

S from AdS

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Outline

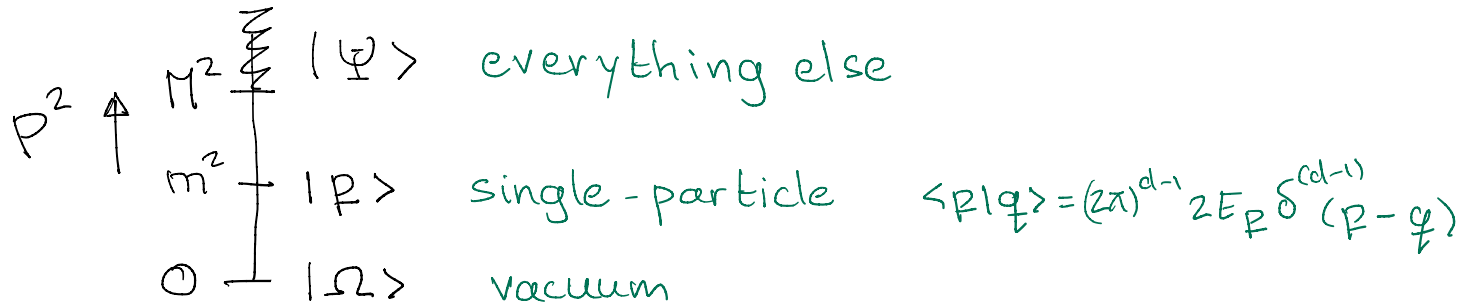
1 disconnected conjecture

2 connected conjecture

3 conformal dispersion relation

Haag-Ruelle theory

assume: Wightman axioms with mass gaps such that



and that $\langle p|\phi(x)|\Omega\rangle \neq 0$.

Theorem: $\exists |p_1, \dots, p_n\rangle_{\text{in/out}}$ with free-field inner products
 (e.g. $\langle p_1, p_2 | p_3, p_4 \rangle_{\text{in}} = \langle p_1 | p_3 \rangle \langle p_2 | p_4 \rangle + (1 \leftrightarrow 2)$)

corollary: $\exists$ S-matrix as $\text{out} \langle \dots | \dots \rangle_{\text{in}}$

(distribution, Poincaré invt., $S^\dagger S \leq 1$) equality if $H = H^{\text{out}}$
 "asymptotic completeness"

proof: reviewed in [Hebbar, Pochart, BvR (to appear)]

Haag-Ruelle theory

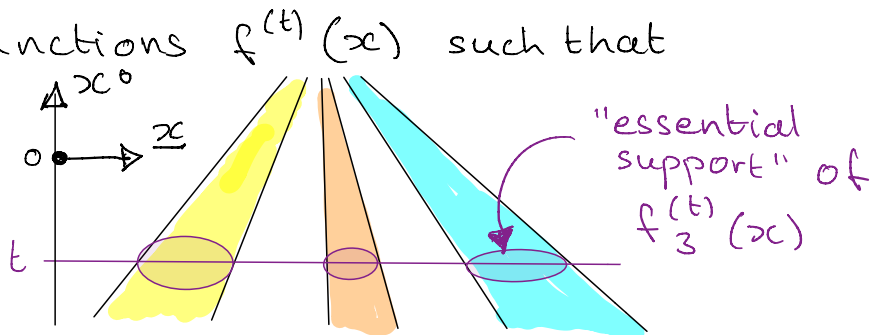
theorem: $\exists |p_1, \dots, p_n\rangle_{\text{in/out}}$ with free-field inner products found as

$$\lim_{t \rightarrow \pm\infty} \varphi(f_1^{(t)}) \dots \varphi(f_n^{(t)}) |\Omega\rangle$$

$$"\varphi(f) = \int dx \varphi(x) f(x)"$$

with t -dependent test functions $f^{(t)}(x)$ such that

- 1 "essential support" is a domain around $x^0 \sim t$ inside "velocity cone" defined by $\text{supp}(f(p))$.



$\Rightarrow$ as $t \rightarrow \pm\infty$, n -point connected correlators vanish by cluster decomposition for $n > 2$.

