The Bootstrap Program for Large N QCD
Part I: Overview and Form Factors

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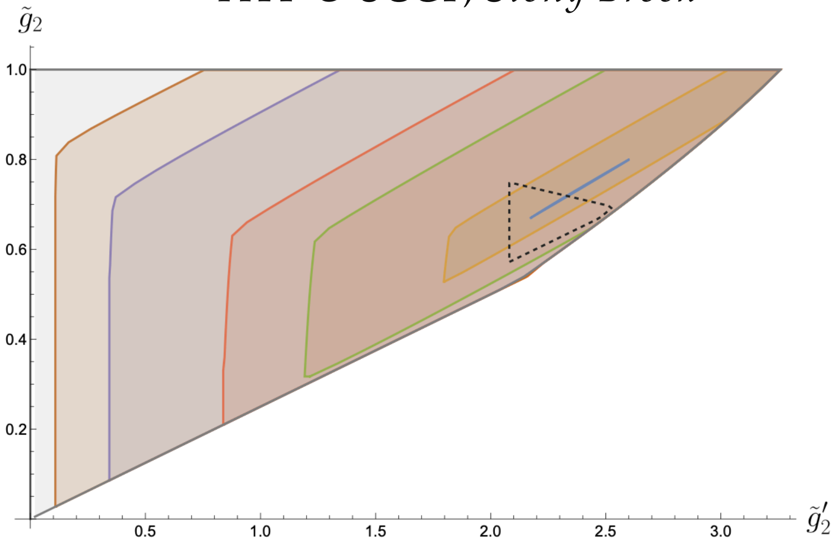


Based on work with

Jan Albert, Johan Henriksson, Dilara Kosva and Alessandro Vichi

Bootstrap 2026
London

Large N QCD

In the real world, $N = 3$ colors (gauge group of QCD is $SU(3)$).

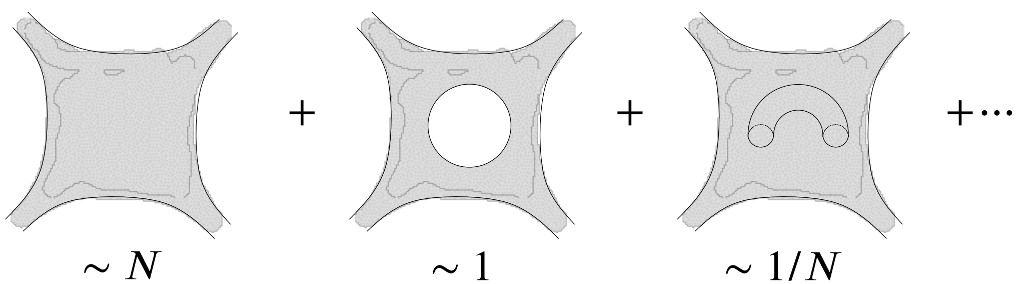
`t Hooft (1974): take the number of colors $N \to \infty$

Finite N

Hadrons unstable (e.g. $\rho \to \pi\pi$)
Branch cuts, resonances.
S has a very complicated analytic structure

Large N

Infinite towers of stable hadrons
Only poles (no cuts)
Meromorphic amplitudes

$\mathcal{M}_{4\text{pt}} =$ 

`t Hooft expansion for meson scattering

Carving out the space of large N theories

Any confining gauge theory at $N = \infty$ has an infinite tower of stable hadrons.
Meromorphic amplitudes.

Consistency constraints on masses, spins and on-shell 3pt couplings $\{m_k, J_k; \lambda_{ijk}\}$
Analogy with conformal bootstrap, with data: $\{\Delta_k, J_k; \lambda_{ijk}\}$.

Large N S-matrix Bootstrap:

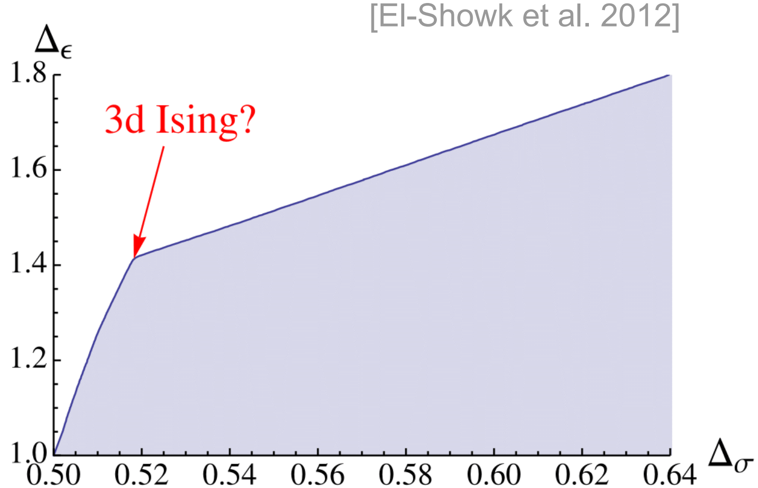
Carve out large N hadronic data from

- Crossing symmetry
- Unitarity
- Regge boundedness

Does large N QCD sit at a special place?

Conformal Bootstrap:

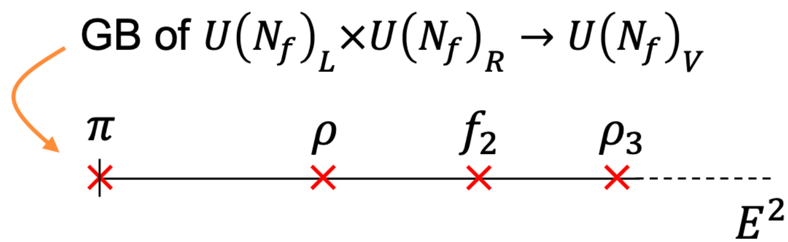
[El-Showk et al. 2012]



Bootstrapping the mesons at large N

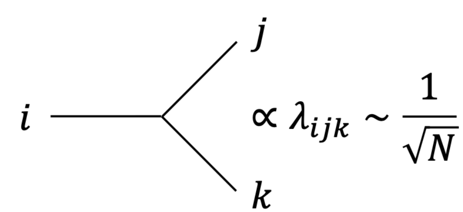
We focus on the mesons ($q\bar{q}$) states) : more constrained, and lots of real world data ($N = 3$).

