S-matrix bootstrap with unequal particles

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Introduction and motivation

Since the seminal papers by Paulos, Penedones, Toledo, van Rees, Vieira [1607.06109,1607.06110], after about 60 years, a renaissance period of the S-matrix bootstrap started, in a time where technology allows numerical implementations.

An impressive number of interesting bounds have been obtained so far, using mostly the so called primal method (construct solutions, no rigorous exclusions).

[Guerrieri, Penedones, Vieira, 1810.12849, 2011.02802; 2102.02847; Cordova et al, 1909.06495; Hebbar, Karateev, Penedones, 2011.11708; Haring et al, 2211.05795; many others …]

In d=4 space-time dimensions, most analysis focused on 2-2 scattering of the lightest particle(s): $AA \to AA$

Good reason: this elastic scattering is the one where we have more control, full crossing is trivial to implement, and we can assume maximal analyticity.

Maximal analyticity: In the principal sheet the amplitude is analytic everywhere outside the normal threshold physical cuts on the real line (in both s and t) + possible poles due to bound states.

It is important to set an S-matrix bootstrap program for different particles to enlarge its applicability. This has been explored in 2d [Homrich et al, 1905.06905], less so in 4d.

Extending the S-matrix bootstrap with particles with different masses m (A) and M (B), M>m, is not a straightforward generalization, because strict maximal analyticity no longer holds.

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In particular, we can have

- Anomalous thresholds
- Pseudothresholds

Moreover, when $M \gg m$, **inelasticities** are generally non-negligible.

Studying the set of amplitudes

$$AA \to AA, AA \to BB, BB \to BB, AB \to AB$$

might no be enough to identify **physical extremal amplitudes**
[Aks, 1965; Correia, Sever, Zhiboedov, 2006.08221]

However, if appropriate global symmetries are imposed, the channel

$$AB \to AB$$

remains elastic for any mass ratio (in the appropriate energy range).

Extremal amplitudes of this kind could be useful for example in pion-Kaon or pion-nucleon scattering in QCD.

Our main aim:

Study the $AB \to AB$ amplitude and shows that numerical bounds can be found (stronger than positivity bounds)

Plan

Introduction

Set-up of the scattering amplitude
and its analytic structure

Numerical results

Conclusions

Set-up of the scattering amplitude and its analytic structure

We take both A and B to be scalar particles.

We assume a $\mathbf{Z}_2 \times \mathbf{Z}_2$ global symmetry with A$(-,+)$ and B$(+,-)$.

Stability of B is ensured, no bound states are possible (all trilinears vanish), no anomalous thresholds from (effective) triangular diagrams. We assume that this is enough to not have anomalous thresholds at the non-perturbative level.

$$A(p_1)B(p_2) \to A(p_3)B(p_4)$$

$$\begin{cases} s = -(p_1+p_2)^2 = \left(\sqrt{M^2+p^2}+\sqrt{m^2+p^2}\right)^2 , \\ t = -(p_1-p_3)^2 = -2p^2(1-z) , \\ u = -(p_1-p_4)^2 = \Sigma - s - t . \end{cases} \qquad \Sigma \equiv 2m^2+2M^2$$

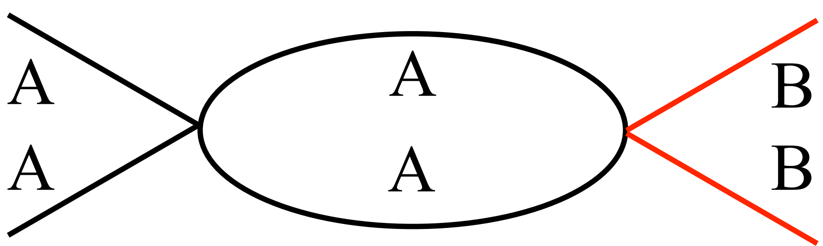
$$z = \cos\theta = 1 + \frac{2st}{(s-m_+^2)(s-m_-^2)} , \qquad m_\pm \equiv M \pm m$$

Physical kinematical region is $s \geq m_+^2\,, \qquad t \leq 0\,, \qquad u \leq m_-^2$

s-u, u-t crossing: $$T_{AB\to AB}(s,t,u) = T_{AB\to AB}(u,t,s)\,,$$ $$T_{AB\to AB}(s,u,t) = T_{AB\to BA}(s,t,u)\,.$$

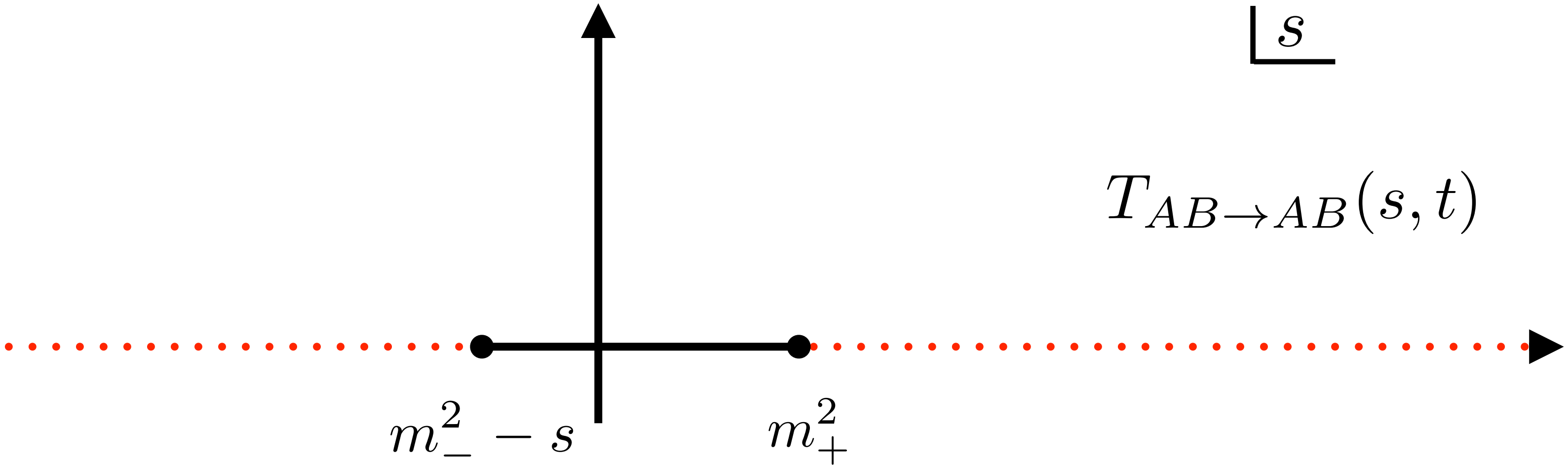
s-t crossing: $T_{AB\to AB}(t,s,u) = T_{AA\to BB}(s,t,u) = T_{BB\to AA}(s,t,u)$

