

Graded Unitarity

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This is a talk about vertex algebras/chiral algebras

A novelty of the past 10+ years is that interesting vertex algebras are now known to play an important role in the study of supersymmetric QFTs in $d \geq 2$.

mostly restrict to this case today $\longrightarrow$ 4d $N \geq 2$ SCFT $(\mathcal{T})$ $\xrightarrow[\text{GLLPRV 2013}]{\text{SCFT/VOA corr.}}$ Vertex operator algebra $\mathcal{V}[\mathcal{T}]$

3d $N \geq 4$ SQFT $(\mathcal{T})$ $\xrightarrow[\text{Costello/Gaiotto 2018}]{\text{"deformable" B.C.}}$ Boundary vertex algebra $\mathcal{V}_\partial[\mathcal{T}_{A/B}]$

These vertex algebras come w/ various bells & whistles inherited from their higher-dim'l origins. It is an ongoing effort to characterise the class of algebras produced in this manner.

One immediate observation is that these vertex algebras are all *non-unitary*, even when $\mathcal{T}$ is unitary.

Vertex Operator Algebras

$$\begin{array}{c}
 \text{state space} \quad \text{vacuum} \\
 \downarrow \quad \downarrow \\
 (V, Y, \mathbb{1}, \omega) \\
 \uparrow \quad \uparrow \\
 \text{state/operator map} \quad \text{conformal vector}
 \end{array}$$

$$Y(a, z) = a(z) = \sum_{n \in \mathbb{Z}} \overset{\text{End}(V)}{a_{-1-n}^\omega} z^n$$

$$Y(\omega, z) = T(z) = \sum_{n \in \mathbb{Z}} L_{-2-n} z^n$$

in this talk

Always assume a vertex operator algebra of "strong CFT type":

- $V = \bigoplus_{n \in \frac{1}{2}\mathbb{N}} V_n$ $\sim$ L_0 -eigenvalues
- $V_0 = \text{span}\{\mathbb{1}\}$
- $L_1 V_i = 0$

Unique invariant bilinear pairing $(x, y) := \text{Res} \left(z \langle 0 | Y(e^{zL_1}(e^{i\pi} z^{-2})^{L_0} x, z^{-1}) y_{-1} | 0 \rangle \right)$
 (for x, y quasiprimary of weight h : $(i)^{2h} \cdot z^{2h} \langle x(z) y(0) \rangle$)

Unitary Vertex Operator Algebras

(formalized by C. Dong & X. Lin, 2013)

Let $\phi: V \rightarrow V$ $\mathbb{C}$ -antilinear involution ($\phi \circ \phi = \text{Id}$) such that

- $\phi(1) = 1$
- $\phi(\omega) = \omega$
- $\phi(a_n b) = (-1)^{|a||b|} \phi(a)_n \phi(b)$

Define $\langle a, b \rangle := (\phi(a), b)$. This is Hermitian.

$(V, Y, 1, \omega, \phi)$ **unitary** if $\langle -, - \rangle$ is positive definite.

Virasoro VOA unitary only if $c = c_{p,p+1}$ ($c_{p,q} = 1 - 6 \frac{(p-q)^2}{pq}$) or $c \geq 1$

Affine Kac-Moody current algebras unitary only for level $k \in \mathbb{N}$

} for $\forall [T] \quad c < 0 \ \& \ k < 0$

Why isn't $V[T]$ unitary (if T is)?

