Gradient Flow in Scalar-Fermion Theories

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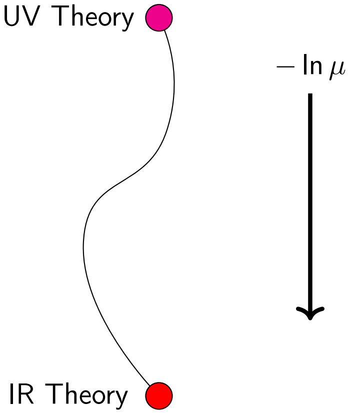
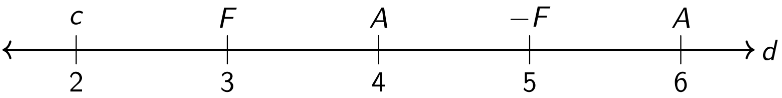
Based on 2511.01971 with W. Ronayne and A. Stergiou

QFT as EFT



Monotonicity theorems



All unitary RG flows in two-dimensions satisfy

$$\beta^I \partial_I C = 12 \beta^I G_{IJ} \beta^J \geqslant 0 \, .$$ [Zamolodchikov (1986)]

In higher dimensions this has been extended to:

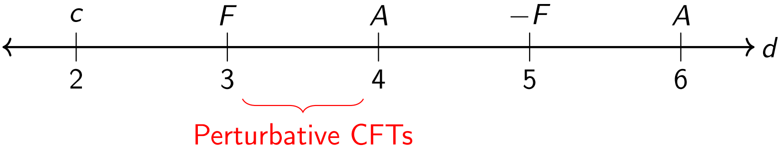
Even d: A, coefficient of Euler anomaly.

[Cardy (1988); Jack and Osborn (1990); Komargodski and Schwimmer (2011)]

Odd d: $(-1)^{\frac{d-1}{2}} \log Z_{S^d}$.

[Jafferis, Klebanov, Pufu and Safdi (2011); Casini and Huerta (2012)]

Overview



Question: Is there a monotonicity theorem in the ε-expansion?

1. Background on gradient flow
2. Scalar-Fermion theories
3. Gradient flow and the $\widetilde{F}$-theorem

Strengths of monotonicity

Not all monotonicity theorems are created equal, and there are three strengths of statement:

- *Weak Monotonicity*: $A_{IR} < A_{UV}$,
- *Strong Monotonicity*: $\frac{dA}{d\ln\mu} > 0$,
- *Gradient Flow*: $\partial_i A = G_{ij}\beta^j$.

For true gradient flow, G_{ij} must satisfy

1. symmetric, $G_{ij} = G_{ji}$, and
2. positive-definite, $G \succ 0$.

Without 1, this becomes strong monotonicity instead.

Gradient flow in other theories

Previously, $\partial_i A = G_{ij}\beta^j$ has been shown to have a solution in:

1. ϕ^4-theory in $d = 4$ and $d = 4 - \varepsilon$ through six loops
   [Jack and Poole (2018); WP and Stergiou (2024,2025)]
2. ϕ^3-theory in $d = 6$ and $d = 6 - \varepsilon$ through three loops
   [Gracey, Jack and Poole (2016); Benfatto and Zanusso (2025)]
3. ϕ^6-theory in $d = 3$ through six loops
   [Jack and Osborn (2026)]

Importantly, β is not gradient!

$\hookrightarrow$ Must shift $\beta \to B = \beta - (Sg)$ to get a solution.

[Jack and Osborn (1990)]

Scalar-Fermion Theories

We will look now for monotonicity theorems in a slightly more general setting: scalar-fermion theories near $d = 4$.

$$\mathscr{L} = \tfrac{1}{2}\,\partial^\mu\phi_i\,\partial_\mu\phi_i + i\bar{\psi}^a\bar{\sigma}^\mu\partial_\mu\psi_a + \tfrac{1}{4!}\lambda_{ijkl}\,\phi_i\phi_j\phi_k\phi_l + \left(\tfrac{1}{2}\,y_{iab}\,\phi_i\psi_a\psi_b + \text{h.c.}\right)$$

General beta functions now known through four loops. [Steudtner (2025)]

Perturbatively describes a number of well-studied theories:

- $y_{iab} = 0$: $O(N)$, Hypercubic
- $y_{iab} \neq 0$: Gross–Neveu–Yukawa, Nambu–Jona-Lasinio–Yukawa

as well as many other CFTs.

The beta shift

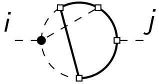
Many of the fixed points points mysteriously stop being solutions at three loops, and instead become *limit cycles*.
[Fortin, Grinstein and Stergiou (2013)]

These flows are fictitious, and to remove them we must introduce a gauge potential S: [Jack and Osborn, 1990]

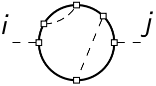
$$\beta \to B = \beta - (Sg)\,, \qquad \gamma \to \Gamma = \gamma + S\,.$$

S appears at 3-loops for scalar-fermion theories and 5-loops for purely scalar theories:

$$S_{ij} = \frac{5}{8}$$



$$+\ \frac{3}{4}$$



$$+ \cdots$$

Gradient Flow in Scalar-Fermion Theories

Want to find a function $A(\lambda, y)$ and a rank-2 tensor $G(\lambda, y)$ such that

$$\begin{pmatrix} \partial_I A \\ \partial_\alpha A \end{pmatrix} = \begin{pmatrix} G_{IJ} & G_{I\beta} \\ G_{\alpha J} & G_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \beta^J \\ \beta^\beta \end{pmatrix}$$

$$I = (ijkl) \qquad\qquad \alpha = i(ab)$$

has a perturbative solution through 4-loops.