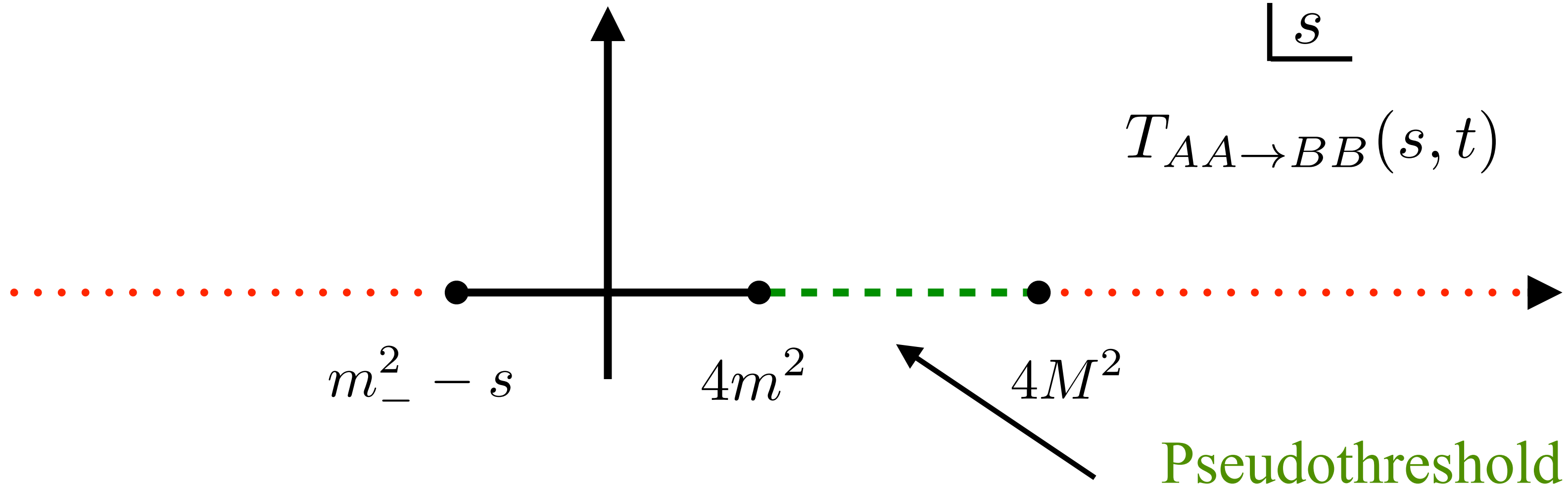
In the s-channel, the first non-analyticity occurs at $s = 4m^2$, **before** the physical threshold at $s = 4M^2$.



We denote by **pseudothreshold** the branch-cut occurring between $4m^2$ and $4M^2$.

We assume that normal and pseudothresholds are the only non-analyticities of the amplitude





Singularities in the s, u, t $AB \to AB$ channels are then

$$s_{\rm th} = m_+^2\,, \qquad u_{\rm th} = m_+^2\,, \qquad t_{\rm th} = 4m^2$$

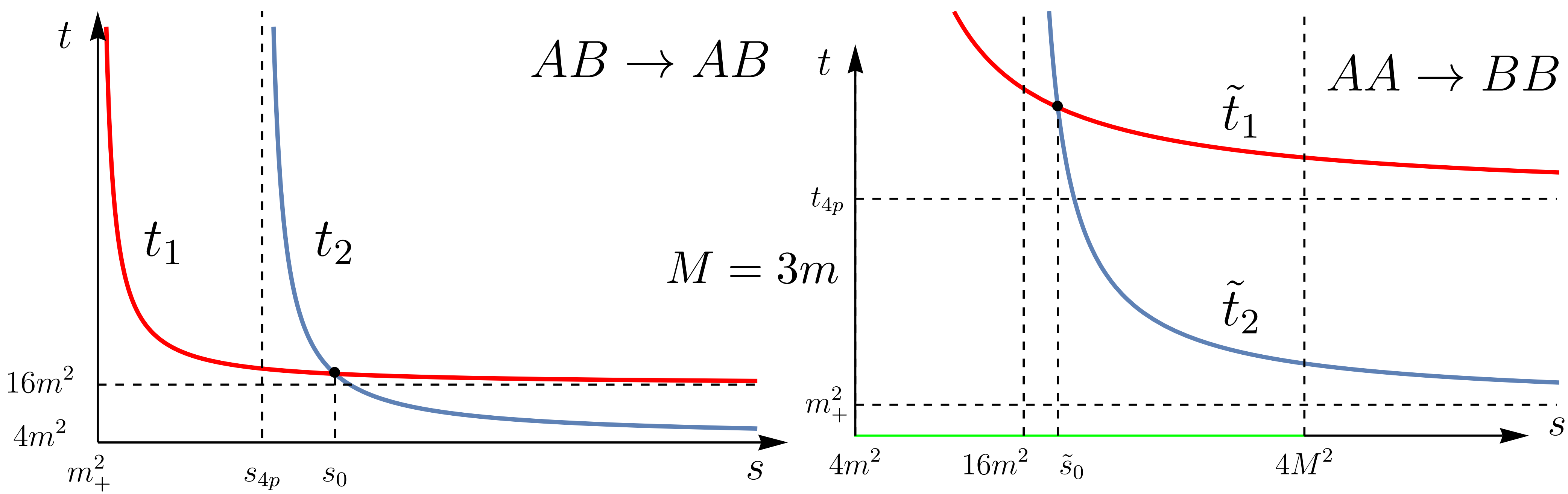
Partial waves: $\quad T^{\ell}_{AB\to AB}(s) \sim \int_{-1}^{1} dz\, T_{AB\to AB}(s, t(z,s)) P_\ell(z)\,.$