- 2 $\varphi(f^{(t)}) |\Omega\rangle$ lies entirely in one-particle subspace
 $\Rightarrow$ 2-point (connected) correlators become those of the free field.

proof: see for example [Hebbar, Pochart, BvR (to appear)]

QFT in AdS

Consider a gapped QFT in AdS

$$ds^2 = dr^2 + \frac{R^2 \sinh^2(r/R)}{\cosh^2(z)} (d\tau^2 + d\Omega_{d-1}^2)$$

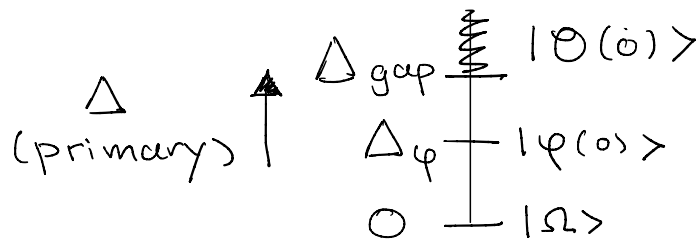
$S^{d-1} : \underline{n} \text{ s.t. } \|\underline{n}\|=1$
 then
 $n^\mu = \left(\tanh(\tau), \frac{\underline{n}}{\cosh(\tau)} \right)$
 has $n^\mu n_\mu = 1$.

We will suppose that the boundary correlators

$$\langle \mathcal{O}(\tau_1, \underline{n}_1) \dots \mathcal{O}(\tau_k, \underline{n}_k) \rangle := \lim_{r \rightarrow \infty} e^{\Delta r} \langle \mathcal{O}(\tau_1, \underline{n}_1, r) \dots \mathcal{O}(\tau_k, \underline{n}_k, r) \rangle$$

obey the CFT axioms (covariance, convergent OPE).

In the flat-space limit $R \rightarrow \infty$ we suppose a spectrum



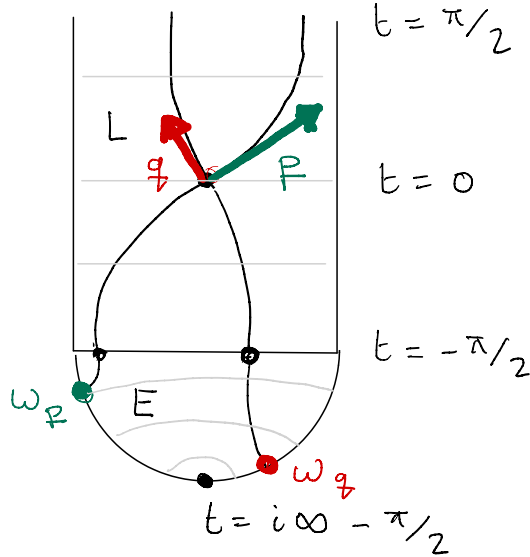
with $\Delta_\psi (\Delta_\psi - d) = m^2 R^2$
 and $\Delta_{\text{gap}} / \Delta_\psi > 1$ fixed.

Q: Can we find the Haag-Ruelle states?

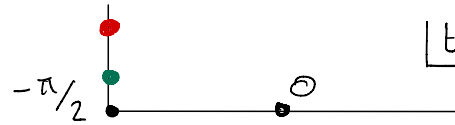
'in' states from AdS

$$t = -iz$$

massive particles move along geodesics:



- 1 interaction per half period $0 \leq t < \pi$
- to reach boundary, add Euclidean AdS



↙ boundary operator

$$\Rightarrow \phi_{in}^>(p) := \phi(t = -\frac{\pi}{2} + i\omega_p, \underline{n} = -\underline{p}/\|\underline{p}\|)$$

$$\omega_p := \operatorname{arctanh}(m/E_p) \in (0, \infty]$$

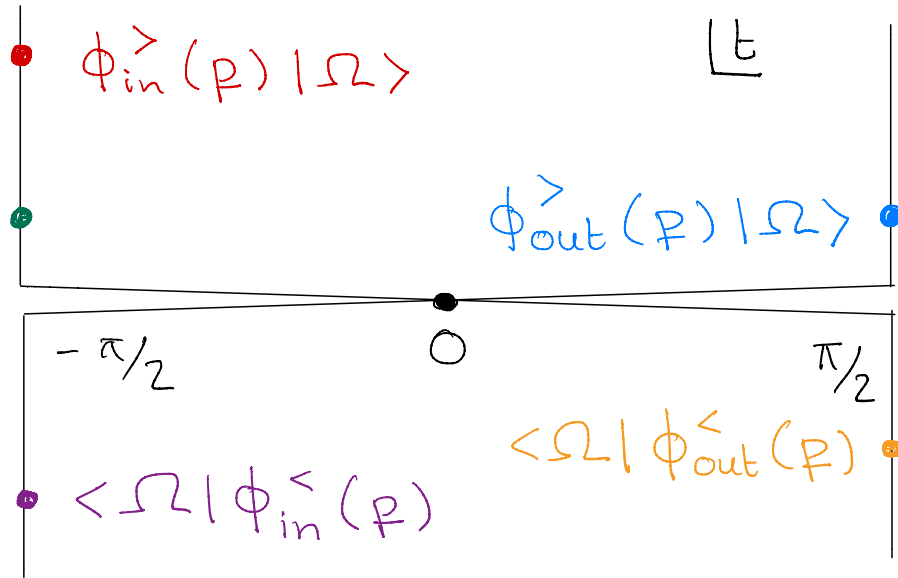
and we can speculate that:

$$|p_1, \dots, p_k\rangle_{in} \stackrel{?}{=} \lim_{R \rightarrow \infty} \phi_{in}^>(p_1) \dots \phi_{in}^>(p_k) |\Omega\rangle$$

