS} = |I|(N - |I|)m_\phi^2/(N-1)$
- singularity-free for large enough $m_{\mathcal{O}}$

4.2 Implications for analyticity: physical region

- $s^\star_{a\bar{b}} = m_\phi(5m_{\mathcal{O}} - 6m_\phi)$ at $N = 5$
- Center-of-mass frame; fix energies $E_{3,4,5} = -k^0_{\bar{3},\bar{4},\bar{5}}$ and angle $\vec{k}_{\bar{3}} \cdot \vec{k}_{\bar{4}}$
- From [BEG'72]: analytic if $-\det_{\{3,4,5\}} k_i \cdot k_j = f(E_3, E_4, \vec{k}_{\bar{3}} \cdot \vec{k}_{\bar{4}}) \geq m_\phi^6$

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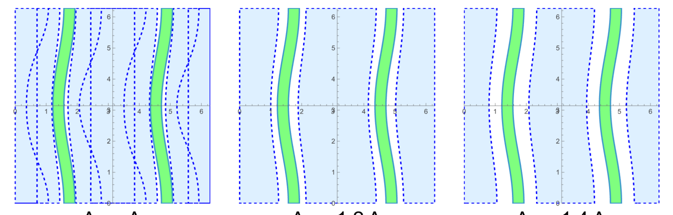
- Examples where $f < m_\phi^6$:

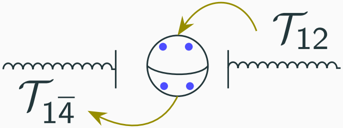


$\Delta_0 = \Delta$ $\Delta_0 = 1.2\,\Delta$ $\Delta_0 = 1.4\,\Delta$

- $s_{12} = 15m_\phi^2$, $s_{34} = 7m_\phi^2$
- unfixed angles $0 \leq \theta_1, \theta_2 \leq 2\pi$
- green = physical region
- blue = crossed channel poles

4.2 Implications for analyticity: crossing

s_{12}, fixed $s_{1\bar{4}}$



Analytic near crossing-symmetric point if

$$m_{\mathcal{O}} > \frac{N - |I|}{N - 1} |I| m_\phi$$

Defines crossing path through *Euclidean* region

$$\tau = t_E \iff i\vec{k} \in \mathbb{R}^d$$

Need "subtracted" Mellin amplitude at higher N

Summary and outlook

Complementary analyticity regions from axiomatic QFT vs. Mellin amplitudes:

local $\mathrm{Im}\, k$-cones near $s_I \in \mathcal{M}_{phys}$ vs. global OPE pole hyperplanes

Simpler structure at low energies:

Landau diagrams/anomalous thresholds vs. crossed-channel poles

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Future directions

- Establish Mellin analyticity and consistent subtractions [Penedones-Silva-Zhiboedov'19] $\rightarrow$ crossing
- *c.f.* extended axiomatic results using asymptotic completeness/unitarity [Symanzik'60,...,Epstein-Glaser-Iagolnitzer'84,...,Bros'84,...]
- (Conformal) multi-Regge theory? [White'70s,...,Stapp-White'82,...,CGSV-B'23]

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