Due to the existence of the pseudothreshold, $f^{\ell}_{AB\to AB}$ have a **circular cut** around the origin with radius $R = m_+ m_-$

The convergence of the partial wave expansion (and its imaginary part) are given in terms of the small and large Lehmann ellipsis in the z-plane, with semi-major axis

$$z_s^{\mathrm{SL}} = z(t = 4m^2, s)\,, \qquad z_s^{\mathrm{LL}} = z(t = \min(t_1, t_2), s)\,, \quad AB \to AB$$

$t_{1,2}$ are the leading Karplus curves delimiting the region where $\mathrm{dDisc}_{st} \neq 0$



$$s_{4p} = t_{4p} = (3m + M)^2$$

Unitarity

In terms of $S^{\ell}_{AB\to AB} = 1 + iT^{\ell}_{AB\to AB}(s)$

we have $|S^{\ell}_{AB\to AB}(s)| \leq 1 \qquad s \geq m_+^2$

Without taking account $AA \to AA$ and $BB \to BB$ processes, unitarity in neutral channels reduce to

$$|S^{\ell}_{AA\to BB}(s)| \leq 1 \qquad s \geq 4M^2$$

For $4m^2 \leq s \leq 4M^2$ the amplitudes satisfy **extended unitarity**

$$\mathrm{Im}\, T^{\ell}_{ij} \sim \sum_k T^{\ell}_{ik} T^{\ell *}_{kj}$$

Its implementation requires the knowledge of other channels

Observables

We set bounds to **couplings**, defined as

$$\lambda_{k,l} \equiv \frac{m^{2k+2l}}{k!l!}\frac{\partial^k}{\partial s^k}\frac{\partial^l}{\partial t^l}T_{AB\to AB}(s_0,t_0,s_0)$$

$$s_0 \equiv \frac{m^2}{3}+M^2\,, \qquad t_0 \equiv \Sigma - 2s_0 = \frac{4}{3}m^2\,.$$

Focus on lowest independent couplings $\lambda_{0,0}, \lambda_{2,0}, \lambda_{2,1}$

Using dispersion relations and linear unitarity $\operatorname{Im} T^{\ell}_{AB\to AB}(s) > 0$

$$\lambda_{2,0} \geq 0\,, \qquad \lambda_{2,1} \geq -\frac{9m}{4(m+3M)}\lambda_{2,0}\,.$$

Positivity bounds

We also consider **scattering lengths**, defined as

$$\operatorname{Re} T^{\ell}_{AB\to AB}(s) = \left(\frac{q}{m}\right)^{2\ell+1}\left[a_\ell + O\Big(\big(\frac{q}{m}\big)^2\Big)\right]$$

Focus on the scalar scattering length a_0

Numerical results

We use the "primal S-matrix bootstrap", easier to handle than the "dual" one.

The ansatz is constructed using rho-variables (rho-ansatz in short) [Paulos et al, 1708.06765]

$$\rho_x(z) = \frac{\sqrt{x^2 - z_0} - \sqrt{x^2 - z}}{\sqrt{x^2 - z_0} + \sqrt{x^2 - z}}$$

$$T_{AB\to AB}(s,t,u) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \alpha_{a,b,c}(\rho_{m_+}(s))^a(\rho_{2m}(t))^b(\rho_{m_+}(u))^c$$

$$\alpha_{a,b,c} = \alpha_{c,b,a}\,, \qquad \alpha_{a,b,c} = 0 \qquad \text{if } abc \neq 0$$

Unitarity constraints are set on the partial waves $T^{\ell}_{AB\to AB}(s)$ and $T^{\ell}_{AA\to BB}(s)$

Truncations: $a+b+c \leq N_{\max}, \qquad \ell \leq L_{\max}$

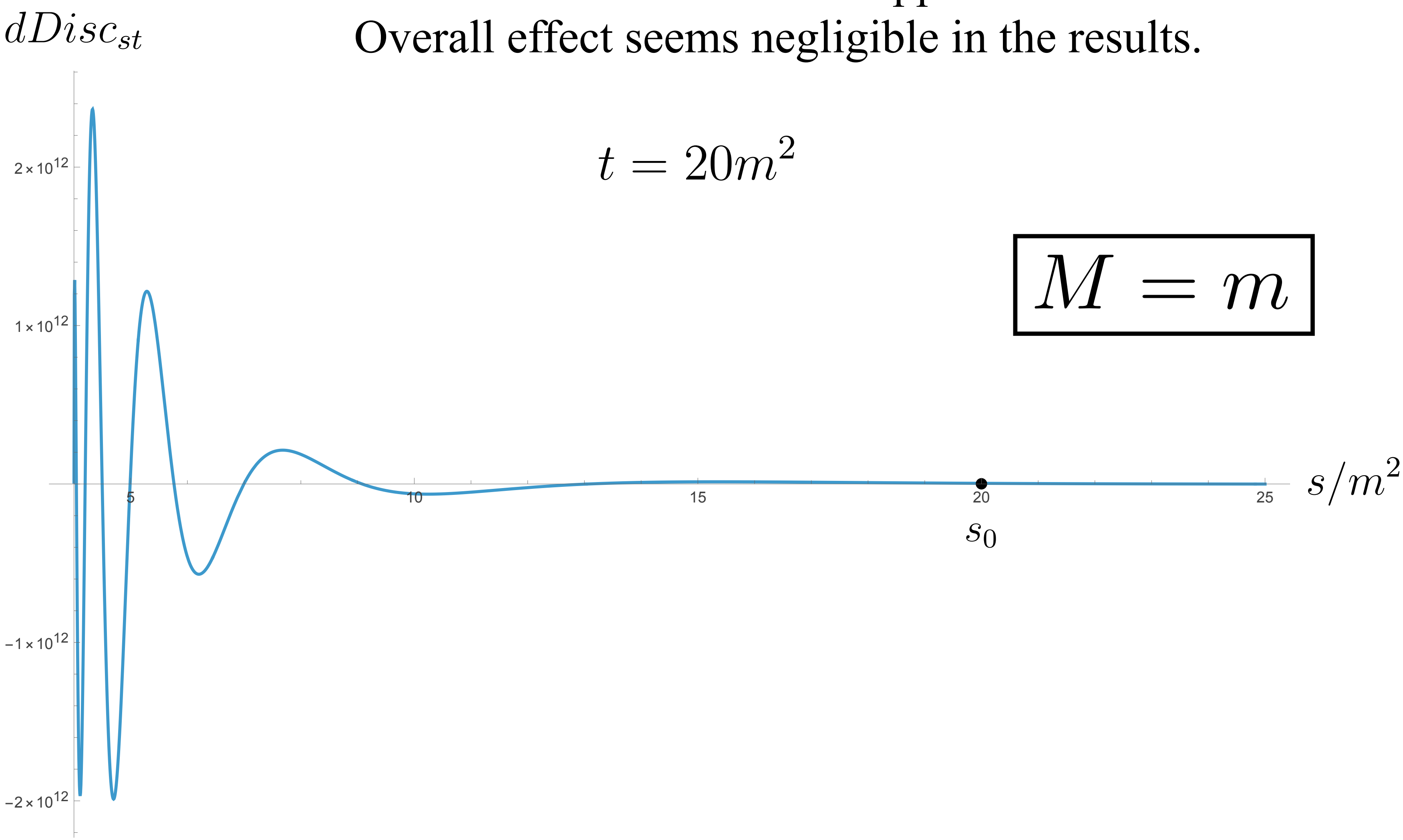
Three values of mass ratios are studied: $M = 1.5m \quad M = 3.554m \quad M = m$