[PPT, RV (2017); Dubovsky, Gorbenko, Mirbabayi (2016); Hijano (2019); ...]

'in' and 'out', bras and kets

repeating this analysis gives the four variations:



* by Euclidean T-ordering
we can only write:

$$\langle \Omega | \phi_{\bullet}^< \dots \phi_{\bullet}^< \phi_{\bullet}^> \dots \phi_{\bullet}^> | \Omega \rangle$$

* $\eta = -p/\|p\|$ for 'in',

$\eta = +p/\|p\|$ for 'out'

* $\omega_p = \pm \operatorname{arctanh}(m/E_p)$

Q: what do the overlaps become as $R \rightarrow \infty$?

single-particle states

Weyl factor

$$\langle \Omega | \phi_{in}^<(p) \phi_{in}^>(q) | \Omega \rangle = Z \left(\frac{\cosh(\tau_1) \cosh(\tau_2) / 2}{\cosh(\tau_1 - \tau_2) - \Omega_1 \cdot \Omega_2} \right)^{\Delta_\phi}$$

almost Eucl. correlator!

$$\begin{aligned}\tau_1 &= -\pi/2 + i\omega_p \\ \tau_2 &= -\pi/2 + i\omega_q \\ \Omega_1 &= -p / \|p\| \\ \Omega_2 &= -q / \|q\|\end{aligned}$$

$$= Z \left(\frac{m^2}{(p+q)^2} \right)^{\Delta_\phi} \xrightarrow{\Delta_\phi \rightarrow \infty} (2\pi)^d 2E_p \delta^{(d)}(p-q)$$

using
 $p^2 = q^2 = m^2$

Full derivation (incl. Z)

in [Komatsu Paulos BvR Zhao '20]

$\leadsto$ confirms that $\lim_{R \rightarrow \infty} \phi_{in}^>(p) | \Omega \rangle = | p \rangle_{in}$

same result for $\langle \phi_{out}^< \phi_{out}^> \rangle$, $\langle \phi_{in}^< \phi_{out}^> \rangle$, $\langle \phi_{out}^< \phi_{in}^> \rangle$

$\leadsto$ single-particle states are stable, $| p \rangle_{in} = | p \rangle_{out}$.

two-particle states

let's try: $\langle \Omega | \phi_{in}^>(p) \phi_{in}^>(q) | \Omega \rangle = \dots = Z \left(\frac{m^2}{(p-q)^2 - i\epsilon} \right)^{\Delta\phi}$

$$= \begin{cases} 0 & |(p+q)^2 - 4m^2| > 4m^2 \\ \infty & \text{if } < \end{cases}$$

includes low energies, e.g. $p=q=(m, 0)$

→ we do a Gram-Schmidt step and improve the speculation:

$$|p_1, p_2\rangle \stackrel{?}{=} \lim_{R \rightarrow \infty} \phi_{in}^>(p) \phi_{in}^>(q) | \Omega \rangle - (\langle \Omega | \phi^> \phi^> | \Omega \rangle) | \Omega \rangle$$

this suffices for GFF:

↑ sometimes necessary
sometimes harmless

$$\langle \Omega | \phi_1^< \phi_2^< \phi_3^> \phi_4^> | \Omega \rangle = \langle \Omega | \phi_1^< \phi_2^< | \Omega \rangle \langle \Omega | \phi_3^> \phi_4^> | \Omega \rangle \quad \text{✗}$$

$$\langle \Omega | \phi_1^< \phi_3^> | \Omega \rangle \langle \Omega | \phi_2^< \phi_4^> | \Omega \rangle + \langle \Omega | \phi_1^< \phi_4^> | \Omega \rangle \langle \Omega | \phi_2^< \phi_3^> | \Omega \rangle \quad \checkmark \quad \checkmark$$

Two-particle states, cont'd

Investigations of three-point functions yields (or will yield):

$$\langle \Omega | \Theta(\infty) \phi_{in}^>(p) \phi_{in}^>(q) | \Omega \rangle \quad \text{defines scaling of } \lambda_{\phi\phi\Theta}!$$

$$\xrightarrow{R \rightarrow \infty} \begin{cases} \text{finite} & | (p+q)^2 - 4m^2 | \text{ large} \\ \infty & \text{otherwise} \end{cases} \quad \text{or} \quad \Delta_{\Theta} > 2\Delta_{\phi}$$

so we subtract these as well (when necessary).