We'll assume N_f massless quarks $\Rightarrow m_\pi = 0$ (pions are Goldstone bosons).



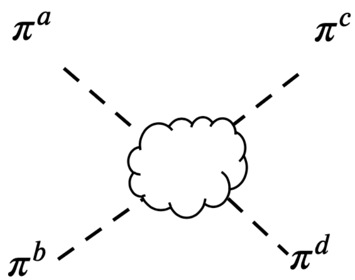
Organize the putative meson spectrum by energy, keeping track states below some cutoff M.

Ultimate goal is to determine full mesonic data $\{m_i, J_i; \lambda_{ijk}\}$



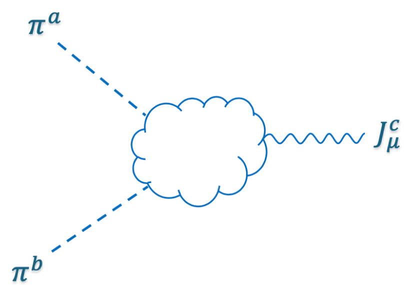
Expect continuous families of solutions.
Which additional input is needed to corner N large QCD?

Outline



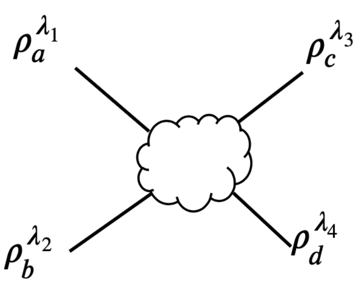
Pion bootstrap

- Universal bounds, $M = m_\rho$
- Integrate in ρ, $M > m_\rho$
- Integrate in ρ and f_2, $M > m_{f_2}$



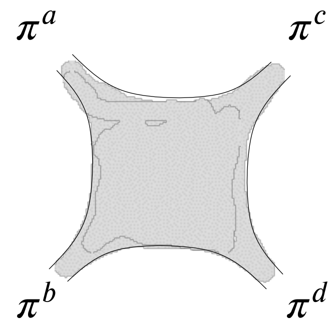
Form factor bootstrap

- Universal rigorous bounds
- Phenomenological bounds



Rho bootstrap: Alessandro's talk

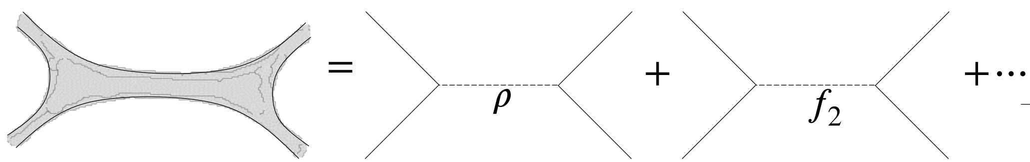
Pion Scattering at Large N



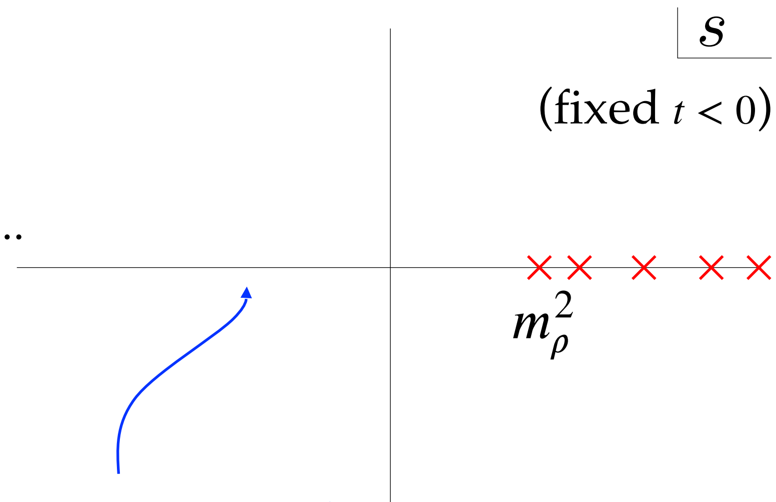
$\sim \mathrm{Tr}(T^a T^b T^d T^c)\; M(s,t)$

Flavored-order amplitude has (only)

$s \leftrightarrow t$ crossing: $M(s,t) = M(t,s)$

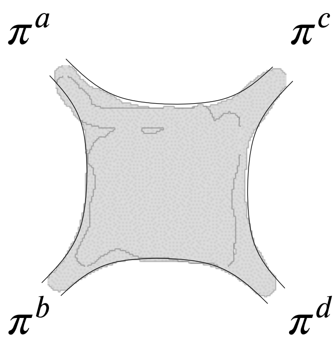


$M(s,t) = \sum$ mesons poles $=$ meromorphic



Only $q\bar{q} \Rightarrow$ no $u-$channel poles

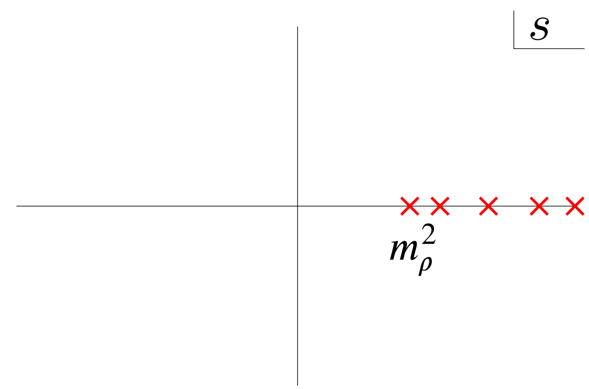
Pion Scattering at Large N



$$\sim \mathrm{Tr}(T^a T^b T^d T^c)\, M(s,t)$$

Flavored-order amplitude has (only)