(Kaon/pion mass ratio)

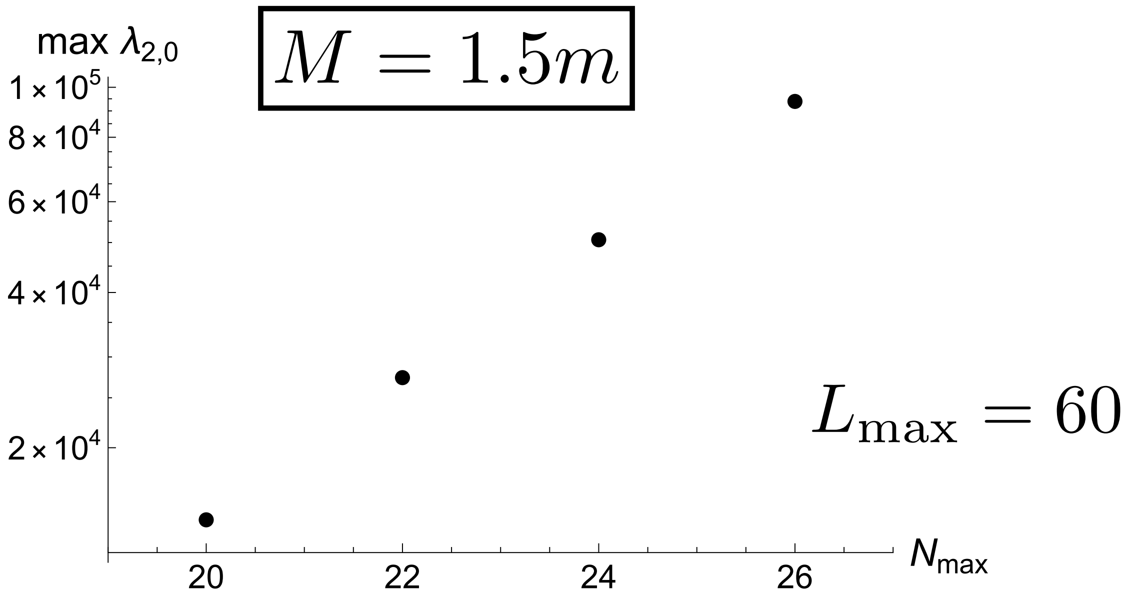
For $s > (M+m)^2 = m_+^2$, $t > 4m^2$, $\quad \rho_s(s) = e^{i\theta_s(s)}$, $\quad \rho_t(s) = e^{i\theta_t(t)}$.

$$\mathrm{dDisc}\, T_{AB\to AB}(s,t,u) = \sum_{a,b} \alpha_{a,b,0} \sin\big(a\theta_s(s)\big) \sin\big(b\theta_t(t)\big)$$

dDisc oscillates where it is supposed to vanish.
Overall effect seems negligible in the results.