We conjecture that

$$\begin{aligned} |p, q\rangle_{in} = \lim_{R \rightarrow \infty} \Big\{ & \phi_{in}^>(p) \phi_{in}^>(q) | \Omega \rangle - \langle \Omega | \phi^> \phi^> | \Omega \rangle \\ & - \sum_{\Delta < 2\Delta_{\phi}} \int dx \langle \tilde{\mathcal{O}}(x) | \phi^> \phi^> | \Theta(x) \rangle \Theta[(p-q)^2 \text{ small}] \Big\} \end{aligned}$$

is the AdS equivalent of Haag-Ruelle.

two-particle states from AClS

If

$$|p, q\rangle_{in} = \lim_{R \rightarrow \infty} \phi_{in}^>(p) \phi_{in}^>(q) |\Omega\rangle - (\text{subtractions})$$

and we want that

$${}_{in} \langle p_1, p_2 | p_3, p_4 \rangle_{in} = \langle p_1 | p_3 \rangle \langle p_2 | p_4 \rangle + \langle p_1 | p_4 \rangle \langle p_2 | p_3 \rangle$$

then we are led to the conjecture:

$$\langle \phi(x_1) \dots \phi(x_4) \rangle = \sum_k \lambda_{\phi\phi k}^2 g_{\Delta_k}^{(l_u)}(x_1 \dots x_4) \Theta(x_1 \dots x_4 \dots)$$
$$0 \leq \tau_k < 2\Delta_\phi$$

$$\xrightarrow{R \rightarrow \infty} \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle.$$

for all Euclidean $(x_1, \dots, x_4)$. Is this always true?
(with our spectrum assumptions: $\Delta_\phi \rightarrow \infty$, τ_{gap}/Δ_ϕ fixed)

Disconnected conjecture

$$\langle 1234 \rangle - (\text{subtractions}) \xrightarrow[\Delta\phi \rightarrow \infty]{?} \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle$$

$\Delta_{\text{gap}} / \Delta\phi$ fixed

- [PPTvRV '16] "spectral density" $\leadsto$ GFF for $\Delta > 2\Delta_\phi$
- [Cordova, He, Paulos '20] 2d analysis (not always true)
- [Paulos, BvR, Zhao '20] subtractions $\leftrightarrow$ AdS Landau diagrams

-
- numerics?
 - derivation? (cf. [Rogelberg, Poland, .. '26])
 - massless cases? (cf. [Buchholz 1971])
 - exact statement for higher-point correlators?

Outline

1 disconnected conjecture

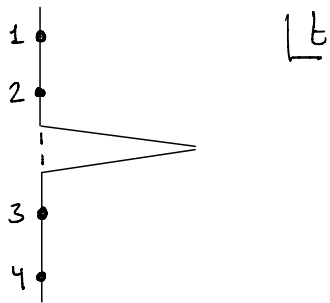
2 connected conjecture

3 conformal dispersion relation

Scattering amplitudes

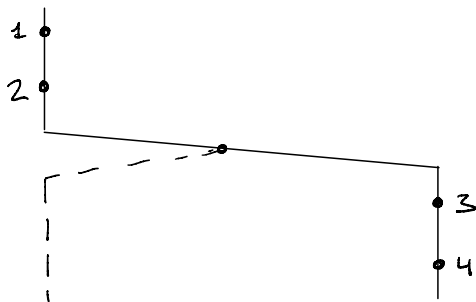
so far we considered

$$\langle \phi_{in}^< \phi_{in}^< \phi_{in}^> \phi_{in}^> \rangle$$



now let us consider

$$\langle \phi_{out}^< \phi_{out}^< \phi_{in}^> \phi_{in}^> \rangle$$



claim: if disconnected conjecture holds then

$$\langle P_4 P_3 | P_2 P_1 \rangle_{in} =$$

$$\lim_{R \rightarrow \infty} \left\{ \langle \phi_{out}^< \phi_{out}^< \phi_{in}^> \phi_{in}^> \rangle - (\text{subtractions}) \right\}$$

obeys $2 \rightarrow 2$ unitarity inequalities.