Recall cohomological construction of $V[T]$ for 4d $N=2$ SCFTs:

$$\mathcal{H}[S^3] \cong \widehat{V}_{loc} : \text{(completed) space of local operators}$$



$$\mathfrak{g} = \mathfrak{sl}(4|2) \cong \mathfrak{su}(2,2|2) \otimes \mathbb{C}$$

$$(\mathbb{R}^4 \cong \mathbb{C}_{z, \bar{z}} \times \mathbb{C}_{\omega, \bar{\omega}}) : \mathfrak{sl}(2)_z \times \overline{\mathfrak{sl}(2)}_{\bar{z}} \times \mathfrak{gl}(1)_{\omega, \bar{\omega}} \times \mathfrak{gl}(1)_r \times \mathfrak{sl}(2)_R$$

$$V_{loc} = \bigoplus_{n=1}^{\infty} \text{Irrep} \left[h_n, \bar{h}_n, \underbrace{l_n}_{j_e - j_r}, r_n, R_n \right]$$

Why isn't $V[T]$ unitary (if T is)?

Recall cohomological construction of $V[T]$ for 4d $N=2$ SCFTs:

[BPL²RvR (2013)]: $\exists$ two supercharges $\begin{cases} Q = Q_- + \tilde{S}_+^\dagger \\ Q^\vee = S_-^\dagger - \tilde{Q}_-^\dagger \end{cases}$ (graded by $gl(1)_r$) such that:

$$\triangleright [Q^{(\vee)}, sl(2)_z] = 0$$

$$\triangleright \{Q^{(\vee)}, *\} = \text{diag}(sl(2)_{\bar{z}} \times sl(2)_z)$$

$$\triangleright [Q^{(\vee)}, d_\omega + d_{\bar{\omega}}] \neq 0$$

Cohomological reductions agree $H_{\mathbb{Q}}^\bullet(Y_{loc}) \cong H_{\mathbb{Q}^\vee}^\bullet(Y_{loc})$

inherits $sl(2)_z$ -covariant $\widehat{sl(2)_{\bar{z}}}$ -invariant OPE algebra (aka vertex algebra) structure.

As vector space, cohomology represented by **Schur operators**:

$$H_{\mathbb{Q}(v)}^{\bullet}(V^{loc}) \cong (V_{Schur}^{loc})^{\bullet} \leftarrow \left. \begin{matrix} \bar{h} = R \\ \bar{\ell} = r \end{matrix} \right\} h = R + j_{\ell} + j_r \in \frac{1}{2}\mathbb{N}$$

$$= \bigoplus_{h, R} (V_{Schur}^{loc, \bullet})_{h, R}$$

Triple grading by h (chiral weight), $2r$ (cohomological degree), R (exotic).

State-operator map comes from **twisted translation**:

$$\mathcal{O}(z) = \left[e^{zL_{-1} + \bar{z}(\bar{L}_{-1} + R^{-})} \mathcal{O}(0) e^{-zL_{-1} - \bar{z}(\bar{L}_{-1} + R^{-})} \right]_{\mathbb{Q}(v)}$$

$$= \left[\mathcal{O}^{h, \omega}(z, \bar{z}) + \bar{z}[R^{-}, \mathcal{O}^{h, \omega}(z, \bar{z})] + \dots + \frac{\bar{z}^{2R_0}}{(2R_0)!} [(R^{-})^{2R_0}, \mathcal{O}^{h, \omega}(z, \bar{z})] \right]_{\mathbb{Q}(v)}$$

R-conservation
violated by OPE

Why isn't $V[\mathbb{T}]$ unitary (if $\mathbb{T}$ is)?

Full theory has a version of ϕ -involution:

$$\phi : V^{\text{loc}} \rightarrow V^{\text{loc}}$$

$$\mathcal{O} \mapsto \mathcal{O}^* \quad \curvearrowright \text{adjoint w.r.t. Hilbert space inner product in planar quantization.}$$

This is not compatible w/cohomological reduction

$$\phi : V_{\text{Schur}} \longrightarrow \overline{V_{\text{Schur}}} \neq V_{\text{Schur}}$$

$\Downarrow$

$$(\mathcal{O}_{h, \bar{h}, \ell, r, R})^* = (\mathcal{O}^*)_{h, \bar{h}, -\ell, -r, -R}$$