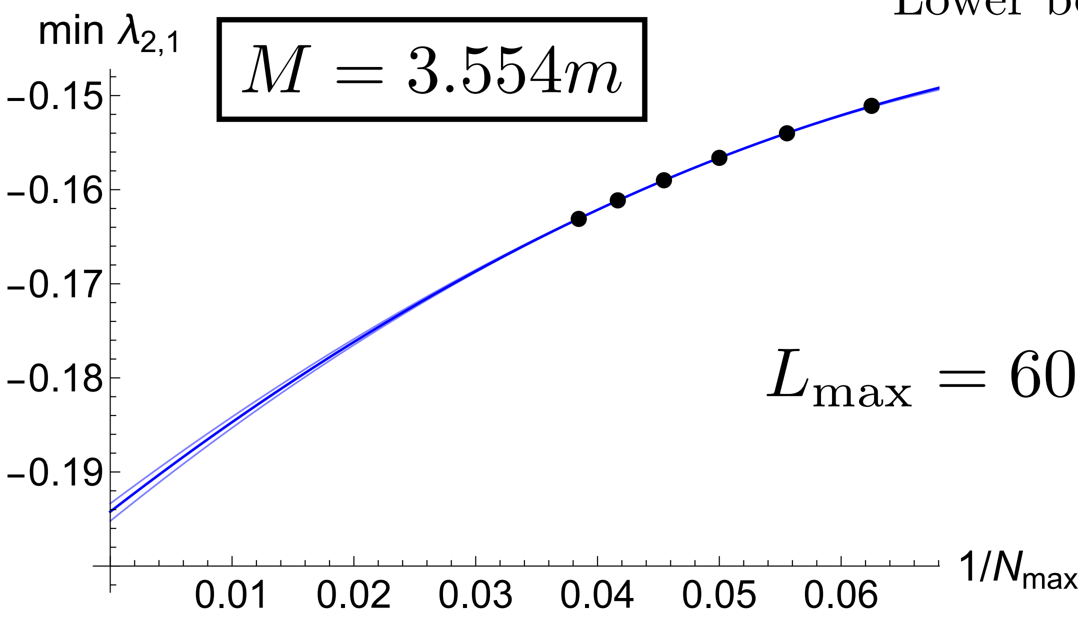


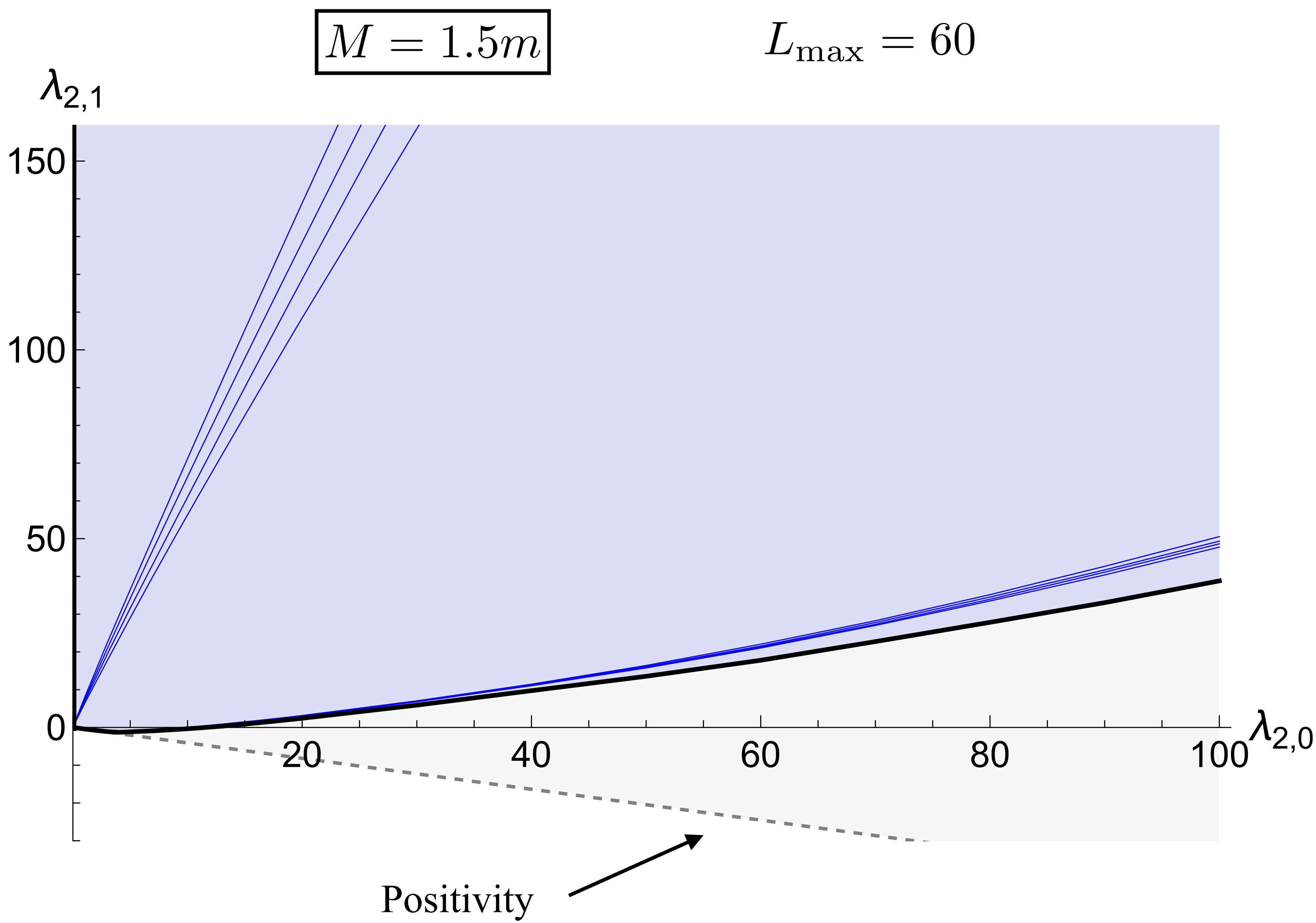
The coupling $\lambda_{0,0}$ is unbounded, $\lambda_{2,0}$ has no upper bound, but it has a lower bound



Similarly for $\lambda_{2,1}$

Lower bound of $\lambda_{2,0}$ compatible with zero.