$$\text{proof: } |\langle \phi_{out}^< \phi_{out}^< \phi_{in}^> \phi_{in}^> \rangle| =$$

$$|\langle \phi_{in}^< \phi_{in}^< e^{iH\pi} \phi_{in}^> \phi_{in}^> \rangle| \leq$$

$$\langle \phi_{in}^< \phi_{in}^< \mathbb{1} \phi_{in}^> \phi_{in}^> \rangle$$

see [BrR Zhao 2023] for details.

scattering amplitudes

$$\langle p_4, p_3 | p_1, p_2 \rangle_{in} = (\text{disconnected}) + \delta^{(d)}(p_1 + p_2 - p_3 - p_4) T(s, t)$$

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2 = 4m^2 - s - t$$

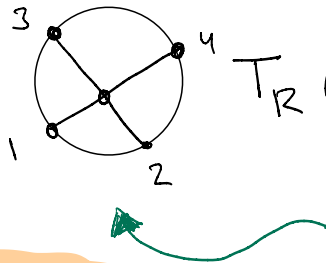
So we write

$$\langle \phi_{out}^<(p_4) \phi_{out}^<(p_3) \phi_{in}^>(p_2) \phi_{in}^>(p_1) \rangle - (\text{subtractions}) =$$

conformal blocks
 $\checkmark$ with $z \leq 2\Delta\phi$.

$$(\text{disconnected}) + \text{diagram} \cdot T_R(u, v)$$

and claim that



$$\lim_{R \rightarrow \infty} T_R(u, v) \Big|_{\text{s-matrix}} \stackrel{?}{=} T(s, t)$$

todo: map from (s, t) to (u, v) .

We know that:

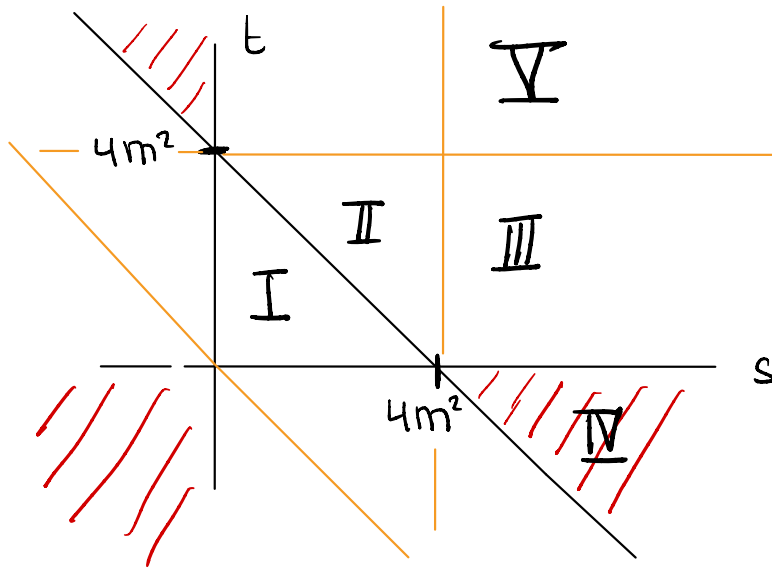
$$\lim_{R \rightarrow \infty} \text{diagram} \Big|_{\text{s-matrix}} = \delta^{(d)}(p_1 + p_2 - p_3 - p_4)$$

Conformal Mandelstam invariants

$$u = \left(\frac{4-s}{s+t} \right)^2$$

$$v = \left(\frac{4-t}{s+t} \right)^2$$

← works also at finite $\Delta\phi$!



I: Euclidean region
 $z \in \mathbb{C}, \bar{z} = z^*$

II: Lorentzian spacelike
 $z, \bar{z} \in (0, 1)^2$

III: $z \odot \circlearrowright, \bar{z} \in (0, 1)^2$

IV: Euclidean second sheet
 $z \odot \circlearrowright \in \mathbb{C}, \bar{z} = (z \odot \circlearrowright)^*$

V: hard: $z \odot \circlearrowright (\pm)$

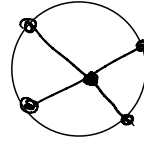
(note: $\theta_s = \theta_{\text{cFT}}$!)