Rectify the situation by defining an "improved" conjugation:

$$\begin{aligned} \rho: \mathcal{V}^{\text{loc}} &\longrightarrow \mathcal{V}^{\text{loc}} \\ \Theta &\longmapsto e^{i\pi R_2} \Theta^* e^{-i\pi R_2} = (e^{i\pi R_2} \Theta e^{-i\pi R_2})^* \\ (\mathcal{V}_{\text{Schur}}^{\bullet})_{h,R} &\longrightarrow (\mathcal{V}_{\text{Schur}}^{-\bullet})_{h,R} \end{aligned}$$

No longer an involution in general:

$$\begin{aligned} \triangleright \rho^2 &= (-1)^{2R} \\ \triangleright \rho^4 &= \text{Id}. \end{aligned}$$

Compatible w/cohomological reduction in an interesting way:

$$\rho \circ \square = \square^{\vee} \circ \rho$$

Gives antilinear vertex algebra automorphism:

$$\rho(a_n b) = (-1)^{|a||b|} (\rho(a))_n \rho(b)$$

This makes it possible to see the imprint of 4d unitarity on VST :

$$\begin{aligned}
 \langle \rho(\Theta)(0) \Theta(z) \rangle &= \frac{1}{(2R_\Theta)!} \langle \rho(\Theta)(0) \rho(\Theta)^*(z, \bar{z}) \rangle \bar{z}^{2R_\Theta} \\
 &\quad \swarrow \text{sl}(2)_R \text{ lowest weight in multiplet } \psi/\Theta \\
 &= \bar{z}^{2R_\Theta} \times (\text{positive for } z \in -i\mathbb{R}_+) \\
 &\quad \swarrow \text{4d reflection positivity} \\
 &=: \frac{K_\Theta}{z^{2h_\Theta}}
 \end{aligned}$$

$$K_\Theta \bar{z}^{2(R_\Theta - h_\Theta)} = (\text{positive for } z \in -i\mathbb{R}_+)$$

$$\begin{aligned}
 (i) \quad & \bar{z}^{2(h_\Theta - R_\Theta)} K_\Theta > 0 \\
 & (-1)^{h-R} K_\Theta > 0 \text{ for bosonic ops}
 \end{aligned}$$

positivity condition depends on $\mathbb{Z}$ -grading

Because $\mathbb{Z}$ -charge is not conserved by OPE, best to introduce as a *Filtration*:

$$G_P V_h^\bullet = \bigoplus_{R \leq P} (V_{\text{Schur}}^\bullet)_{h,R} \quad V_h^\bullet = \bigcup_P G_P V_h^\bullet$$

$$\mathbb{C}^{\delta_{0,h}} \cong G_0 V_h^\bullet \subset G_1 V_h^\bullet \subset \dots$$

$$G_{1/2} V_h^\bullet \subset G_{3/2} V_h^\bullet \subset \dots$$

This is a *good increasing Filtration*:

$$\begin{array}{l} a \in G_P V \\ b \in G_Q V \end{array} \implies a_n b \in \begin{cases} G_{P+Q} V & n < 0 \\ G_{P+Q-1} V & n \geq 0 \end{cases}$$

$$a(z)b = \underbrace{\dots + \frac{a_{-h_a+z} b}{z^2} + \frac{a_{-h_a+1} b}{z^2}}_{R \leq P+Q-1} + \underbrace{a_{-h_a} b + z a_{-h_a-1} b + \dots}_{R \leq P+Q} \quad \mathcal{O}(z^2)$$

This filtration will be **non-degenerate**, in the sense that $(-, -)$ is nondegenerate on $g_p V$ for each p .

