Two applications of SSB in AdS

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Outline

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symmetries in QED$_3$ from perturbation theory in $e^2 L$
w/ Stefano Lanza, Pierluigi Niro

 

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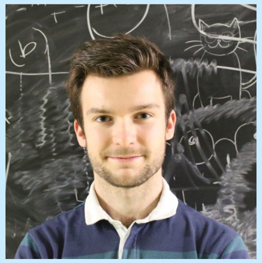
strong-coupling expansion of **Bremsstrahlung** in SYM
from Goldstone bosons in AdS$_2$
w/ Federico Castellani, Valentina Forini, Roman Stemplowski

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  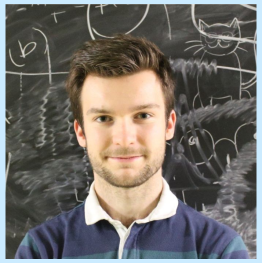

Spontaneous Symmetry Breaking in Anti-de Sitter

Consider a QFT in *rigid* Anti-de Sitter space

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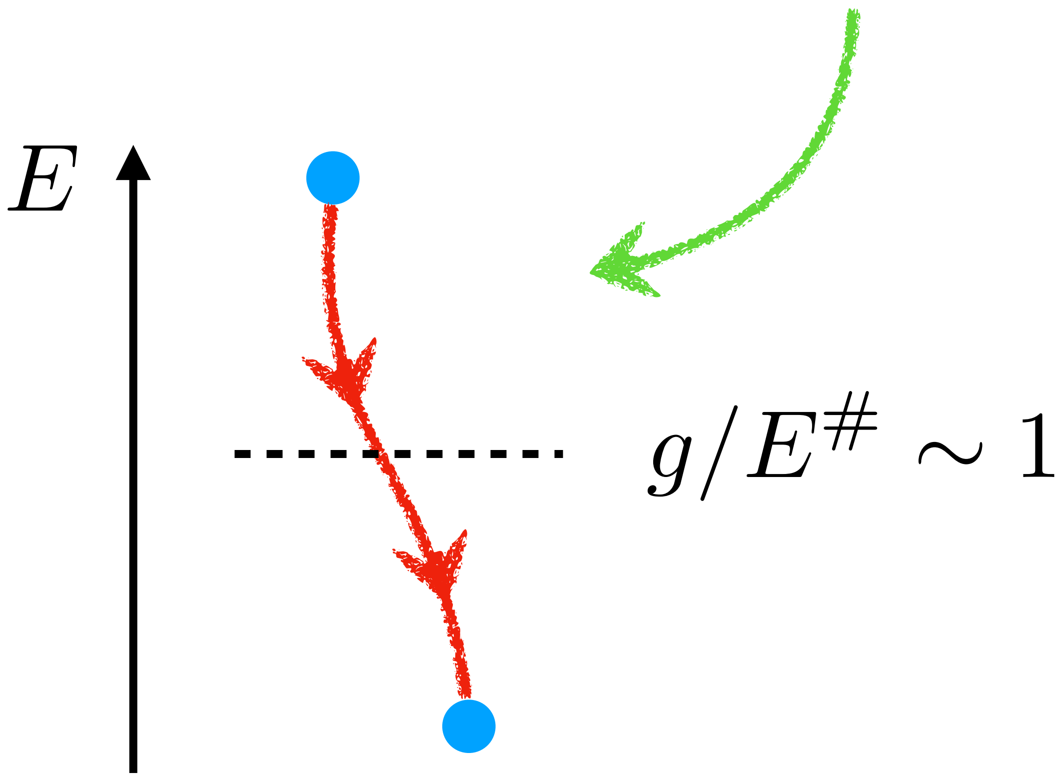

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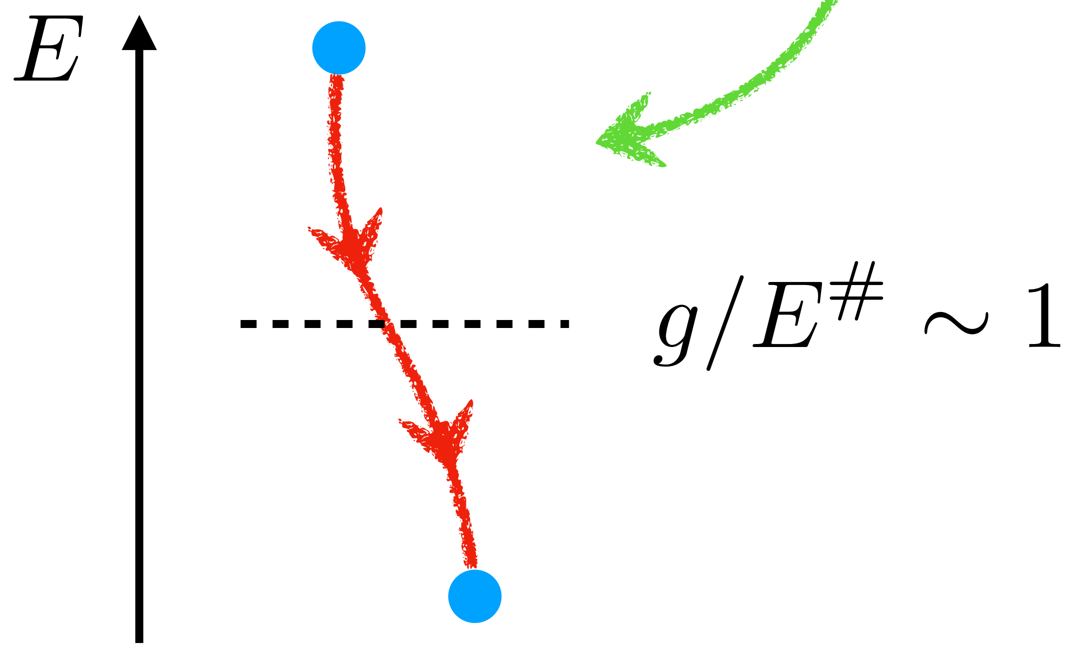
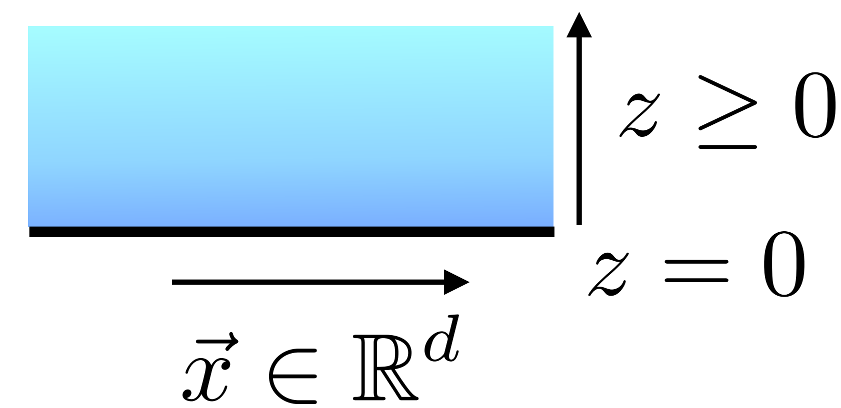