$s \leftrightarrow t$ crossing: $M(s,t) = M(t,s)$



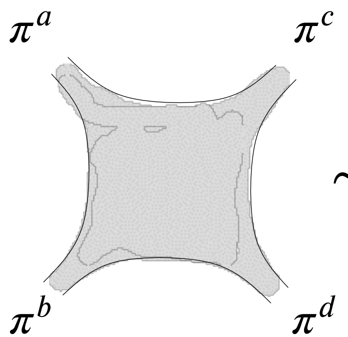
At low energies ($E \ll M = m_\rho$), we can expand

$$M_{\mathrm{low}}(s,t) \quad = \underline{g_{1,0}}\,(s+t) + \underline{g_{2,0}}\,(s^2+t^2) + 2\underline{g_{2,1}}\,(st) + \ldots$$

Coefficients $\{g_{n,\ell}\}$ parametrize our ignorance about the UV theory. All $g_{n,\ell} \sim \dfrac{1}{N}$

Equivalently, $\mathscr{L}_{\mathrm{Ch}} = -\dfrac{f_\pi^2}{4}\left(\partial_\mu U^\dagger \partial_\mu U\right) +$ higher derivatives $\qquad g_{1,0} \sim \dfrac{1}{f_\pi^2}$

Pion Scattering at Large N

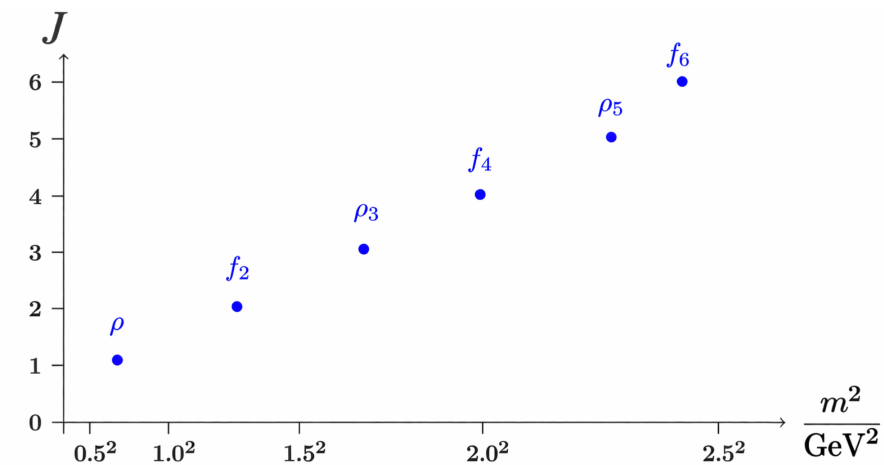


$\sim \mathrm{Tr}(T^aT^bT^dT^c)\, M(s,t)$

Finally we need the large $|s|$ behavior.

We assume

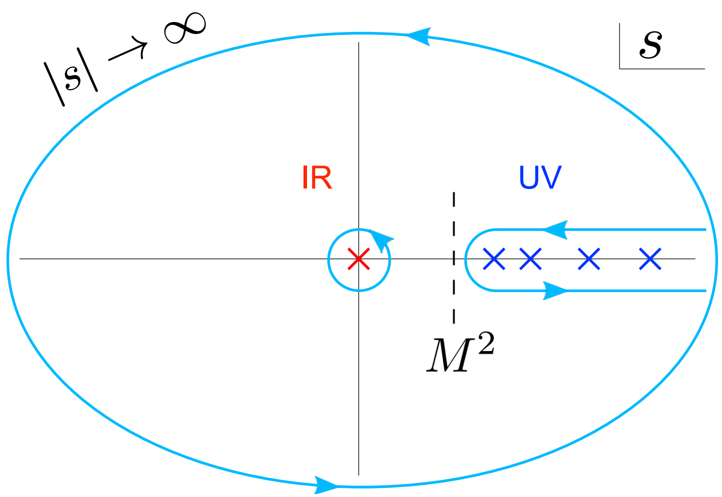
$$\lim_{|s|\to\infty} \frac{M(s,t)}{s} = 0 \qquad \text{for fixed } t < 0$$



This is better than generic behavior (slower than $O(s^2)$). Justified by Regge theory: $M(s,t) \sim s^{\alpha(t)}$ with $\alpha(0) \approx 0.5$ for the trajectory of the ρ, which is leading at large N.

Adams Arkani-Hamed Dubovsky Nicolis Rattazzi   Arkani-Hamed T-C Huang Y-t Huang   Bellazzini Mirò Rattazzi Riembau Riva

IR = UV



can drop piece at ∞

$$\oint_\gamma \frac{M(s,t)}{s^n} = 0$$

IR couplings $g_{n,\ell}$ get expressed in terms of UV spectral density $\sim$ Im M ≥ 0 (unitarity)

There are also **"null constraints"** for UV density that encode crossing symmetry.