Remark : A non-deg. filtration on V can be refined to a grading via Gram-Schmidt orthogonalisation.

$$V_h = \bigoplus V_{h,p}^\bullet$$

Define $\sigma: V \rightarrow V$ by $\sigma|_{V_{h,p}^\bullet} = (i)^{2p}$ ($\mathbb{Z}_4$ refinement of $\mathbb{Z}_2$ $\mathbb{Z}$ -parity)

$\langle x | y \rangle := ((\sigma \circ \rho)(x), y)$ is non-degenerate & Hermitian

Def : $(V, g_\bullet, \rho)$ s.t. $\langle -, - \rangle$ is positive definite is called a **graded unitary vertex algebra**.

Adjoint

Adjoint w.r.t. $\langle - | - \rangle$ (for x quasi-primary, weight h) given by

$$x_n^* = \sigma \circ \rho \circ x_{2h-n} \circ \rho^{-1} \circ \sigma^{-1}$$

We define homogeneous (in $\mathbb{Z}$ -charge) components of x_n :

$$x_n = \sum_{p' \leq p} x_n^{[p']}, \quad x_n^{[p]} : \mathcal{V}_{m,q} \rightarrow \mathcal{V}_{m+h-n-1, q+p'}$$

Then adjoints are $(x_n^{[p]})^* = \underline{(i)}^{2p} (\rho(x))_{2h-n}^{[-p]}$

Some technical remarks:

- If we relax the goodness criterion on $G_\bullet$ (so only impose $a_n b \in G_{p+q-1} V$), rest of the construction still makes sense.

We call these *weak graded-unitary vertex algebras*.

- Weak graded unitary vertex algebras are what appears in general for boundary VOAs in 3d $N=4$ theories.

True for A-twisted gauge theories; First principles physics derivation lacking.

- *Unitary vertex algebras* are weak graded unitary vertex algebras with $G_0 V = V$.

Example - Symplectic Boson VOA (arising from free hypers)

(V, ω) a symplectic vector space/ $\mathbb{C}$ and $\{e_a\}$ basis w/ $\omega(e_a, e_b) =: \omega_{ab}$ unitary symplectic form

$\text{Sb}[V]$ freely generated by bosonic fields $q^{a=1, \dots, \dim(V)} \rightsquigarrow h=1/2$

$$q^a(z)q^b(w) = \frac{\omega^{ab}}{z-w} + :q^a(z)q^b(w):$$

$$q^a(z) = \sum_{n \in \mathbb{Z}} q_n^a z^{-n-1} \quad \text{w/} \quad [q_n^a, q_m^b] = \omega^{ab} \delta_{n+m+1, 0}$$

Graded Unitary structure

- $\mathcal{G}_{P/2} \text{Sb}[V] := \text{span} \{ q_{-1-n_1}^{a_1} \cdots q_{-1-n_k}^{a_k} | 0 \rangle ; n_i \geq 0, k \leq p \}$
- $\rho(q^a) = i\omega_{ab} q^b$
- $(q_{-1-n}^a)^* = \begin{cases} \omega_{ab} q_n^b & n \geq 0 \\ -\omega_{ab} q_n^b & n < 0 \end{cases}$ here each mode already homogeneous w.r.t. $\mathbb{Z}$ -grading

Positivity of $\langle -1- \rangle$ verified directly.

Example - Symplectic Fermion VOA (arising from free vectors)

(V, ω) a symplectic vector space/ $\mathbb{C}$ and $\{e_a\}$ basis $\omega/\omega(e_a, e_b) =: \omega_{ab}$ unitary symplectic form

$\text{Sf}[V]$ freely generated by fermionic fields $\eta^{a=1, \dots, \dim(V)} \rightsquigarrow h=1$

$$\eta^a(z)\eta^b(w) = \frac{\omega^{ab}}{(z-w)^2} + :\eta^a(z)\eta^b(w):$$

$$\eta^a(z) = \sum_{n \in \mathbb{Z}} \eta_n^a z^{-n-1} \quad \omega / [\eta_n^a, \eta_m^b] = n \omega^{ab} \delta_{n+m, 0}$$