$$ds^2_{\rm AdS} = L^2 \frac{dz^2 + d\vec{x}^2}{z^2}$$



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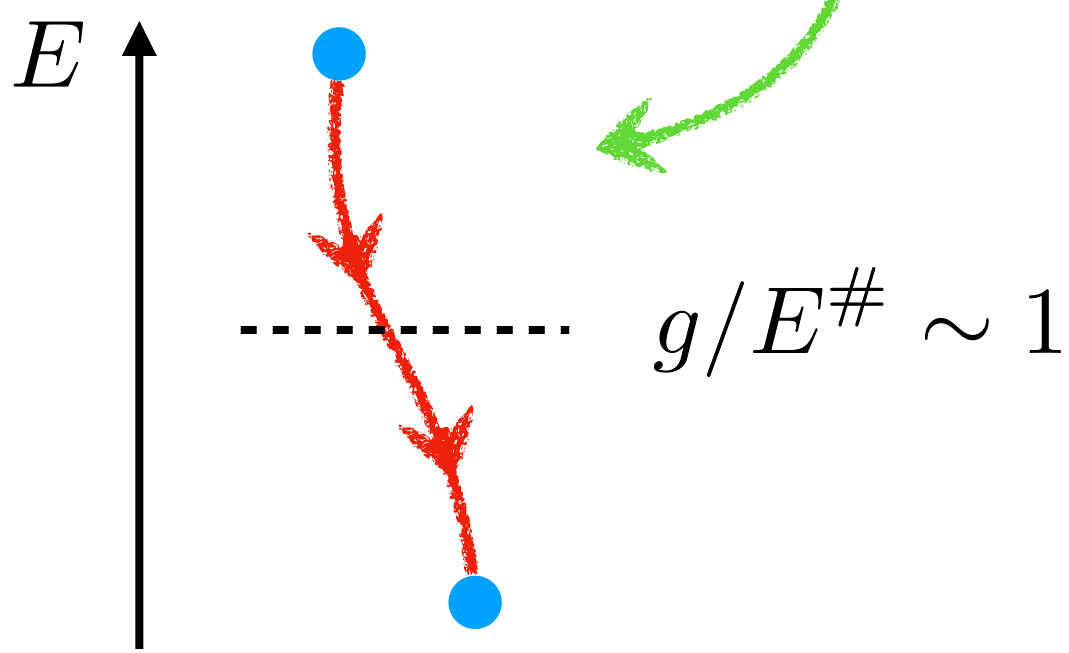
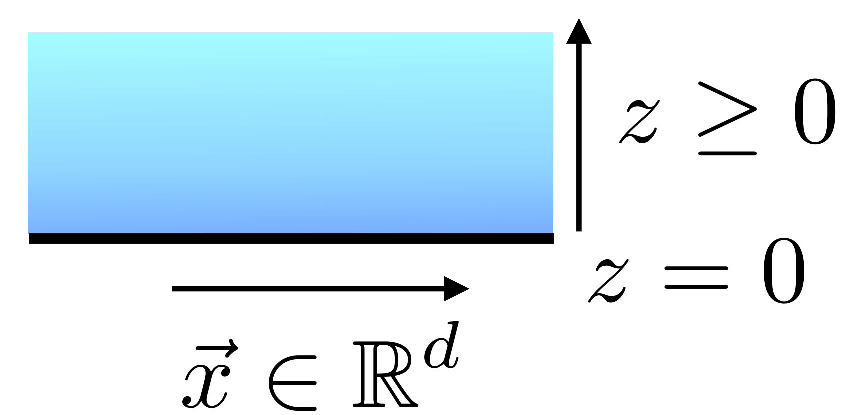


$$ds^2_{\mathrm{AdS}} = L^2 \frac{dz^2 + d\vec{x}^2}{z^2}$$

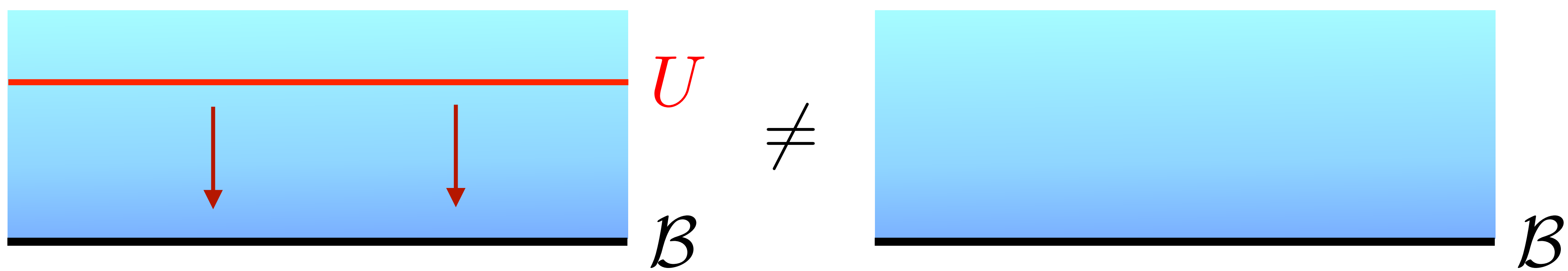


- Radius L : Dimensionless parameter $gL^{\#}$, allowing perturbative expansion
- Boundary correlators covariant under $SO(1, d+1)$, like CFT$_d$ (but no stress tensor) [Paulos, Penedones, Toledo, van Rees, Vieira (2016)]
- Data (e.g. dimensions, OPE coeff.) fixed by $gL^{\#}$ but not uniquely: choice of **<u>boundary condition</u>**

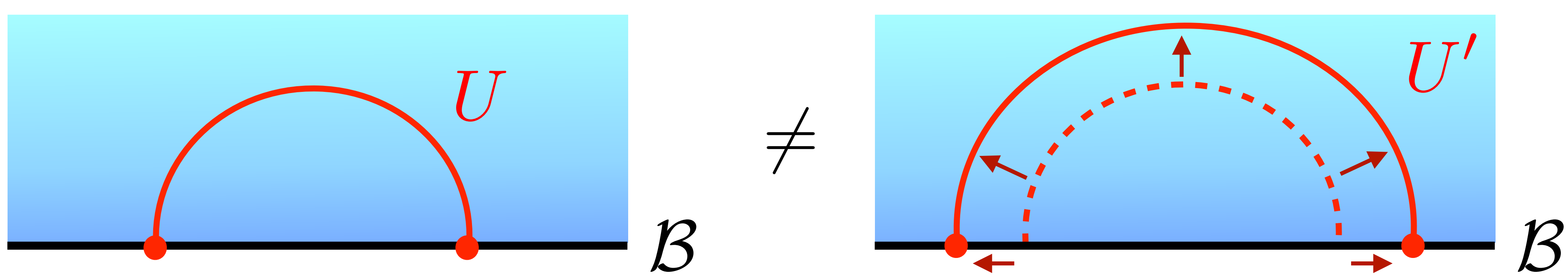
Spontaneous Symmetry Breaking in Anti-de Sitter

Possible equivalent definitions:

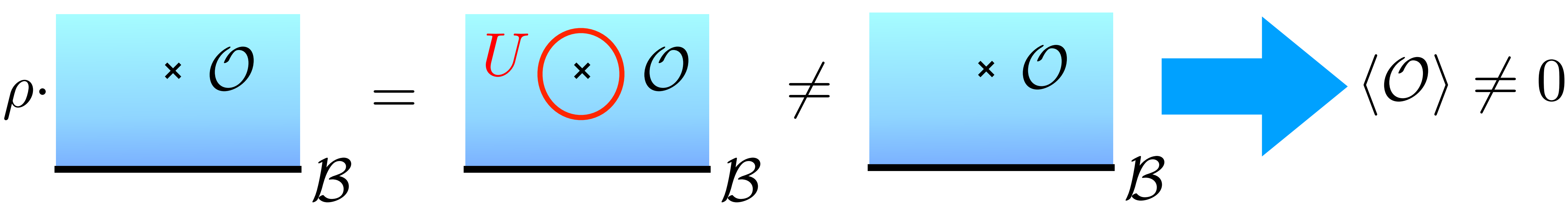
1. Symmetry breaking boundary condition



2. Absence of symmetry in the boundary correlators



3. Charged operator with nonzero 1-pt function



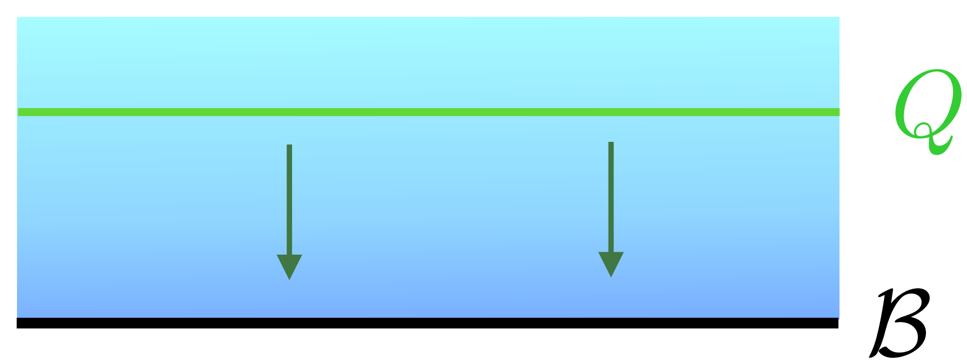
Spontaneous Symmetry Breaking in Anti-de Sitter

Specify to **continuous symmetries**

Bulk conserved current: $J_\mu(z,\vec{x})$, $\nabla^\mu J_\mu = 0$

$$Q = -i\int_{\Sigma_d} \star J \quad , \quad U = e^{i\alpha Q}$$

SSB means that:



$\neq 0$

$$Q|_\partial = i\int d^d\vec{x}\lim_{z\to 0} z^{1-d} J_z(z,\vec{x}) \neq 0$$

flux of charge to the boundary

Spontaneous Symmetry Breaking in Anti-de Sitter

More precisely phrased in terms of boundary operator expansion (BOE)

$$J_z(z, \vec{x}) \underset{z \to 0}{\sim} z^{d-1}\, t(\vec{x})$$

only power compatible with current conservation

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$\exists$ boundary scalar primary operator $t(\vec{x})$ of scaling dimension $\Delta = d$, i.e. marginal:

tilt operator

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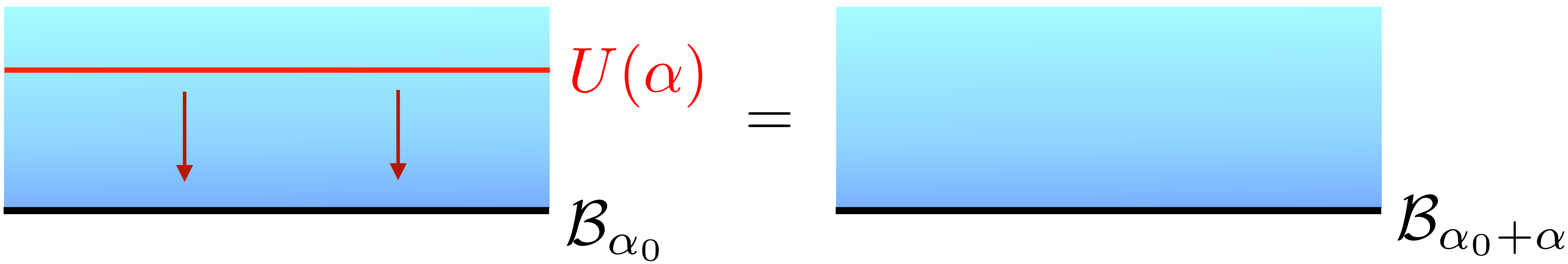
Well-known in BCFT and DCFT
[Bray, Moore (1977); Herzog, Huang (2017); Cuomo, Mezei, Raviv-Moshe (2021); Padayasi, Krishnan, Metlitski, Gruzberg, Meineri (2022)]

Adapted to massive QFT in AdS in [Copetti, Di Pietro, Ji, Komatsu (2023)]

Spontaneous Symmetry Breaking in Anti-de Sitter

$$Q|_\partial = i\int d^d\vec{x}\, t(\vec{x}) \qquad , \qquad U(\alpha)|_\partial = e^{-\int d^d\vec{x}\,\alpha\, t(\vec{x})}$$



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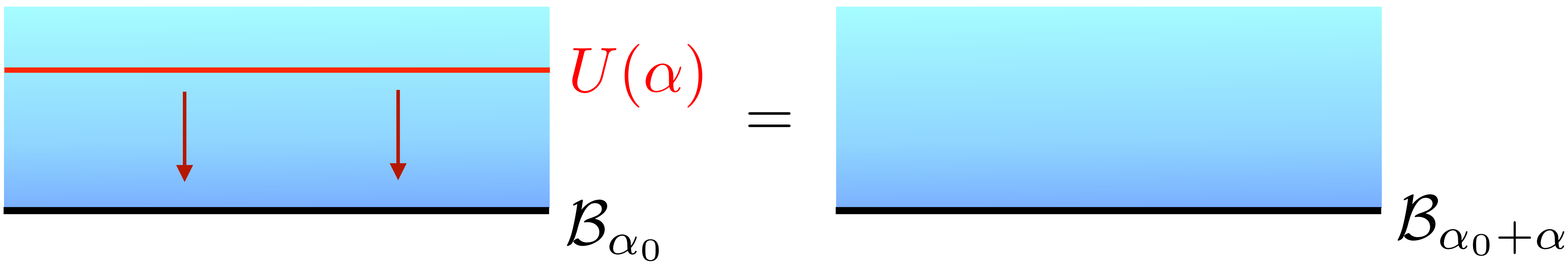

Continuous family of boundary conditions:

$$Z[\mathcal{B}_{\alpha_0+\alpha}] = Z[\mathcal{B}_{\alpha_0}] \left\langle e^{-\int d^d\vec{x}\, \alpha\, t(\vec{x})} \right\rangle$$

related by shift of the boundary action via marginal tilt operator

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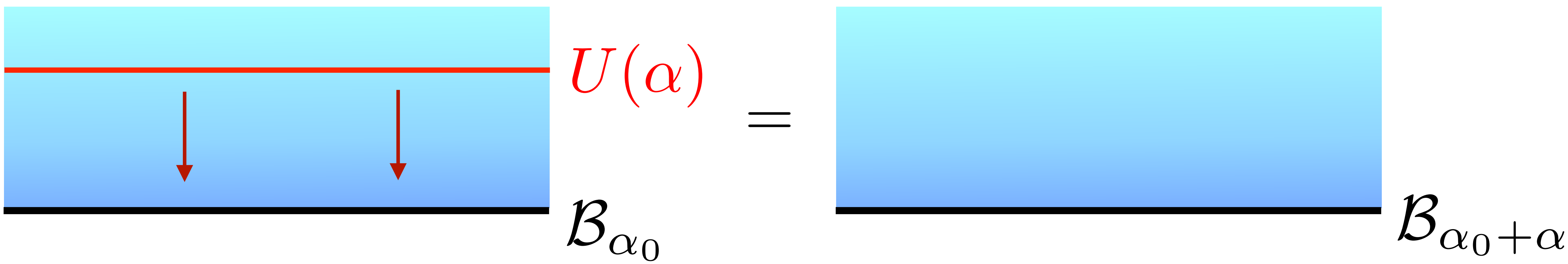

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SSB $G \to H$ gives **conformal manifold** G/H
("trivial": same boundary data)