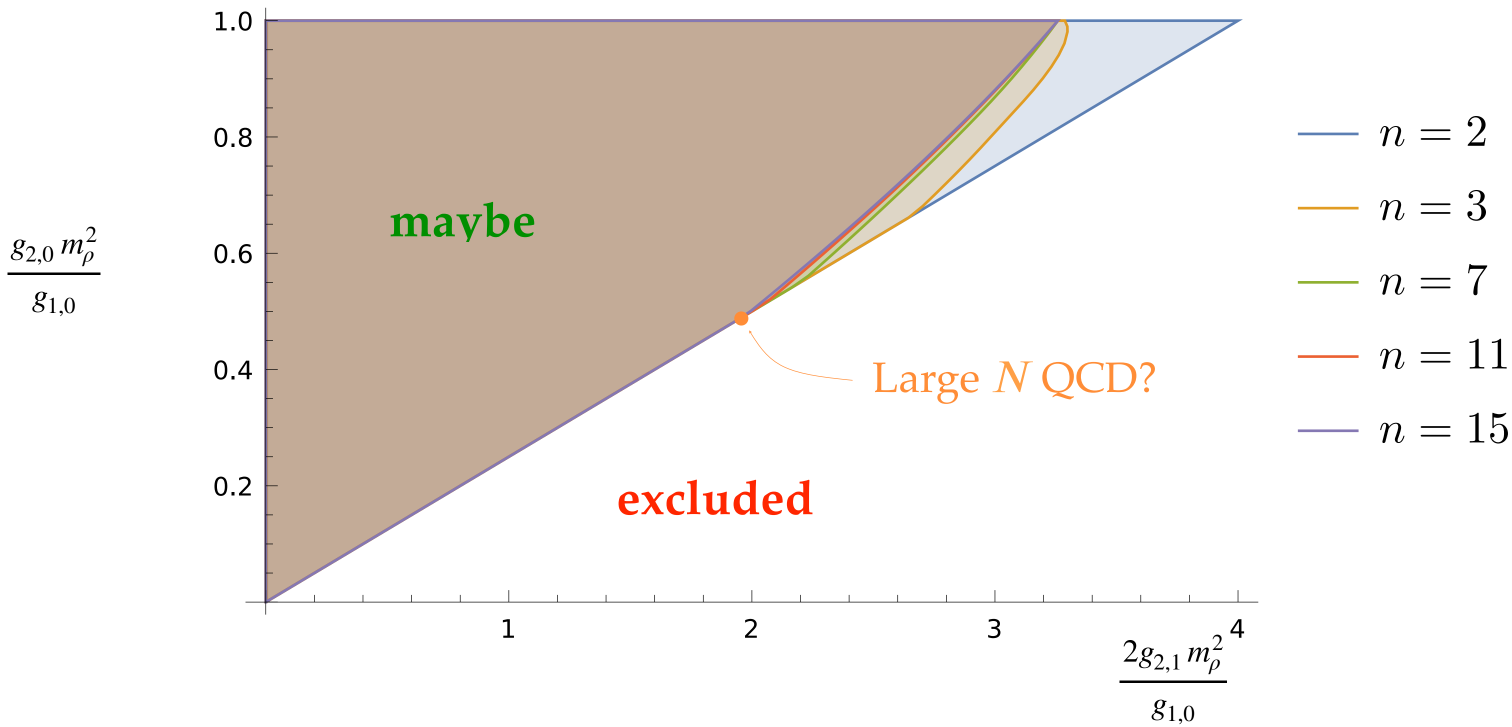
Caron-Huot & Van Duong   Tolley, Wang & Zhou

Semidefinite programming can be used to derive **two-sided bounds** for homogeneous ratios of the couplings, in units of the cutoff $M = m_\rho$, e.g.

$$\frac{g_{2,0}M^2}{g_{1,0}}, \quad \frac{g_{2,1}M^2}{g_{1,0}}$$

Universal Bounds

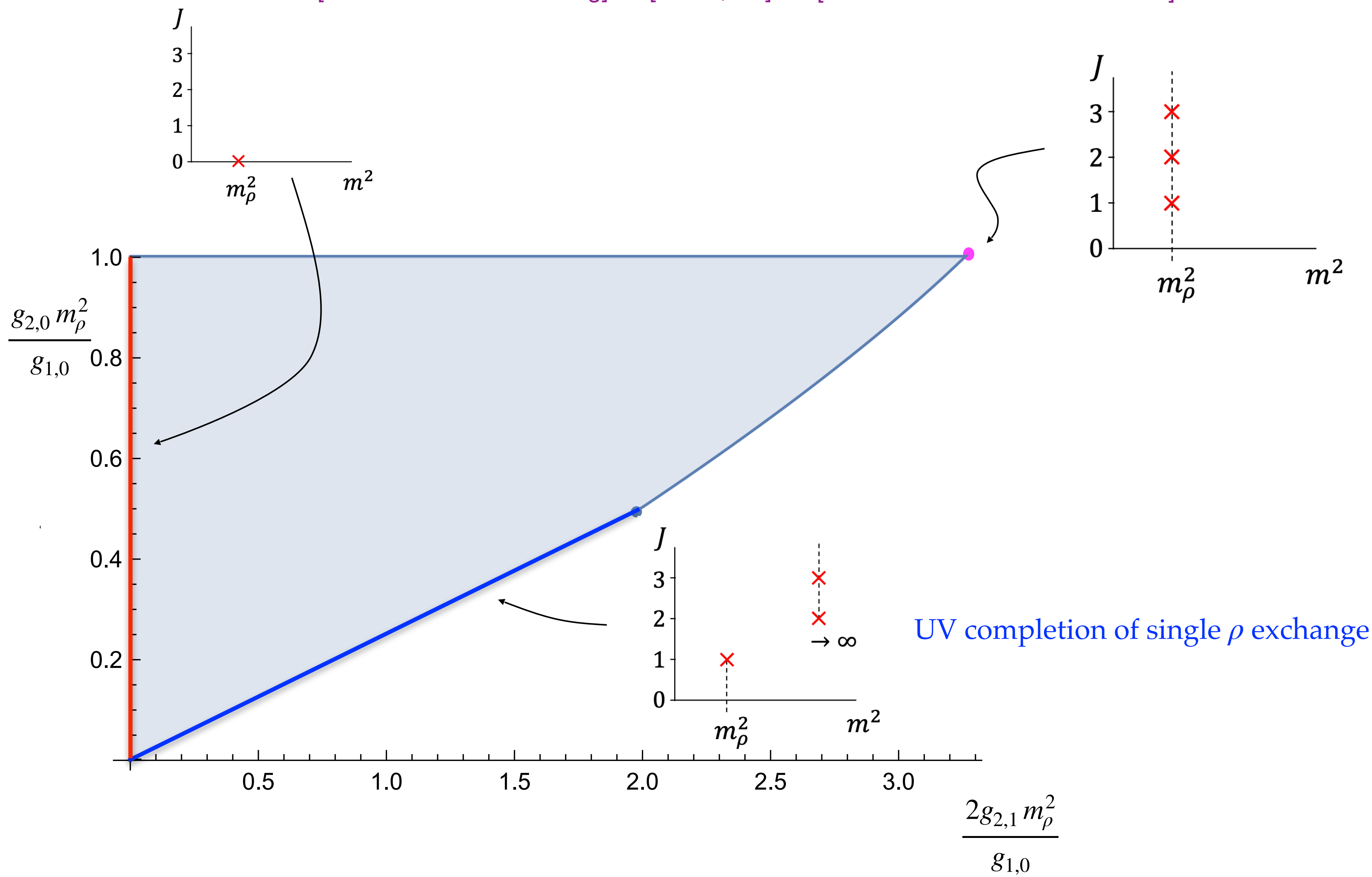
[Albert, LR, arXiv:2203.11950]