Graded Unitary structure

- $\mathcal{H}_{P/2} \text{Sf}[V] := \text{span} \{ \eta_{-1-n_1}^{a_1} \cdots \eta_{-1-n_k}^{a_k} | 0 \rangle ; n_i \geq 0, k \leq p \}$
- $\rho(\eta^a) = i\omega_{ab}\eta^b$
- $(\eta_{-1-n}^a)^* = \begin{cases} \omega_{ab}\eta_n^b & n \geq 0 \\ -\omega_{ab}\eta_n^b & n < 0 \end{cases}$ here each mode already homogeneous w.r.t. $\mathbb{Z}$ -grading

Positivity of $\langle - | - \rangle$ verified directly.

Graded Unitarity for Virasoro VOA

Suppose $V[T] \cong \text{Vir}_c$ for some central charge $c \in \mathbb{R}$. How is c constrained by ~~four dimensional~~ graded unitarity?

Facts from 4d:

- $R[T] = 1$
- $\rho(T) = T$
- No other quasi-primaries w/ $R=1$

$$\text{Level 2: } \Theta = T, \quad (-1)^{\overset{h}{\downarrow} \underset{2-1}{\downarrow} \overset{R}{\downarrow}} K_T > 0 \quad \text{w/ } K_T = \frac{1}{2}c \longrightarrow c < 0$$

$$\text{Level 4: } \Theta = (TT) - \frac{3}{10}T'', \quad (-1)^{\overset{h}{\downarrow} \underset{4-2}{\downarrow} \overset{R}{\downarrow}} K_\Theta > 0 \quad \text{w/ } K_\Theta = \frac{c(22+5c)}{10} \longrightarrow c \leq -\frac{22}{5} = c_{2,5}$$

$\uparrow R=2$

Continuing requires more input/assumptions about R -filtration.

Graded Unitarity for Virasoro VOA

Def: the standard $\mathbb{Z}$ -filtration on Vir_c is given by

$$\mathcal{G}_p^{(\text{st})} \text{Vir} = \text{span} \{ L_{-2-n_1} \dots L_{-2-n_k} | 0 \rangle \mid k \leq p \}$$

Thm : $(\text{Vir}_c, \mathcal{G}_\bullet^{(\text{st})}, \rho = \text{Id})$ is not graded unitary for $c \neq c(2, 2n+3)$ $n \in \mathbb{N}$

Proof amounts to checking sign of Kac determinant against dictates of graded unitarity.

At level $2n$: $c \notin (c_{2, 2n+1}, c_{2, 2n-1}) \implies$ all values except $c_{2,p}$ ruled out.

Cases not ruled out are exactly those realized by (A_1, A_{2n}) Argyres-Douglas SCFTs.

Graded Unitarity for $\widehat{\mathfrak{sl}(2)}$ AKM

Now suppose $\forall [\tau] \in (\widehat{\mathfrak{sl}(2)})_k$ for some $k \in \mathbb{R}$ $\left(J^a(z) J^b(\omega) \sim \frac{k \delta^{ab}}{(z-\omega)^2} + \frac{i J^a_b J^c(\omega)}{(z-\omega)} + \dots \right)$

Facts from 4d:

- $\mathbb{R}[J^a] = 1$
- $\rho(J^a) = -J^a$
- $\mathbb{R}[T_{\text{sug}} \sim J^a J_a] = 1$

Level 1: $\mathcal{O} = J^A$, $(-1)^{\overset{h}{\downarrow} \overset{R}{\downarrow} 1-1} K_{J^A} > 0$ where $K_{J^A} = -K \rightarrow K < 0$

Level 2 (weight zero): $\mathcal{O} = \frac{1}{k+2} (J^A J_A)$, $(-1)^{2-1} K_{\mathcal{O}} > 0$ where $K_{\mathcal{O}} = C_{\text{sug}} = \frac{3K_{2d}}{K_{2d}+2} \rightarrow -2 < K < 0$

Again continuing requires more input/assumptions about $\mathbb{R}$ -filtration.