Spontaneous Symmetry Breaking in Anti-de Sitter

Zamolodchikov metric on G/H : two-pt of the tilt

$$\langle t(\vec{x})t(\vec{y})\rangle = C_t \, \frac{1}{|\vec{x}-\vec{y}|^{2d}}$$

- related to size of the space of vacua, in units of the AdS radius L
- nicely encoded in the harmonic decomposition of **two-pt function of the bulk current**

[Costa, Goncalves, Penedones (2014)]

$$\langle J_{\mu_1}(X_1)J_{\mu_2}(X_2)\rangle = \int_{-\infty}^{+\infty} d\nu \, F_J(\nu)(\Omega_\nu^{(1)})_{\mu_1\mu_2}(X_1,X_2)$$

$$\boxed{C_t = -\frac{\Gamma(d+1)}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}F_J(i(d/2-1))}$$

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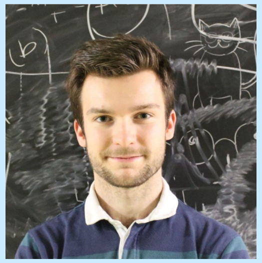


Symmetries in QED$_3$

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\Psi}_I \, \not{D} \, \Psi^I$$

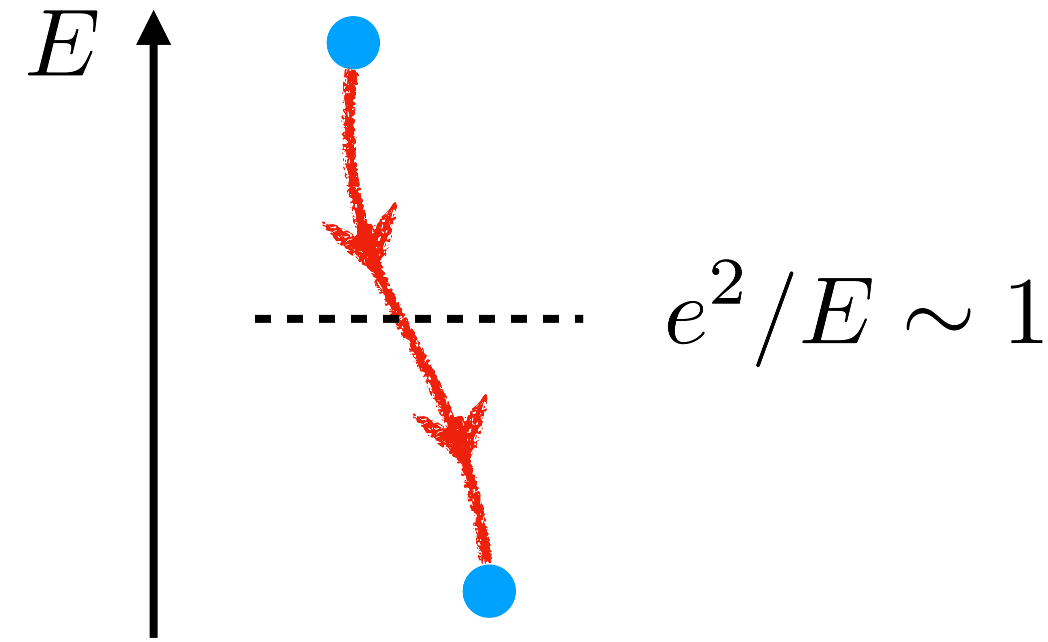
3d two-component Dirac fermion

$$I = 1, \ldots, N$$

- $(J_\mu)_I{}^J = \bar{\Psi}_I \gamma_\mu \Psi^J -$ trace , $\Psi^I \to U^I{}_J \Psi^J$
- $J_\mu^m = \frac{i}{4\pi} \epsilon_\mu{}^{\nu\rho} F_{\nu\rho}$, $\mathcal{M} \to e^{i\alpha} \mathcal{M}$
- when N is odd the path integral over Dirac fermions breaks reflections R (i.e. $x^3 \to -x^3$): assume N even

all together: $\left(\frac{SU(N) \times U(1)_m}{\mathbb{Z}_N} \right) \rtimes (\mathbb{Z}_2^R \times \mathbb{Z}_2^C)$

Symmetries in QED_3



What happens in the IR?

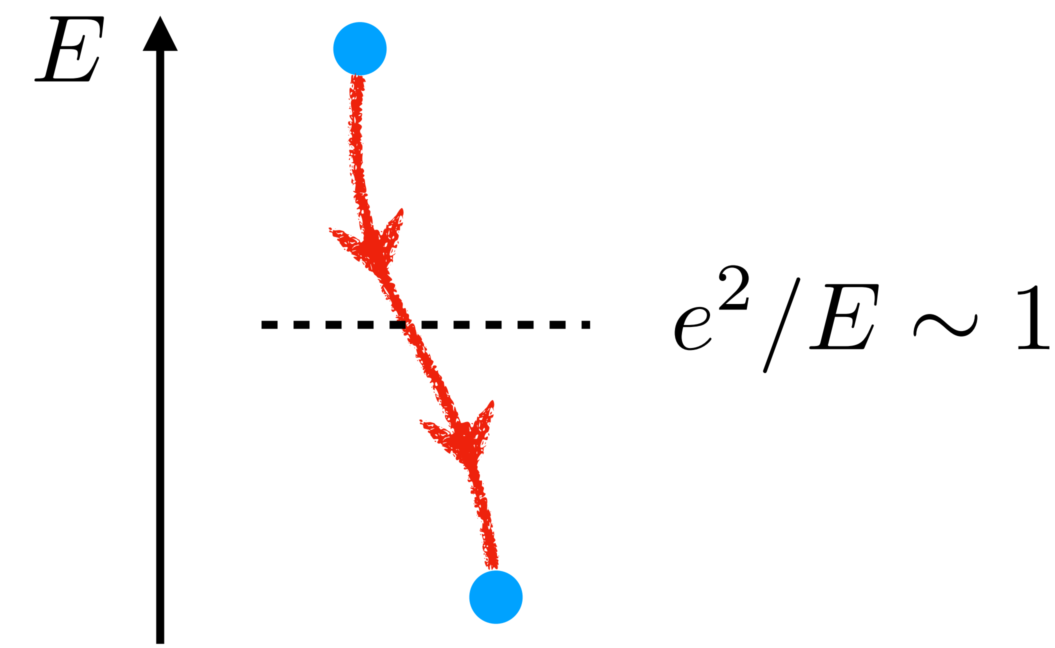
One possibility is: SSB

flow to Goldstone bosons

Original proposal:
[Pisarski (1984)]

$$\langle \bar{\Psi}_I \Psi^J \rangle \propto e^4 \begin{pmatrix} \mathbf{1}_{N/2} & 0 \\ 0 & -\mathbf{1}_{N/2} \end{pmatrix}$$

<u>Symmetries in QED_3</u>



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(Some of the) modern progress:

- from conformal bootstrap: evidence that $N \geq 4$ flows instead to *interacting CFT*

  [Albayrak, Erramili, Li, Poland, Xin (2021)]

- proposal for $N = 2$: <u>**monopole condensate**</u>

  [Dumitrescu, Niro, Thorngren; Chester, Komargodski (2024)]

Symmetries in QED$_3$

Monopole with charge $q_m = 1$ is in fund. of $SU(2)$

$$\langle \mathcal{M}_1^I \rangle \neq 0 \quad \Longrightarrow \quad U(2) = \frac{SU(2) \times U(1)_m}{\mathbb{Z}_2} \to U(1)_+$$

mix between $U(1)_3 \subset SU(2)$ and $U(1)_m$

- Mixing implies off-diagonal terms in the metric of the Goldstone target space: squashed sphere