Semidefinite programming ⇒ **two-sided bounds** for coupling ratios, in units of the cutoff $M = m_\rho$.

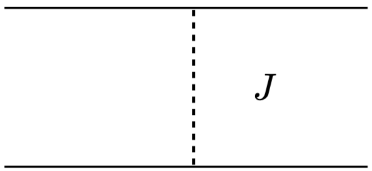
Analytically ruling in

[Caron-Huot & Van Duong] [Albert, LR] [Fernandez Pomarol Riva Sciotti]



Simple unphysical solutions to crossing saturate (some of) the bounds.

Analyticity in Spin



$$\sim \frac{s^J}{t - M_J^2}$$

A massive exchange of spin $J > 1$

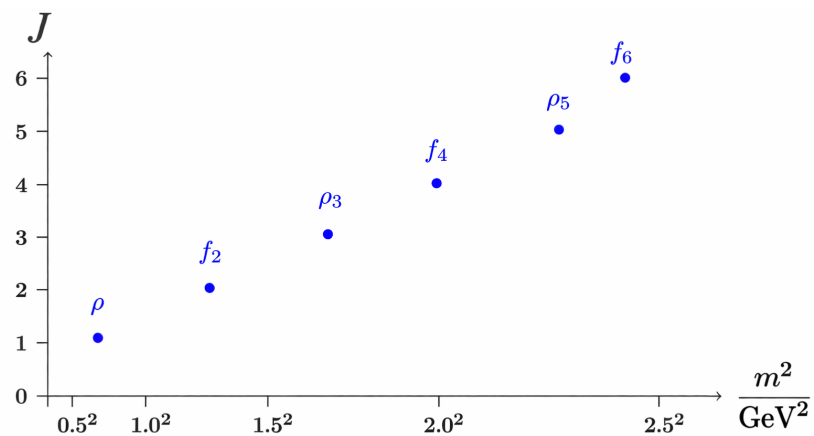
violates large $|s|$ asymptotics

ρ exchange can be UV completed by states at $M \to \infty$ because $J = 1$ is "marginal"

No *s*uch simple UV completion if we include states with $J > 1$.

Need an *infinite* fine-tuned tower of higher spins!

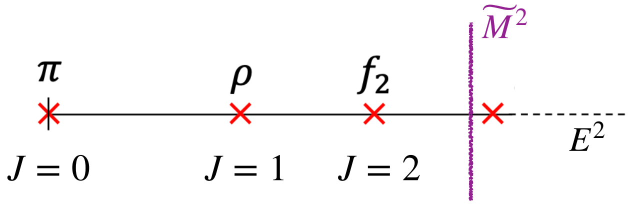
$$\sum_J c_J \frac{s^J}{t - M_J^2} \sim s^{J(t)}$$



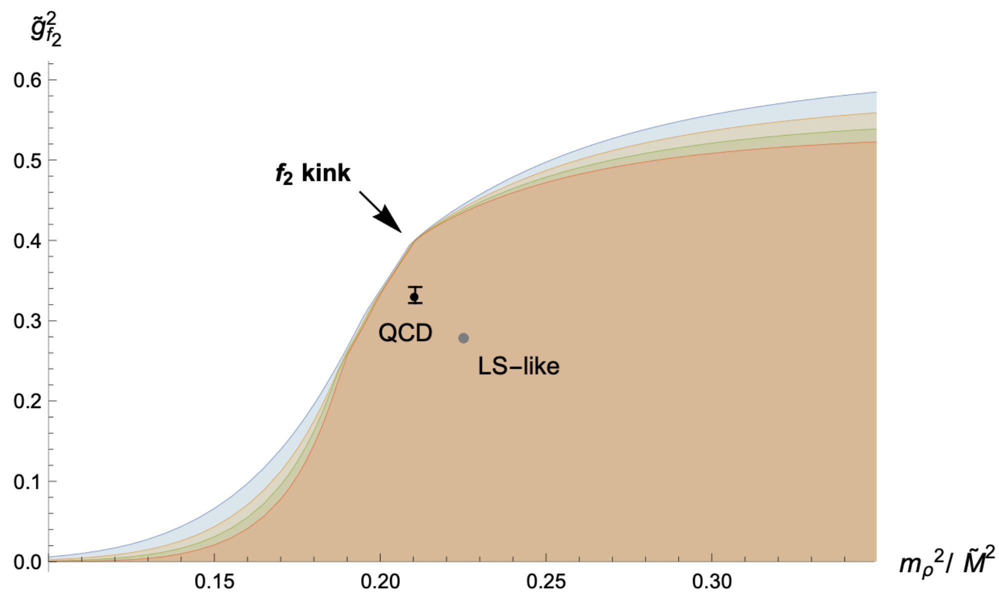
$J(t) =$ analytic continuation in *spin*