Graded Unitarity for $\widehat{\mathfrak{sl}(2)}$ AKM

Def: the standard $\mathbb{Z}$ -filtration on $V_K(\mathfrak{sl}_2)$ is given by

$$\mathcal{G}_p^{(st)}(\widehat{\mathfrak{sl}(2)}) = \text{span} \left\{ J_{-l-m_1}^{A_1} \cdots J_{-l-m_K}^{A_K} \underbrace{L_{-2-n_1} \cdots L_{-2-n_\ell}}_{\text{Sugawara stress tensor modes}} |0\rangle \mid K+\ell \leq p \right\}$$

Thm: $(V_K(\mathfrak{sl}_2), \mathcal{G}_\bullet^{(st)}, \rho(J^A) = -J^A)$ is not graded unitary for $K \neq K(2, 2n+1) = -2 + \frac{2}{2n+1}$

Proof again amounts to checking sign of Kac determinant against dictates of graded unitarity.

At level $2n+1$ ($\mathfrak{sl}_2$ -weight zero) $K \notin (K_{2,2n+1}, K_{2,2n-1}) \implies$ all values except $c_{2,p}$ ruled out.

Cases not ruled out are exactly those realized by (A_1, D_{2p+3}) Argyres - Douglas SCFTs.

Comments

- ① Similar exclusions for $\widehat{\mathfrak{sl}}(3)$, $\widehat{\mathfrak{sl}}(4)$ AKM algebras, W_N algebras, though at higher rank some not all "allowed" levels have known realisations.
- ② Though surely true, we do not have a **positive proof** of graded unitarity in the non-excluded examples. Seems much harder!
- ③ Allowed AKM/ W_N algebras related by quantum Drinfel'd-Sokolov reduction. Tracking graded unitarity seems challenging.
- ④ Assuming the $\mathbb{R}$ -filtration is not optimal. In fact, there are examples where Virasoro/AKM VOAs at other levels will be compatible with graded unitarity (w/different $\mathbb{R}$ -filtration)!

In all such examples I know of, the VOA can be extended (Vir/AKM realised by finite gauging). Relevance of this observation is unclear.

For gauge theories, corresponding VOAs are realised via BRST/semi-infinite cohomology:

- G_c : compact, semi-simple group
- $\mathfrak{g}_c = \text{Lie}(G_c)$
- $\{t_a\}$ be a basis for $\mathfrak{g} = \mathfrak{g}_c \otimes \mathbb{C}$ with $[t_a, t_b] = f_{ab}^c t_c$

Suppose V_{mat} is vertex algebra w/morphism $V^{K=-2h^\vee}(\mathfrak{g}_c) \xrightarrow{\mu} V_{\text{mat}}$

(relative) BRST complex: $V = (V_{\text{mat}} \otimes \text{St}(\mathbb{C}^2 \otimes \mathfrak{g}))^{G_c}$
 $\uparrow$
 outer $\text{USp}(2)$ commutes w/ G_c

Differentials: $Q^\pm = \sum_{n \neq 0} \frac{1}{n} : \eta_{-n}^{\pm A} J_{A,n} : + \sum_{\substack{m,n \neq 0 \\ m \neq n}} \frac{f_{ABC}}{2mn} : \eta_{-n}^{\pm A} \eta_{-m}^{\pm B} \eta_{n+m}^{\mp C} : \quad \text{outer USp}(2)$

Relative semi-infinite cohomology: $H^\bullet(V, Q^-) = H^{\bullet + \frac{\dim}{2}}(\hat{\mathfrak{g}}_{-2h^\vee}, \mathfrak{g}, V_{\text{mat}})$

Suppose V_{mat} is graded-unitary. We say $\hat{g}$ -action is *good* if $\mu(J_{A,-1}|0\rangle) \in \mathfrak{g}_\perp V_{\text{mat}}$.