[Komatsu Paulos BvR Zhao '20]

connected conjecture

We are led to conjecture that

$$T_R(u,v) := \frac{\langle \phi_{\text{out}}^< \phi_{\text{out}}^< \phi_{\text{in}}^> \phi_{\text{in}}^>^{\text{conn}} \rangle - (\text{subtr})^{\text{conn}}}{\quad}$$



is (locally) bounded in a big domain, i.e.

$$|T_R(u,v)| \leq f(u,v)$$

which means that, if

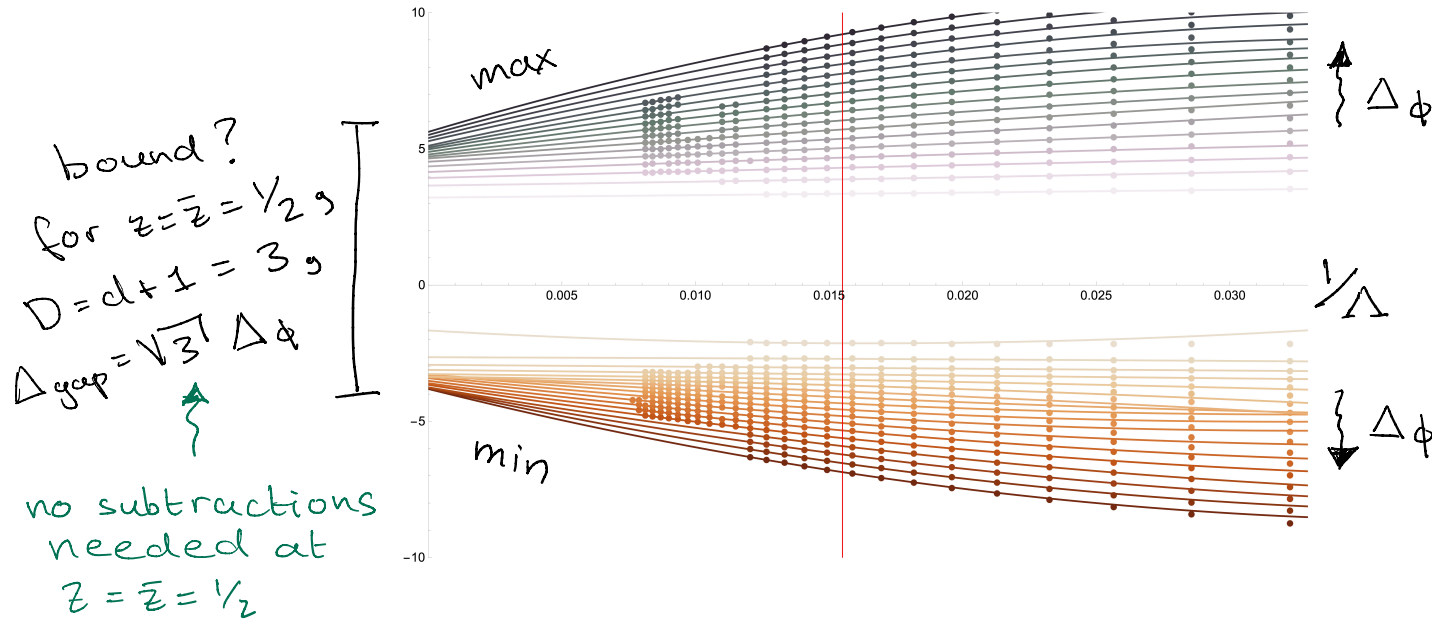
$$\lim_{R \rightarrow \infty} T_R(u,v) \Big|_{\text{s-matrix}} = T(s,t)$$

exists, then $T(s,t)$ is finite and analytic.

connected conjecture

Q: can we find a functional bounding $T_R(u, v)$?

Numerically [Hebbar, Pochart, BvR (wip)] following [Paulos Zheng '21]
we have some indication:



Analytic progress would be very interesting!

Outline

1 disconnected conjecture

2 connected conjecture

3 conformal dispersion relation

Connected correlators

claim: for $\langle \phi \phi \phi \phi \rangle^{\text{conn}}$ must use conformal dispersion relation

$$g^{(\text{conn})}(u, v) = \int_{\mathcal{R}} du' dv' K(u, v; u', v') d\text{Disc}_u [g^{(\text{conn})}](u', v') + (u \leftrightarrow v)$$

$$d\text{Disc}_u [g](u, v) := g(u, v) - \frac{1}{2} g(u \odot, 1) - \frac{1}{2} g(u \uparrow \odot, 1)$$

[Carmi, Caron-Huot ('19); Caron-Huot ('17)]

(reason: positive "Polyakov-Regge" block decomposition, dispersive functionals, ...)