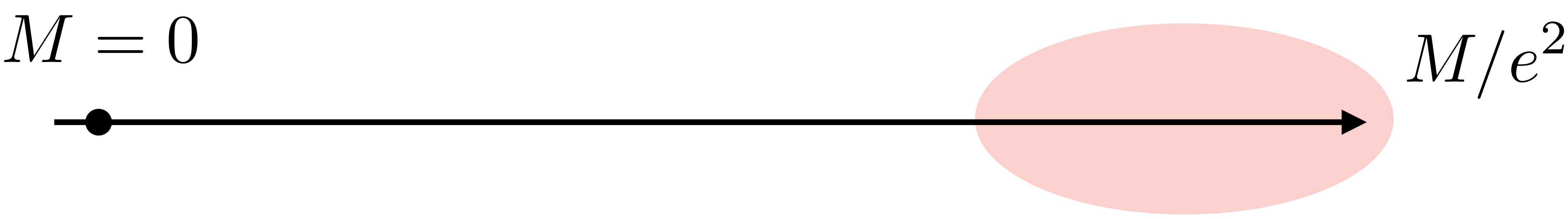
- Argued by deforming with a mass

$$M_I{}^J = M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and studying the controllable regime: $e^2/M \ll 1$

Symmetries in QED_3

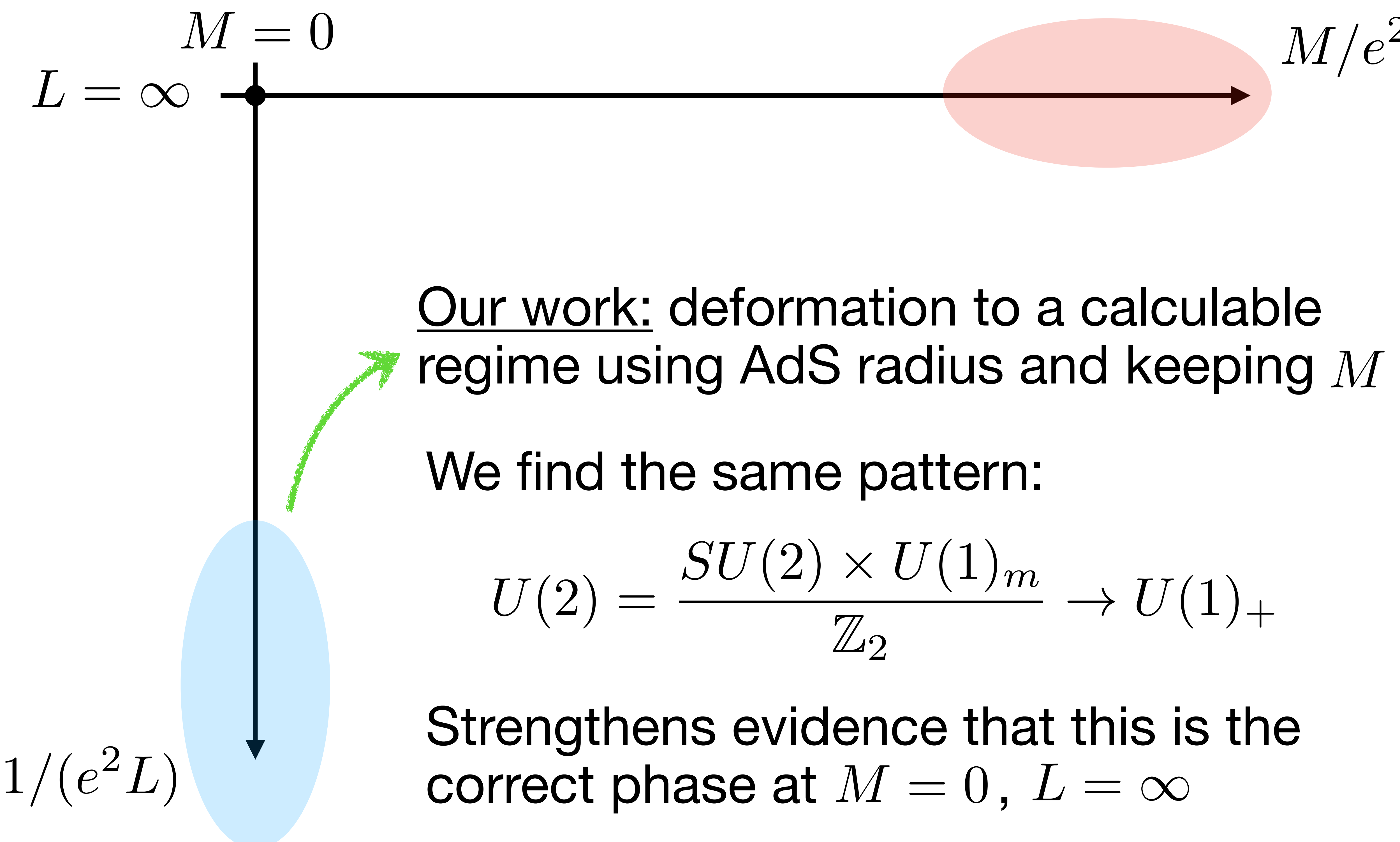
[Dumitrescu, Niro, Thorngren (2024)]



Reliable calculation of symmetry breaking
+ argument for no phase transition for any $M \neq 0$

Symmetries in QED$_3$

[Dumitrescu, Niro, Thorngren (2024)]



Our work: deformation to a calculable regime using AdS radius and keeping $M = 0$

We find the same pattern:

$$U(2) = \frac{SU(2) \times U(1)_m}{\mathbb{Z}_2} \to U(1)_+$$

Strengthens evidence that this is the correct phase at $M = 0$, $L = \infty$

Symmetries in QED$_3$

Tree level:

$$\gamma_z \Psi^I|_\partial = B^I{}_J \Psi^J|_\partial \ , \ B^I{}_J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad SU(2) \to U(1)_3$$

$$F_{zi}|_\partial = 0 \ \text{ i.e. } \ J^m_z|_\partial = \epsilon^{ij} F_{ij}|_\partial \neq 0 \quad : \quad U(1)_m \to 1$$

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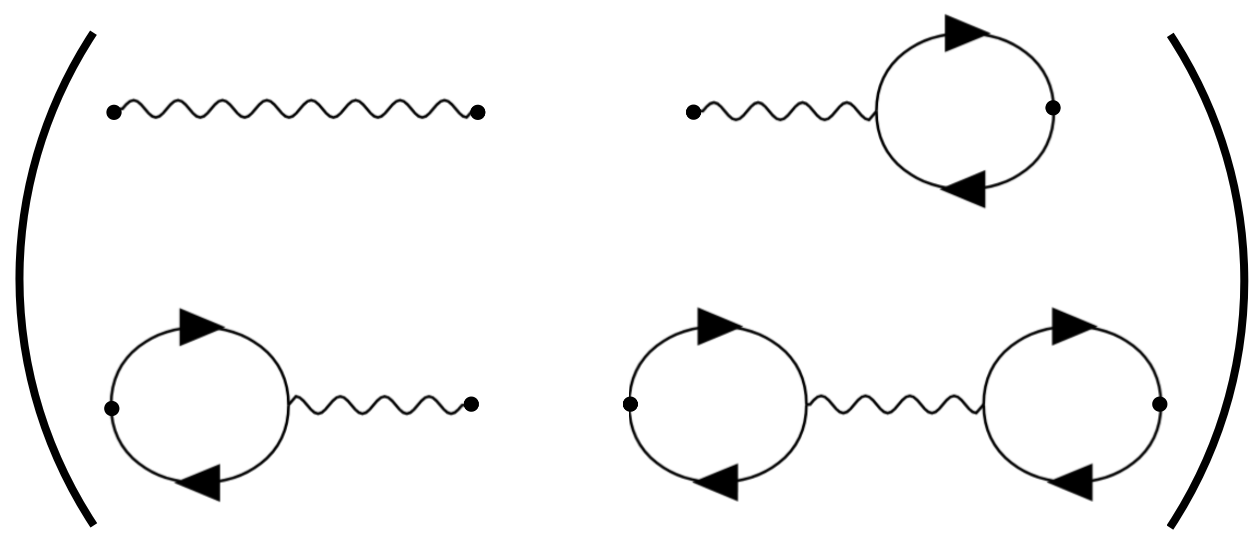
We extract the geometry of the space of vacua/conformal manifold at 1 loop from the 2-pt function of the bulk current:

J^m $\qquad$ J^3

$$C_t = -\frac{2}{\pi} F_J(0) \underset{\mathcal{O}(e^2)}{=}$$



J^m

J^3

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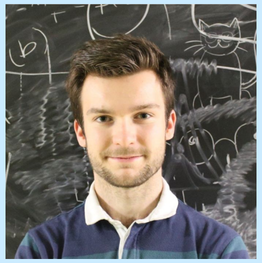

Bremsstrahlung function from AdS_2

1/2-BPS Wilson–Maldacena line in $\mathcal{N} = 4$ SYM:

$$W = \mathrm{tr} P e^{\int d\tau (i A_\tau + \Phi^6)}$$

breaks $SO(6) \to SO(5)$: tilt t^p , $p = 1, \ldots, 5$

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Interesting observable, known from localization:

$$\langle t^p(x) t^q(x') \rangle = \frac{C_t(\lambda)\delta^{pq}}{|x - x'|^2} \quad , \quad C_t(\lambda) = 2\,B(\lambda)$$

[Correa, Henn, Maldacena, Sever (2012)]

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<u>Goal</u>: match strong-coupling expansion with holographic calculation

$$C_t(\lambda) = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Bremsstrahlung function from AdS_2

Holographic dual: string on $AdS_2 \times \mathrm{pt} \subset AdS_5 \times S^5$

At large N : QFT on AdS_2

$\mathrm{x}^i,\ i = 1, 2, 3$	$\Delta = 2\ (m^2 = 2)$	$SO(3)$ (AdS_5 transverse)
$\mathrm{y}^p,\ p = 1, \ldots, 5$	$\Delta = d = 1$ (massless)	$SO(5)$ Goldstone bosons
θ	$\Delta = \frac{3}{2}\ (\lvert m \rvert = 1)$	8 Majorana fermions

with Green-Schwartz action [Giombi, Roiban, Tseytlin (2017); Beccaria, Kurlyand, Tseytlin (2026)]

$$S = T \int d^2\sigma \sqrt{-g}\left(\mathcal{L}_B^{(2)} + \mathcal{L}_B^{(4)} + \mathcal{L}_F^{(2)} + \mathcal{L}_F^{(4)} + o(\theta^4) + \ldots\right)$$

Bremsstrahlung function from AdS$_2$

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sufficient for LO

$$C_t(\lambda) = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

LO calculation: y^p "dual" to t^p, i.e. $\mathrm{y}^p(z,x) \underset{z\to 0}{\sim} -\frac{1}{T}\, z\, t^p(x)$

Matches general definition of the tilt:

$$J^p_\mu = -T\partial_\mu \mathrm{y}^p + \ldots \qquad J^p_z(z,x) \underset{z\to 0}{\sim} t^p(x)$$

Bremsstrahlung function from AdS$_2$

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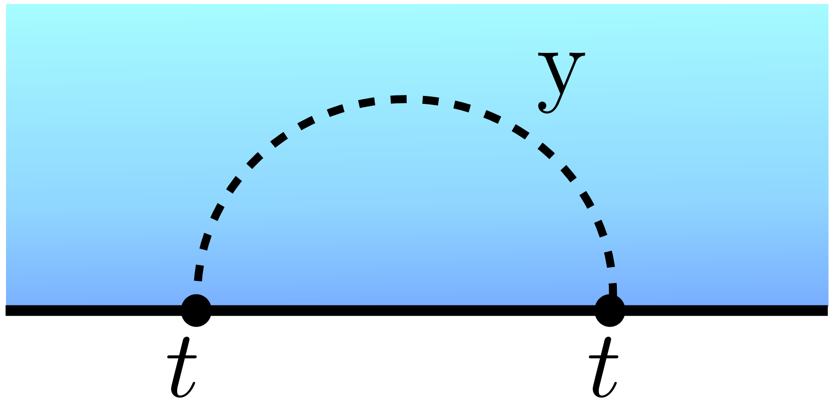


correct result upon identification

$$T = \frac{\sqrt{\lambda}}{2\pi}$$

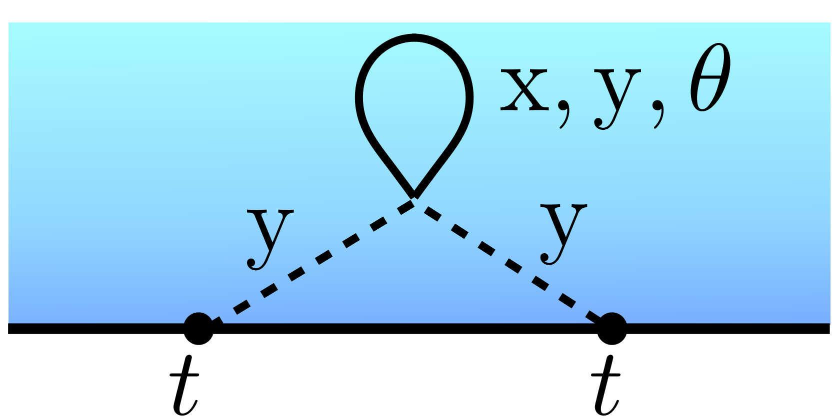
Bremsstrahlung function from AdS_2

$$S = T\int d^2\sigma\sqrt{-g}\,\left(\mathcal{L}_B^{(2)} + \mathcal{L}_B^{(4)} + \mathcal{L}_F^{(2)} + \mathcal{L}_F^{(4)} + o(\theta^4) + \ldots\right)$$

needed for NLO

$$C_t(\lambda) = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

NLO calculation: puzzle



UV divergence does not cancel, seems to require mass counterterm, see similar discussion for x in [Beccaria, Kurlyand, Tseytlin (2026)]

Bremsstrahlung function from AdS$_2$

Resolution: rely on the *bulk current* to properly identify / normalize the tilt operator

$$J^p_\mu = -T\partial_\mu \mathrm{y}^p + \text{ cubic terms}$$

fixed from $\mathcal{L}^{(4)}$

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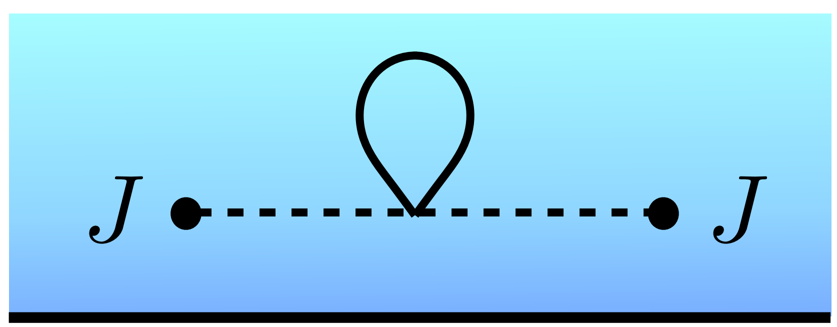
We use: $$C_t = -\frac{1}{\pi} F_J(-i/2)$$