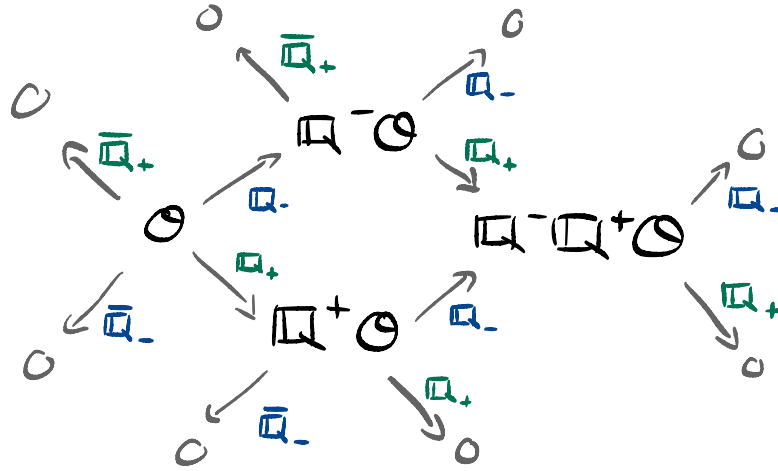
Then decomposition of modes takes the form $J_{A,n} = \underbrace{J_{A,n}^{[1]}}_{=0 \text{ for } n \geq 0 \text{ (goodness)}} + J_{A,n}^{[0]} + \underbrace{J_{A,n}^{[-1]}}_{=0 \text{ for } n \leq 0 \text{ (adjoint of } J_{A,n}^{[1]})}$

This simplifies decomposition of differentials $Q^\pm = \underbrace{Q^\pm}_{Q^\pm}^{[1/2]} + \underbrace{Q^\pm}_{S^\pm}^{[-1/2]}$ $(Q^\alpha)^* = \epsilon_{\alpha\beta} S^\beta$
 $(S^\alpha)^* = -\epsilon_{\alpha\beta} Q^\beta$

Letting $\overline{Q}_\alpha = (Q^\alpha)^*$ (i) $\{Q^\alpha, Q^\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0$ (ii) $\{Q^\alpha, \overline{Q}_\beta\} = \delta_\beta^\alpha \Delta$ } $V = \text{Ker } \Delta \oplus \text{im } Q^1 \oplus \text{Ker } Q_1$
 $= \text{Ker } \Delta \oplus \text{im } Q^2 \oplus \text{Ker } Q_2$

analogous to algebra of $\{\partial, \bar{\partial}, \partial^\dagger, \bar{\partial}^\dagger\}$ on $\Omega^{\bullet,\bullet}(M)$ for M a compact Kähler manifold.

Quartet mechanism / $\mathbb{Q}^+ \mathbb{Q}^-$ lemma



V decomposes as direct sum of quartets and singlets $\cong H^*(V, \mathbb{Q}^\pm)$.

(this exactly parallels consequence of "recombination rules" for unitary representations of 4d $\mathcal{N}=2$ superconformal algebra.)