We express A, G and β as sums over graphs with arbitrary coefficients:

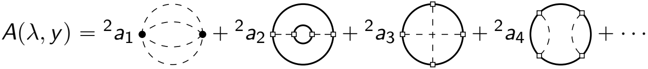
$$A = \sum_n \sum_m {}^n a_m \, {}^n A_m \,,$$

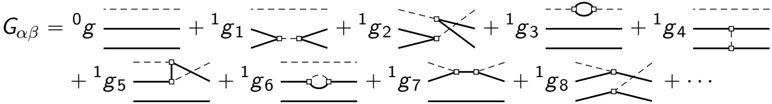
$$G_{\circ\star} = \sum_n \sum_m {}^n g_m ({}^n G_m)_{\circ\star} \,,$$

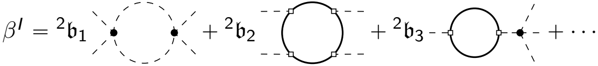
$$\beta^\star = \sum_n \sum_m {}^n \mathfrak{b}_m ({}^n \beta_m)^\star \,.$$

Loop order:	1	2	3	4
Terms in A	4	12	53	309
Terms in $G_{\circ\star}$	–	8	59	593
Terms in $\beta^\star$	4	22	168	1739

For instance,







Constraints on the beta function

Loop order:	1	2	3	4
Constraints on β	–	4	63	888
Constraints on S	–	–	4	63

We find a total of 955 scheme-invariant constraints through 4 loops.

Plugging in the $\overline{MS}$ values for the ${}^{n}\mathfrak{b}_m$, one finds that these constraints are **not** satisfied!

Including the beta shift, however, it seems that B **is** gradient, where β is not!

Extension to $d = 4 - \varepsilon$

We now want to continue our $d = 4$ solution to a solution in $d = 4 - \varepsilon$.

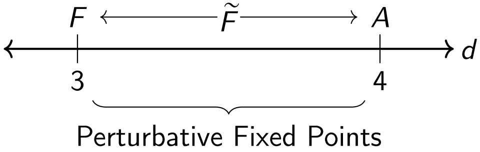
$$\begin{aligned}
\beta^\star &= \sum_n \sum_m ({}^n\mathfrak{b}_m + \varepsilon\, {}^n\mathfrak{e}_m)({}^n\beta_m)^\star\,, \\
A &= \sum_n \sum_m ({}^n a_m + \varepsilon\, {}^n d_m(\varepsilon))\, {}^n A_m\,, \\
G_{\circ\star} &= \sum_n \sum_m ({}^n g_m + \varepsilon\, {}^n h_m(\varepsilon))({}^n G_m)_{\circ\star}\,.
\end{aligned}$$

We then proceed as before, using the ${}^n d_m$ and ${}^n h_m$ to remove the new ε-dependent terms.

This comes at a cost:

$\hookrightarrow$ 61 constraints through 4 loops $\longrightarrow$ 2 constraints on S

The $\widetilde{F}$-conjecture



It has been proposed that a good monotone across dimensions is

$$\widetilde{F} = \sin\left(\frac{\pi d}{2}\right) \log Z_{S^d}$$

[Giombi and Klebanov (2015); Fei, Giombi, Klebanov and Tarnopolsky (2015)]

For purely scalar theories, $\widetilde{F} = \frac{\pi}{288} A$ at fixed points.
[WP and Stergiou (2025)]

Does the matching with $\widetilde{F}$ continue once fermions are added in?

Matching A to $\widetilde{F}$

The value of $\widetilde{F}$ has been computed at:

- the $O(N)$ and Hypercubic fixed points,
  [Fei, Giombi, Klebanov and Tarnopolsky (2015)]
- the GNY and NJLY fixed points,
  [Fei, Giombi, Klebanov and Tarnopolsky (2016)]

Can we choose the freedom in A such that it matches $\widetilde{F}$ there?
Yes!

$$\widetilde{F}|_{CFT} - N_s \widetilde{F}_s - N_f \widetilde{F}_f = \frac{\pi}{288} A_{CFT}$$

We need to use two of the three degrees of freedom to match.

Unlike $\widetilde{F}$, our A is also defined away from fixed points.
$\hookrightarrow$ Strongest version of $\widetilde{F}$-conjecture

A constraint from $\widetilde{F}$

Does this matching with $\widetilde{F}$ come for free, or is it a non-trivial statement?

Consider the lowest order contributions:

$$A^{(2)} = -\tfrac{1}{8}\,{}^{0}g\,\varepsilon\,\mathrm{Tr}(y_i y_i)\,, \qquad \widetilde{F}^{(2)} = -\frac{\pi}{96}\varepsilon\,\mathrm{Tr}(y_i y_i)\,,$$

$$A^{(3)} = -\tfrac{1}{6}\varepsilon\lambda_{ijkl}\lambda_{ijkl}\,, \qquad \widetilde{F}^{(3)} = -\frac{\pi}{1728}\varepsilon\lambda_{ijkl}\lambda_{ijkl}\,.$$

We can only match both of these expressions simultaneously if we take

$$\widetilde{F}_{\mathrm{CFT}} = \frac{\pi}{288}A_{\mathrm{CFT}} \qquad \text{and} \qquad {}^{0}g = 24\,.$$

This provides a non-trivial check of the $\widetilde{F}$-conjecture!

Outlook

Through four-loops we have*

$$\partial_\circ \widetilde{F}(\lambda, y) = G_{\circ\star}(\lambda, y) B^\star \,.$$

for a general scalar-fermion theory.

*Up to a four-loop computation of S.

Two main future approaches for strengthening our results:

- The $\widetilde{F}$-theorem? Understand $\widetilde{F}$ in more detail.

  [Fraser-Taliente (2026)]

- Systematize the constraints.

Thank you!