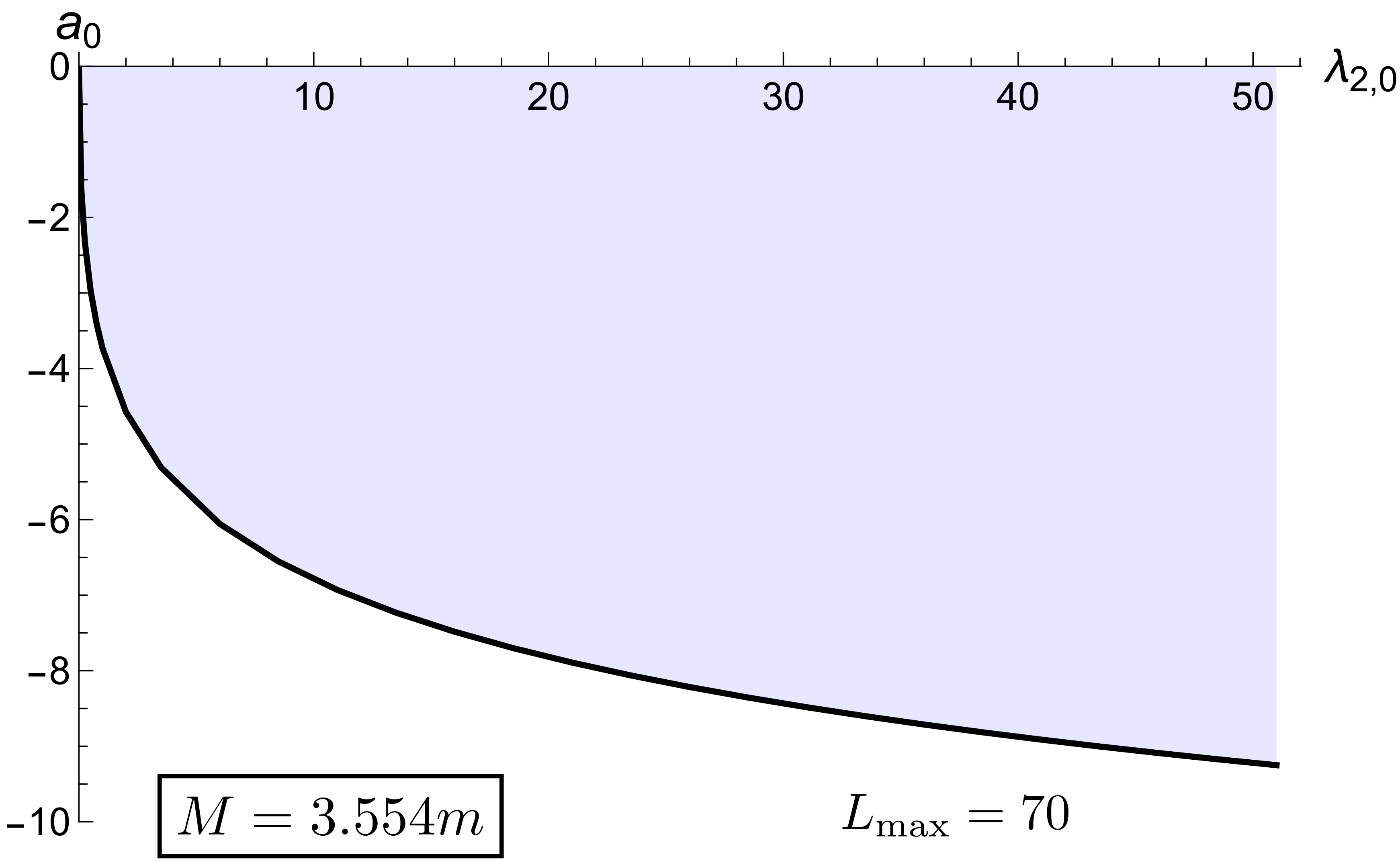




$\boxed{M = 1.5m}$
$L_{\max} = 60$
$\lambda_{2,1}$
150
100
50
0
20
40
60
80
100
$\lambda_{2,0}$
Positivity

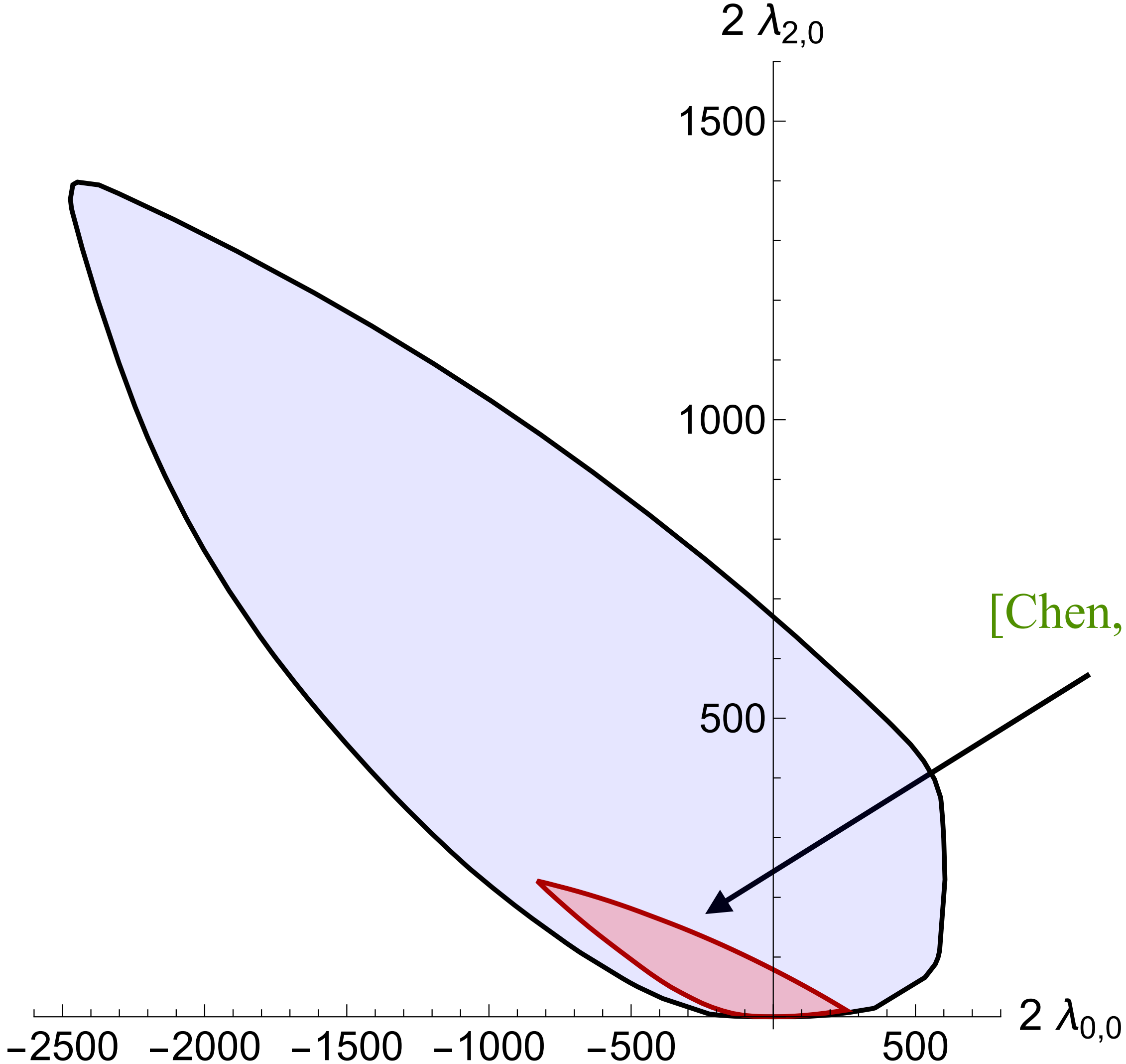
No evidence of upper and lower bounds on a_0 in general

Instead, at fixed $\lambda_{2,0}$ we get an **exponentially** convergent lower bound on a_0.