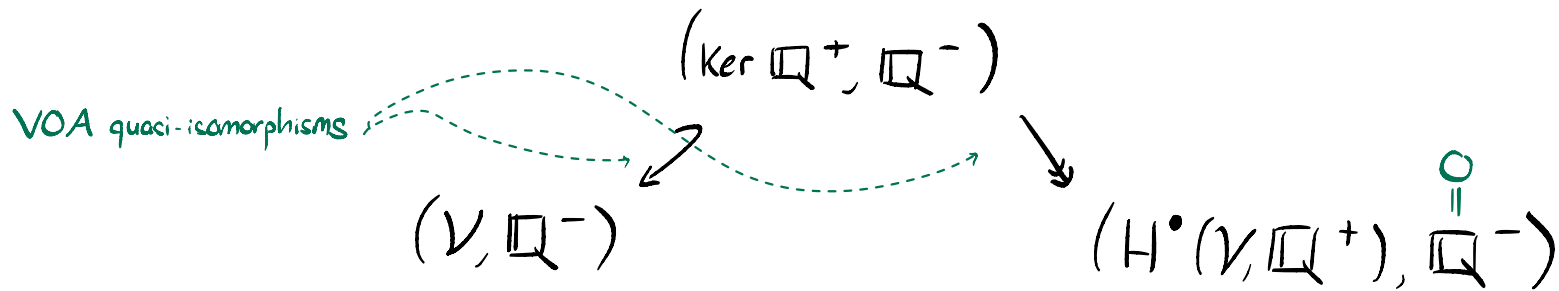
$$\frac{\text{Ker}(\mathbb{Q}^-)}{\text{im}(\mathbb{Q}^-)} \cong \frac{\text{Ker}(\mathbb{Q}^-) \cap \text{Ker}(\mathbb{Q}^+)}{\text{im}(\mathbb{Q}^- \mathbb{Q}^+)} \quad \hookrightarrow \text{USp}(2) \cong \text{SL}(2)$$

outer aut. of $\text{St}(\mathfrak{g} \otimes \mathbb{C}^2)$

Thm: $H^{\frac{3}{2}+*}(\hat{\mathfrak{g}}, \mathfrak{g}; V_{\text{mat}})$ admits outer "Lefschetz" $\text{SL}(2)$ w/ Cartan given by cohomological degree.

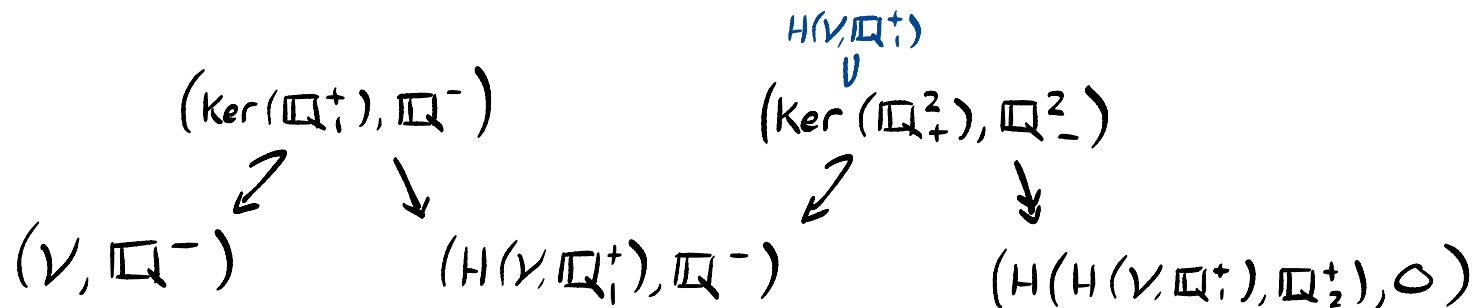
Thm: $H^{\frac{3}{2}+*}(\hat{\mathfrak{g}}, \mathfrak{g}; V_{\text{mat}})$ inherits a graded unitary structure from V_{mat} .

"Formality"



Analogous to formality of compact Kähler manifolds [Deligne-Griffiths-Morgan-Sullivan '75]
 (Passing to $gr_g V$ this implies formality of certain finite-type derived symplectic reductions)

For $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathbb{Q}^\pm = \mathbb{Q}_1^\pm + \mathbb{Q}_2^\pm$



So gauging "in stages" matches gauging "all at once".

Final Remarks / Open Directions

- At present, classification results require $\mathbb{R}$ -filtration as input. Is there a better first principles theory?
- Do non-Lagrangian examples admit manifestly graded-unitary presentation?
- Behaviour under DS-reduction?
- Extension to modules/surface operators (in progress w/ N. Garner)
- $4d \rightarrow 3d$ reduction & connection to short, positive star products on symplectic singularities?

...and so on...

Thanks for your attention