[Gopakumar, Sinha, Kaviraj, Sen, Den ('16-'17); Mazac, Paulos ('19); Caron-Huot, Mazac, Rastelli, Simmons-Duffin ('20)]

also: becomes single-var DR in FSL [BrR, Zhao ('21)]

CDR

$$g(u, v) = \int du' dv' K(u, v; u', v') d\text{Disc}[g](u', v') + (u \leftrightarrow v)$$

with

$$K(u, v; u', v') = K_B(u, v; u', v') \theta(\sqrt{v'} > \sqrt{u'} + \sqrt{u} + \sqrt{v}) + K_C(u, v; u') \delta(\sqrt{v'} - \sqrt{u'} - \sqrt{u} - \sqrt{v}). \quad (2.11)$$

The contact term is localized at the boundary of the bulk region. The kernels are respectively

$$K_B(u, v; u', v') = \frac{u - v + u' - v'}{64\pi(uvu'v')^{\frac{3}{4}}} x^{\frac{3}{2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; 1 - x\right), \quad (2.12)$$

$$K_C(u, v; u') = \frac{1}{4\pi(uvu'v')^{\frac{1}{4}}(\sqrt{u} + \sqrt{u'})},$$

where x is the following combination of the four cross-ratios

$$x = \frac{16\sqrt{uvu'v'}}{[(\sqrt{u} + \sqrt{v})^2 - (\sqrt{u'} + \sqrt{v'})^2][(\sqrt{u} - \sqrt{v})^2 - (\sqrt{u'} - \sqrt{v'})^2]}. \quad (2.13)$$

[Caron-Huot, Mazzeo, Rastelli, Simmons-Duffin (2020)]

CDR

$$g(u, v) = \int_{\mathcal{R}} du' dv' K(u, v; u', v') d\text{Disc}[g](u', v') + (u \leftrightarrow v)$$

with

$$K(u, v; u', v') = K_B(u, v; u', v') \theta(\sqrt{v'} > \sqrt{u'} + \sqrt{u} + \sqrt{v}) + K_C(u, v; u') \delta(\sqrt{v'} - \sqrt{u'} - \sqrt{u} - \sqrt{v}). \quad (2.11)$$

why two terms?

The contact term is localized at the boundary of the bulk region. The kernels are respectively

$$K_B(u, v; u', v') = \frac{u - v + u' - v'}{64\pi(uvu'v')^{\frac{3}{4}}} x^{\frac{3}{2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; 1 - x\right), \quad (2.12)$$

$$K_C(u, v; u') = \frac{1}{4\pi(uvu'v')^{\frac{1}{4}}(\sqrt{u} + \sqrt{u'})},$$

is this some conformal block?

where x is the following combination of the four cross-ratios

WTB?

$$x = \frac{16\sqrt{uvu'v'}}{[(\sqrt{u} + \sqrt{v})^2 - (\sqrt{u'} + \sqrt{v'})^2][(\sqrt{u} - \sqrt{v})^2 - (\sqrt{u'} - \sqrt{v'})^2]}. \quad (2.13)$$

[Caron-Huot, Mazzeo, Rastelli, Simmons-Duffin (2020)]

Derivation from LIF in $d=2, 4$ - other d ? Why need LIF?

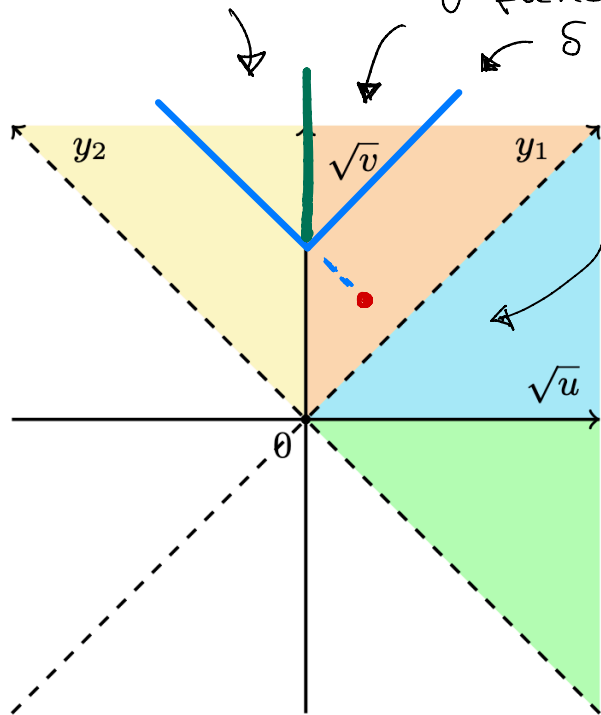
CDR

$$dDisc[g](\sqrt{u}, \sqrt{v}) = g(\sqrt{u}, \sqrt{v}) - \frac{1}{2}g(-\sqrt{u} + i\varepsilon, \sqrt{v}) - \frac{1}{2}g(-\sqrt{u} - i\varepsilon, \sqrt{v})$$

probed by dDisc

θ function

δ function



not drawn: image under $u \leftrightarrow v$.