Including ρ and f_2 as internal states

[Albert Henriksson LR Vichi]



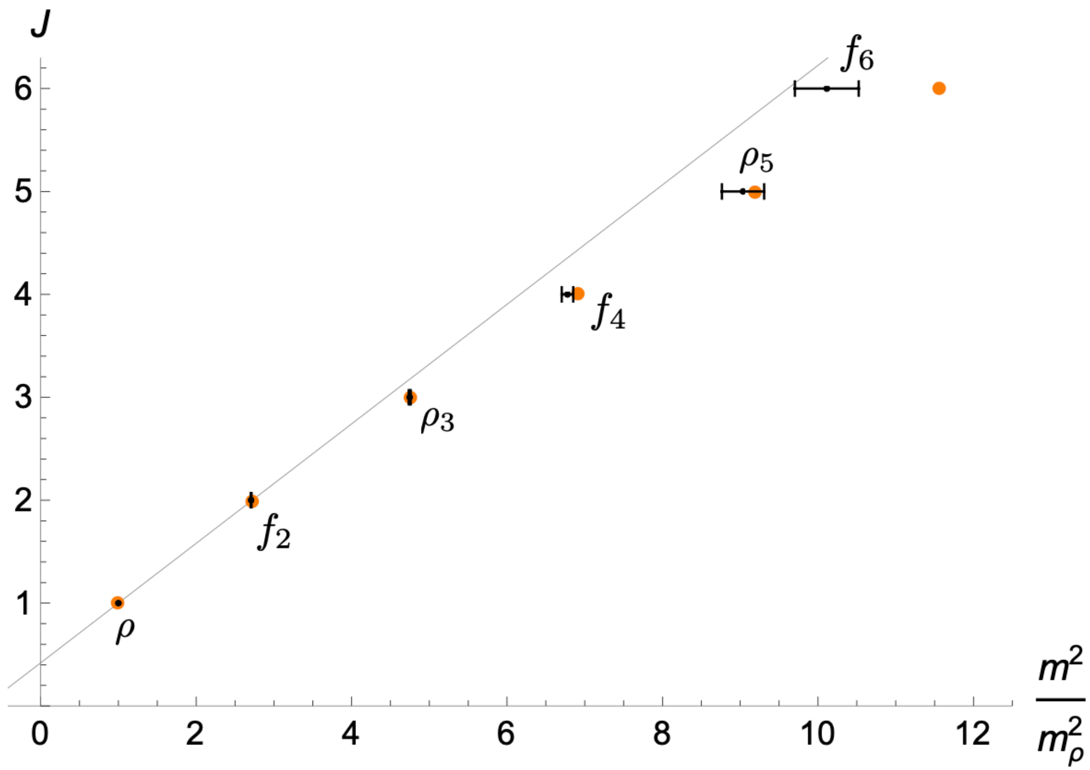
For definiteness pick the physical value $\dfrac{m_{f_2}^2}{m_\rho^2} = (1.65)^2$.