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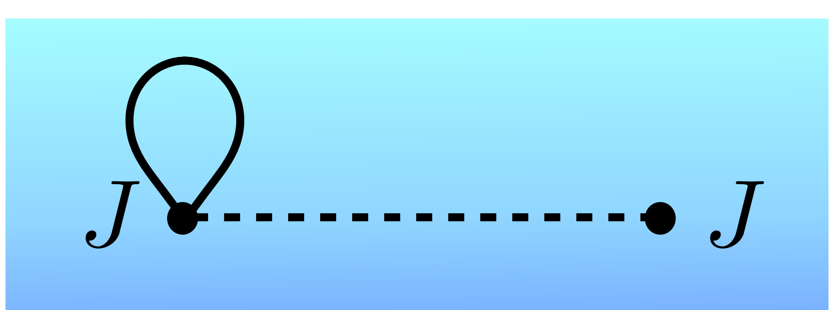

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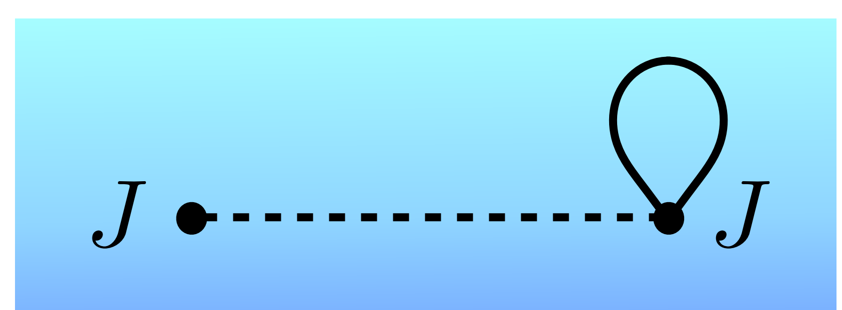

Bremsstrahlung function from AdS_2

Finite coefficient: the contributions from the various tadpoles are ($D = 2 - 2\epsilon$)

$$\mathrm{y}: \quad (N_{\mathrm{y}} - 1)\, G^{\mathrm{y}}\big|_0 \qquad\qquad G^{\mathrm{y}}\big|_0 = \frac{1}{4\pi\varepsilon}$$

$$\mathrm{x}: \quad \left(1 - \tfrac{2}{D}\right) N_{\mathrm{x}}\, G^{\mathrm{x}}\big|_0 \qquad \text{with} \qquad G^{\mathrm{x}}\big|_0 = \frac{1}{4\pi\varepsilon} + \frac{1}{2\pi}$$

$$\theta: \quad \frac{1}{4} N_\theta\, G^\theta\big|_0 \qquad\qquad G^\theta\big|_0 = -\frac{1}{4\pi\varepsilon} + \frac{1}{4\pi}$$

$$C_t(\lambda) = \frac{\sqrt{\lambda}}{2\pi^2} - \frac{3}{4\pi^2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Correct result if we take: $\qquad N_{\mathrm{y}} = 5 + 2\epsilon$

This is the way SUSY-preserving DRED is typically implemented in $\mathcal{N} = 4$

Outlook

- More general breaking patterns in QED$_3$

$$\gamma_z \Psi^I|_\partial = B^I{}_J \Psi^J|_\partial$$

Differences in the flat-space limit? Presumably only

$$B^I{}_J = \begin{pmatrix} \mathbf{1}_{N/2} & 0 \\ 0 & -\mathbf{1}_{N/2} \end{pmatrix}$$

survives as allowed bc. Similar question for QCD$_4$

- Large N limit

- Estimate of critical N from boundary conditions?

Outlook

- Calculation of $C_t(\lambda)$ at NLO in ABJM

- NNLO for $\mathcal{N}=4$ seems attainable

- Similar logic for 2-pt function of the displacement operator from $\mathbb{x}$. Caveat: more complicated non-linearly realized symmetry, mixing global with spacetime

Thank You

Spontaneous Symmetry Breaking in Anti-de Sitter

Data (e.g. dimensions, OPE coeff.) fixed by $gL^{\#}$ but not uniquely: choice of **boundary condition**

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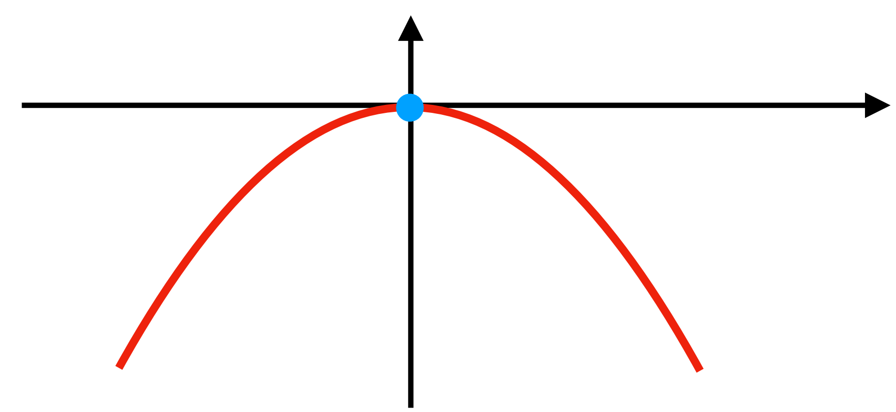
Examples:

(1) Free massive scalar field

$$\phi(z,\vec{x}) \underset{z\to 0}{\sim} z^{\Delta}\, O(\vec{x})$$



$$m^2L^2 \in [m_{\mathrm{BF}}^2 L^2, m_{\mathrm{BF}}^2 L^2 + 1]$$

$$\Delta(\Delta - d) = m^2 L^2 \;\Rightarrow\;$$ two possibilities: $\Delta_\pm$

Boundary correlators: GFF of O

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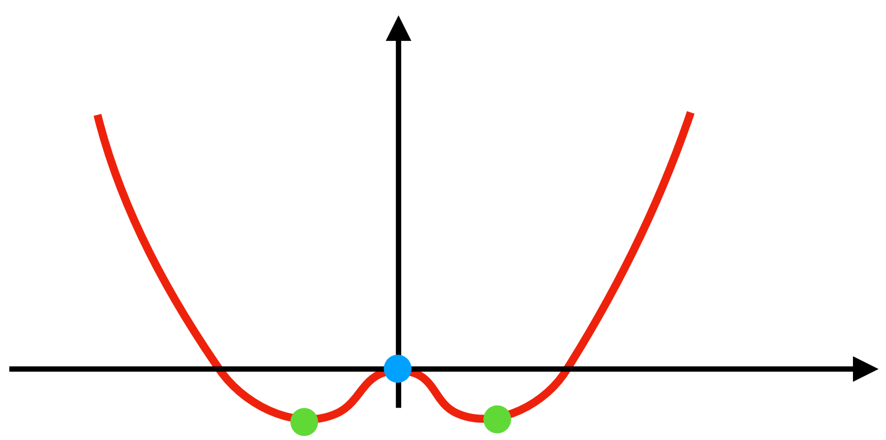
Examples:

(2) Real scalar with SSB potential

$$\phi(z,\vec{x}) \underset{z\to 0}{\sim} z^{\Delta}\, O(\vec{x}) \quad +\ldots$$