- better variables:

$$y_1 = \sqrt{v} + \sqrt{u}$$

$$y_2 = \sqrt{v} - \sqrt{u}$$

- region of interest: $y_{\pm} > 0$

crossing: $y_2 \leftrightarrow -y_2$

$\Rightarrow$ even better variables:

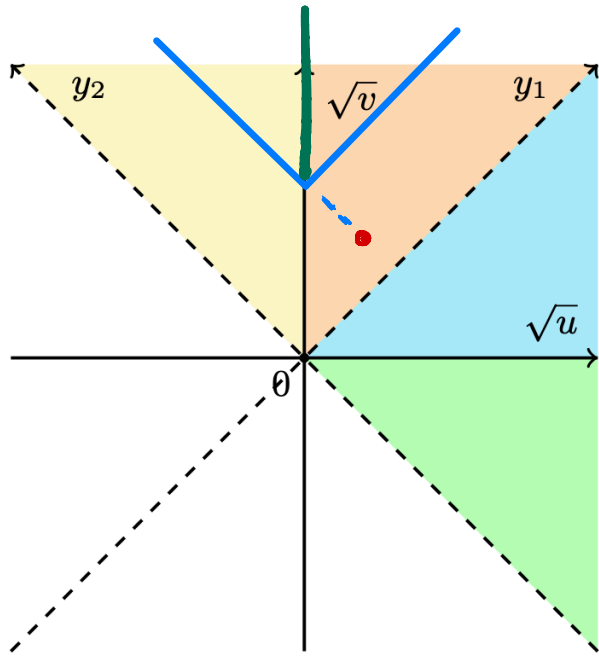
$$x_1 = y_1^2$$

$$x_3 = (\sqrt{v} + \sqrt{u})^2$$

$$x_2 = y_2^2$$

$$x_4 = (\sqrt{v} - \sqrt{u})^2$$

CDR - new derivation



1 start from single-var dispersion relation at fixed x_1 :

$$g(x_1, x_2) = \int_{x_1}^{\infty} \frac{dx_3}{2\pi i} \frac{\text{disc}[g](x_1, x_3)}{x_3 - x_2}$$

2 rewrite in terms of dDisc:

$$= \int \dots d\text{Disc}[g] - \int dx_3 \dots g(x_3, x_4)$$

3 add bulk term to fix error term:

$$\int dx_3 \dots g = \int dx_3 \int dx_4 K_B(\dots) d\text{Disc}[g]$$

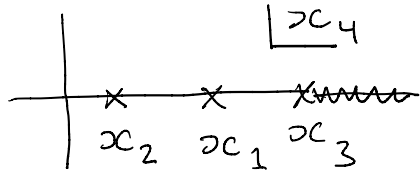
4 unwrap dDisc & demand pointwise cancellation:

$$g(31) = \int dx_4 K_B(1234) g(34) - \int dx_4 K_B(1243) \text{disc}[g](34)$$

$\Rightarrow$ single $\mathbb{C}$ variable (x_4), 3 special pts (x_1, x_2, x_3)!

[Ferragatta, BvR (to appear)]

CDR - new expressions



we obtain:

$$K_B = \frac{F(x)}{x_{12} x_{34}}$$

↙ not quite dDisc

$$"dDisc[F](x) = \# \delta(x-1)"$$

⇓ (almost uniquely)

- $F(x) = \Psi_{1/2}^{(-1,0)}(x)$
- $\Psi_h^{(a,b)}(x) = \# K_h^{(a,b)}(x) + \# K_{(1-h)}^{(a,b)}(x)$
- $K_h^{(a,b)}(x) = x^{h+a} {}_2F_1(h+a, h+b; 2h; x)$

$$x_{ij} := x_i - x_j$$

$$x = \frac{x_{12} x_{34}}{x_{13} x_{24}}$$

cross ratio of
cross ratios!

- * only $\mathbb{C}$ analysis
- * single var. cf. η or t
- * (almost) no freedom
- * Q: unequal ops?
- * Q: higher points?

[Ferragatta, BvR (to appear)]

Conclusion

- 1 disconnected conjecture: "Euclidean correlators $\leadsto$ GFF"
- 2 connected conjecture: " $\frac{\text{connected correlators}}{\bigotimes} \Big|_{S\text{-matrix}} \leadsto T(s, t)$ "
- 3 conformal dispersion relation: simple derivation, magic variables are cross ratios (of cross ratios)

Thank you!