Numerically stable kink at a special value of $\widetilde{M}$

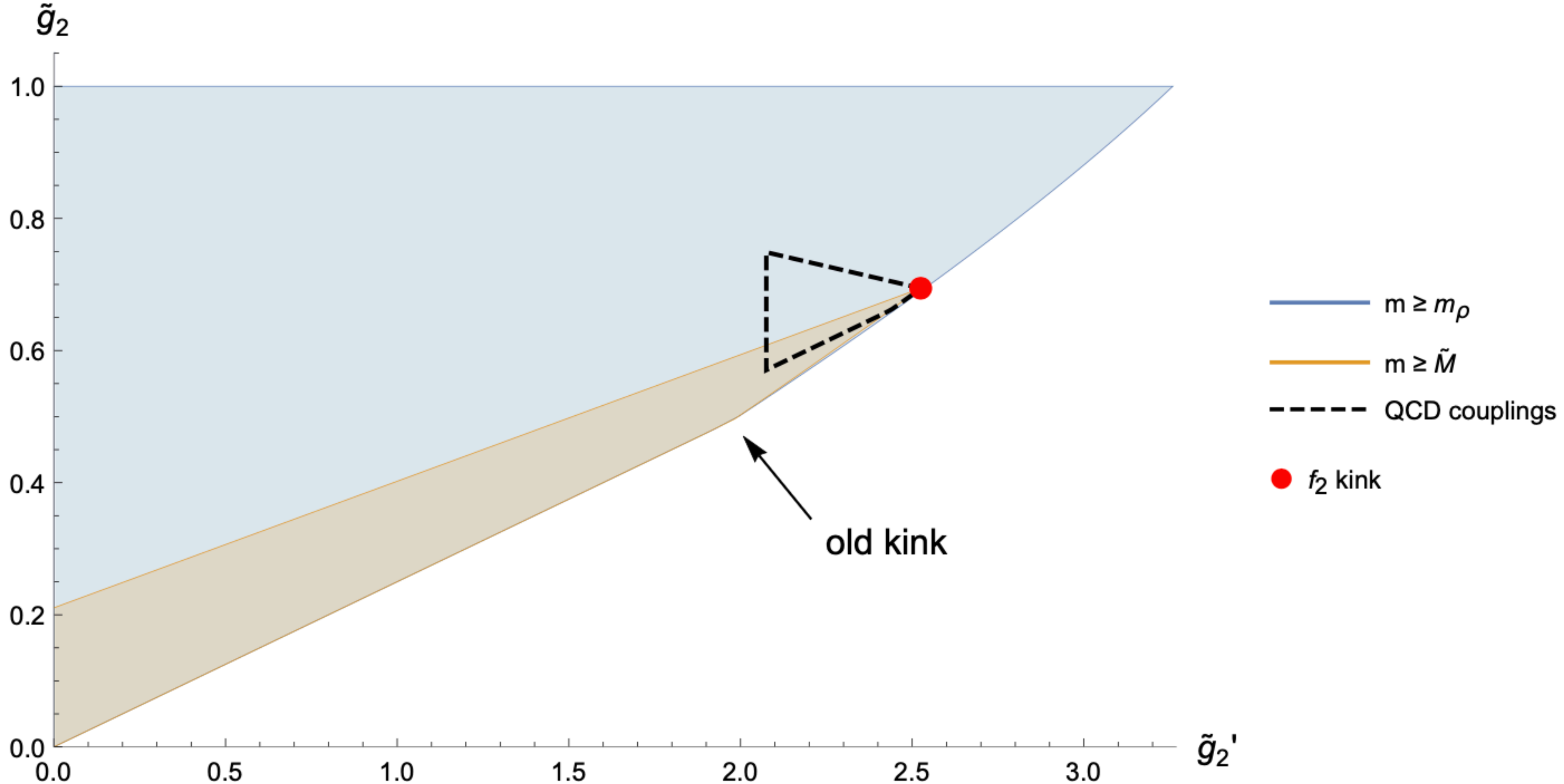
Novel extremal bootstrap solution

A Regge trajectory!

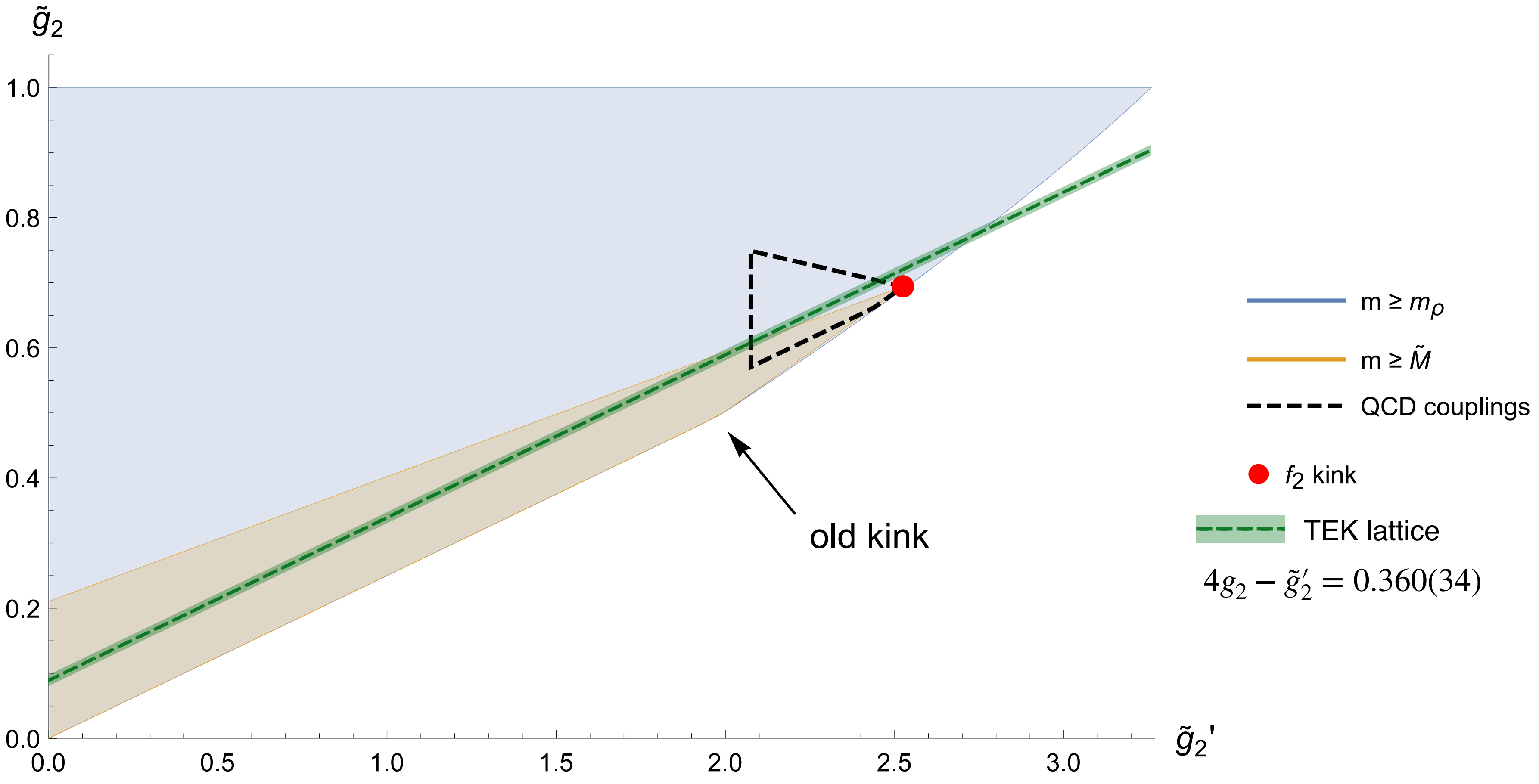


[Albert Henriksson LR Vichi]

Bootstrap solution at the f_2 kink exhibits one full Regge trajectory.
Low-lying states in surprising coincidence with real world data.