M=m

Pseudo threshold shrinks to a point and better bounds are found, despite particles are unequal.



Identical particles

[Chen, Fitzpatrick, Karateev, 2207.12448]

Taming the pseudothreshold

Assumptions on the amplitude in the pseudothreshold region change the picture.

Dominance of scalar resonance (M<2m).

Analytically implemented using the Muskhelishvili-Omnès method.

Extended unitarity requires

$$T^\ell_{AA\to BB}(s) = \left|T^\ell_{AA\to BB}(s)\right| e^{i\delta^\ell(s)}\,, \qquad 4m^2 \leq s \leq 4M^2$$

where $$S^\ell_{AA\to AA}(s) = e^{2i\delta^\ell(s)}$$

$$T^\ell_{AA\to BB}(s) = P^\ell(s)\Omega^\ell(s) \qquad \Omega^\ell(s) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{+\infty} ds'\,\frac{\delta^\ell(s')}{s'(s'-s)}\right)$$

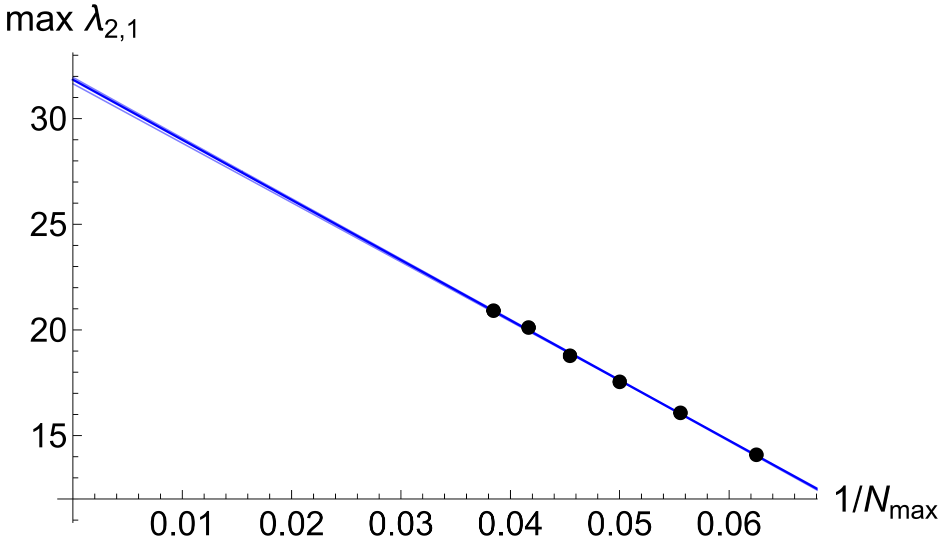
For scalar resonance $$\Omega^0(s) = \frac{m_0^2}{s - m_0^2 + g^2 C(s)}$$

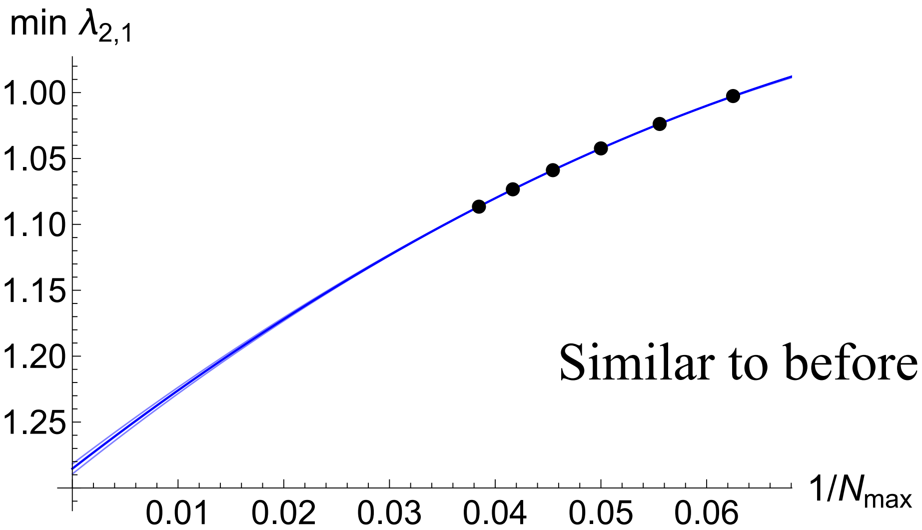
$$C(s) = \frac{1}{\pi}\left(2 + r(s)\log\frac{r(s)-1}{r(s)+1}\right)\,, \qquad r(s) = \sqrt{1 - \frac{4m^2}{s}}\,.$$

Rho-ansatz is now

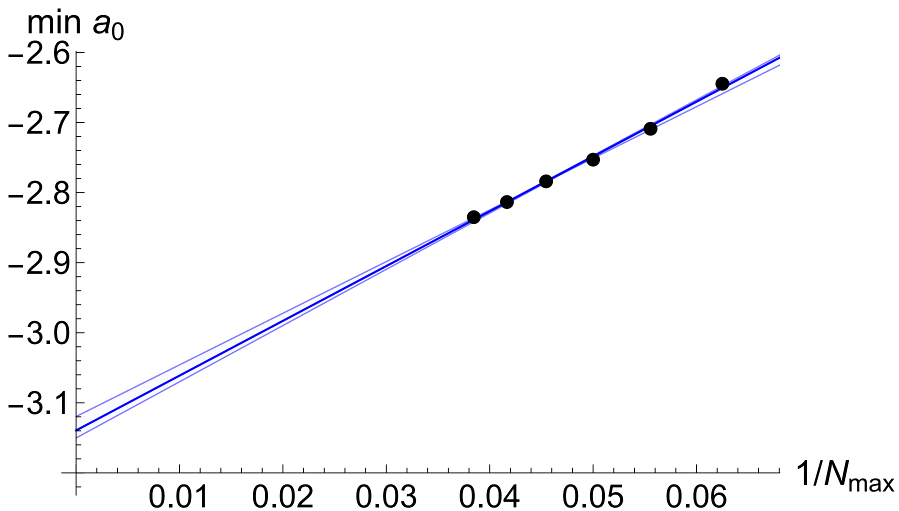
$$T_{AB\to AB}(s,t,u) = \alpha_0\,\Omega^0(t) + \sum_{a,b,c} \alpha_{a,b,c}(\rho_{m_+}(s))^a(\rho_{2M}(t))^b(\rho_{m_+}(u))^c$$