Additional possibility:

$$\phi(z,\vec{x}) \underset{z\to 0}{\sim} \Phi_{\pm} + z^{\Delta'}\, O'(\vec{x}) \quad +\ldots \qquad \langle \phi(z,\vec{x})\rangle = \Phi_{\pm}$$

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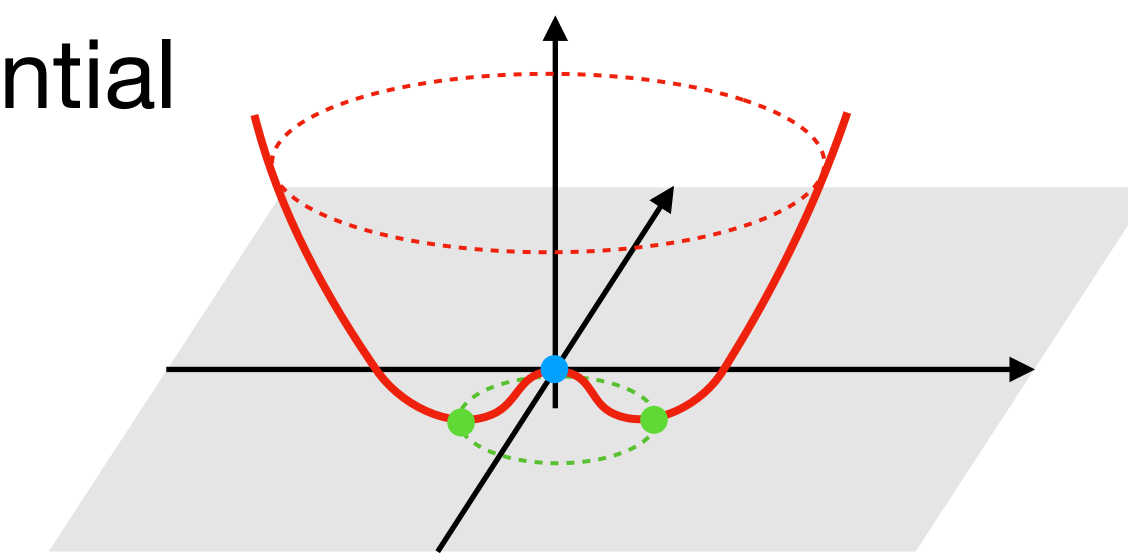
Examples:

(3) Complex scalar with SSB potential

$$\phi(z,\vec{x}) \underset{z\to 0}{\sim} z^{\Delta}\, O(\vec{x}) + \ldots$$



Additional possibility:

$$\phi(z,\vec{x}) \underset{z\to 0}{\sim} \rho e^{i\theta} + z^{d}\, t(\vec{x}) + \ldots \qquad \langle \phi(z,\vec{x})\rangle = \rho e^{i\theta}$$

marginal operator

Spontaneous Symmetry Breaking in Anti-de Sitter

Spectral representation : [Costa, Goncalves, Penedones (2014)]

$$\langle J_{\mu_1}(X_1) J_{\mu_2}(X_2) \rangle = \int_{-\infty}^{+\infty} d\nu \, F_J(\nu) (\Omega_\nu^{(1)})_{\mu_1 \mu_2}(X_1, X_2)$$

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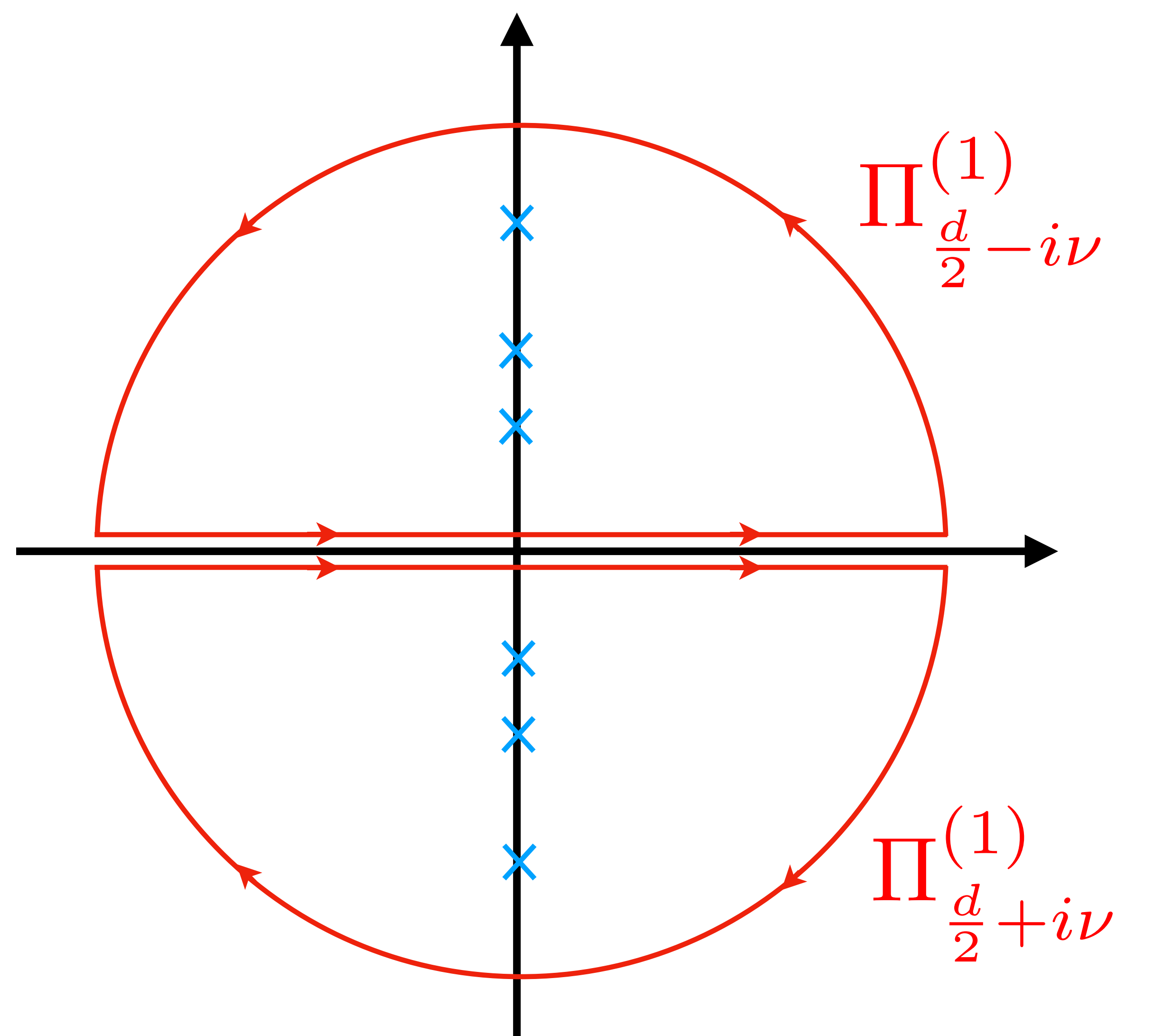
$$\Omega_\nu^{(1)} = \frac{i\nu}{2\pi}(\Pi^{(1)}_{\frac{d}{2}+i\nu} - \Pi^{(1)}_{\frac{d}{2}-i\nu})$$

$\Pi^{(1)}_\Delta$: BOE block with spin 1, and scaling dimension Δ



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To *avoid* SSB it must be compensated by a zero:

$$F_J(i(d/2-1)) = 0$$

If *there is* SSB: $\boxed{C_t = -\frac{\Gamma(d+1)}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})} F_J(i(d/2-1))}$