upper and lower bounds for couplings, lower bounds for scattering length





Similar to before



$$\boxed{M = 1.5m}$$

$$L_{\max} = 60$$

Conclusions

We have shown that the S-matrix bootstrap can set model-independent bounds on 4d amplitudes $AB \to AB$ which feature non-analyticities beyond physical thresholds.

Bounds weaker than those for identical particles.

Stronger bounds are expected if one considers the full system of 2-2 amplitudes involving A and B states. Using $AA \to AA$ and extended unitarity, in particular, amplitudes can be bounded also in the pseudothreshold region.

Future developments

If one is interested to look for a specific theory as an **extremal** solution, there is a theoretical advantage of neglecting other channels. Extremal amplitude is generally elastic and the inelasticities in the full system are bigger that those in the single channel.

Next natural targets are

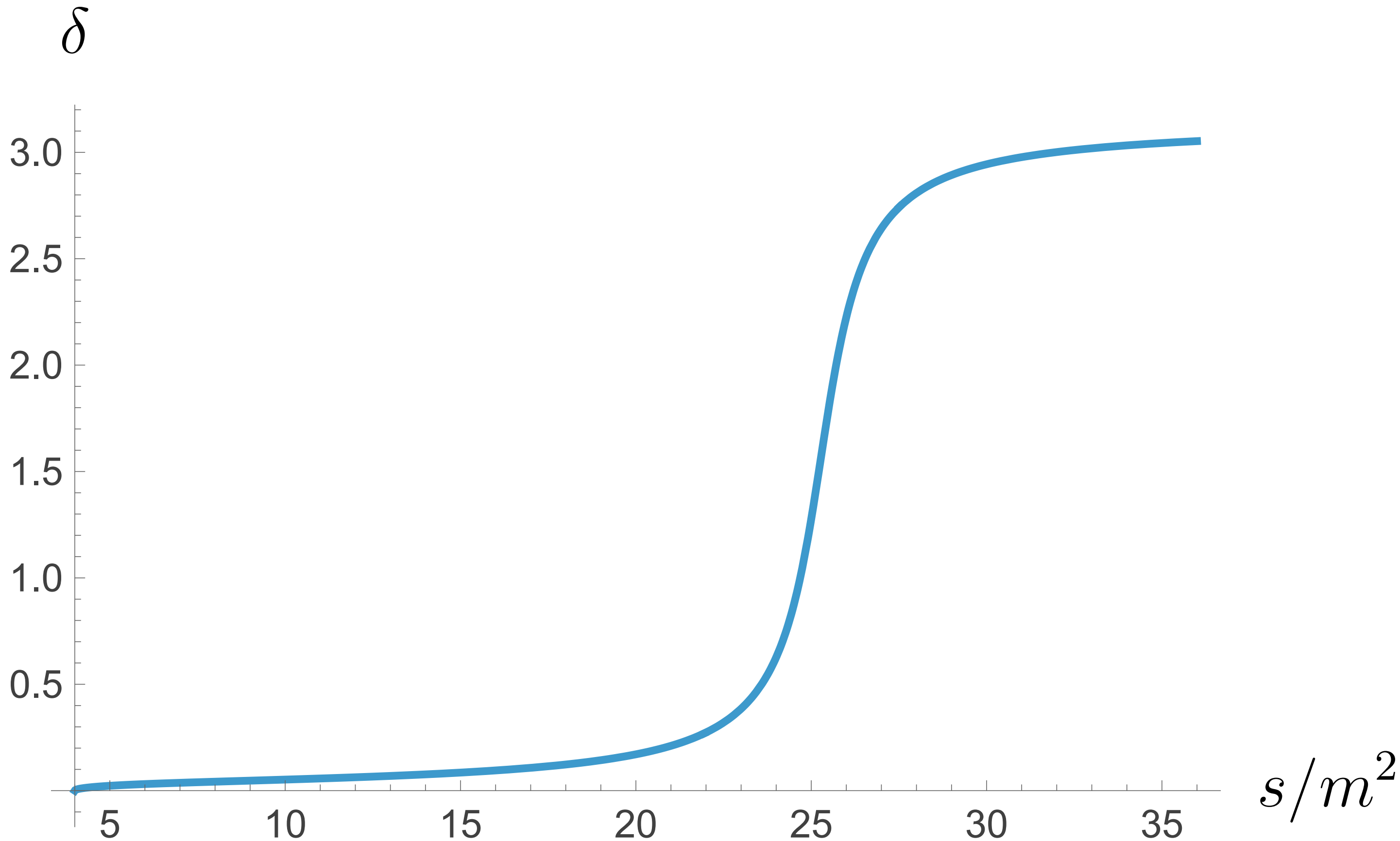
$\pi K \to \pi K$, $\pi N \to \pi N$ scattering in QCD

Input data to make QCD extremal in some parameter and study the resulting amplitude.

Thank You

Back up slides

$$m_{\rm res} = 5m \qquad\qquad g = 1$$



Non-analyticity of partial waves

$z \in [-1, 1]$ $$t = \frac{z-1}{2s}\left(s - m_+^2\right)\left(s - m_-^2\right)$$

As s varies, integral over z can hit a t where the amplitude is non-analytic.

$$t = t_s = 4m^2,\ s = r\exp(i\omega)$$

$$\operatorname{Im} z(-1) = 0 \implies \sin\omega\left(r - \frac{(M^2 - m^2)^2}{r}\right) = 0\,.$$

$r = M^2 - m^2$ is the circular cut (due to pseudothreshold)