Dashed triangle: region compatible with experimental m_{f_2}/m_ρ , $g_{\pi\pi\rho}$, $g_{\pi\pi f_2}$



Dashed triangle: region compatible with experimental m_{f_2}/m_ρ , $g_{\pi\pi\rho}$, $g_{\pi\pi f_2}$

Green band: Bonanno Garcia Pérez González-Arroyo Ishikawa Okawa Romaro-Lopez @ Lattice 2026

How close are we?

What we have

Regge trajectory from bootstrap bounds

Uncanny spectral agreement with real world

[Coupling ratio $g_{\pi\pi\rho}/g_{\pi\pi f_2}$ also matches]

What's missing

No daughter trajectories*

No scalar mesons in extremal solution

No control over fixed-angle scattering

Tantalizingly close — but something still missing to corner large N QCD

*No-go theorems that forbid single trajectory solutions [Eckner Figueroa Tourkine]

Presumably seeing the limit of a *sequence* of good amplitudes, with daughters at $M \to \infty$.

Several ways forward

- Bootstrap $2 \to 2$ scattering of more general states, starting with ρ scattering

  Much more constraining! $\longleftarrow$ Alessandro's talk

  [Albert Henriksson LR Vichi, to appear]

- Bootstrap *higher-point* pion scattering

  [Ambrosino Guerrieri LR, in progress]

- Include local gauge invariant observables:

  Large N form factor bootstrap $\longleftarrow$ Rest of this talk

  Allows to inject UV boundary conditions (asymptotic freedom).

  [See also He Kruczenski]

Large N Form Factor Bootstrap

[Albert Kosva LR]

Result so far insensitive to short distance asymptotics, probed by *fixed-angle* hard-scattering. Unclear how to handle fixed t/s with dispersive methods

We *can* however incorporate form factors of gauge invariant local operators

$$\langle \pi(p_1) \ldots \pi(p_n) \,|\, \mathcal{O}_i(x) \rangle$$

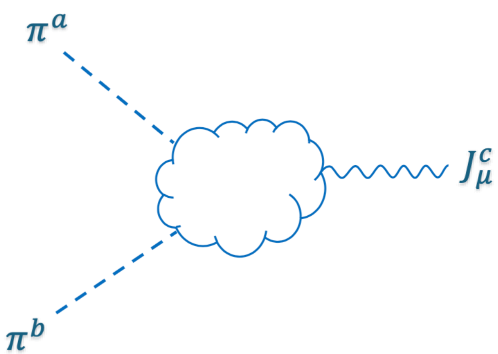
[Karateev Kuhn Penedones]

and input UV physics through OPEs $\lim_{x \to y} \mathcal{O}_i(x)\, \mathcal{O}_i(y) \sim \ldots$ from perturbative QCD.

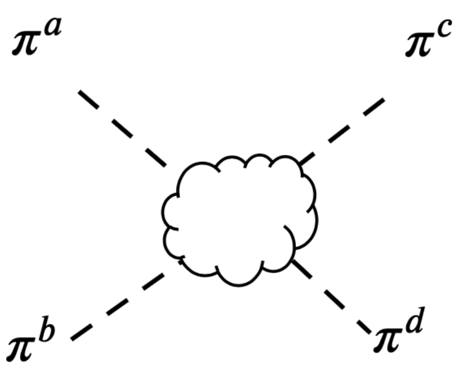
We take as local operator the conserved vector current $J^a_\mu(x)$ of $U(N_f)_V$

Our bootstrap systems comprises:

$$\langle 0 | J^\mu_a(p) J^\nu_b(-p) | 0 \rangle = (p^\mu p^\nu - p^2 \eta^{\mu\nu}) \delta_{ab} \, \Pi(-p^2)$$



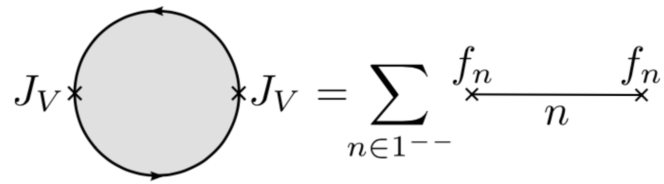
$$\langle 0 | J^\mu_c(0) | \pi_a(p_1) \pi_b(p_2) \rangle = i(p_1 - p_2)^\mu f_{abc} \, F(-(p_1 + p_2)^2)$$



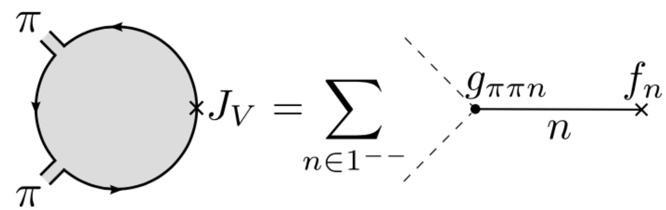
$$\sim \mathrm{Tr}(T_a T_b T_d T_c) \, M(s,t) + \text{other flavor orderings}$$

At large N, spectral functions are saturated by *single-meson* states:

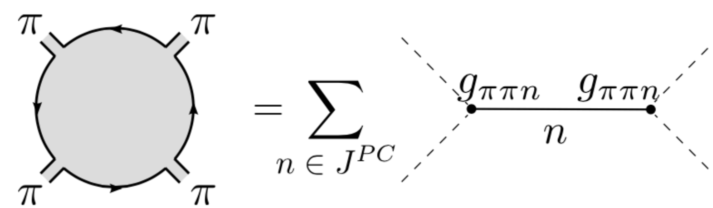
$$\rho^{(\gamma\gamma)}(s) \equiv \mathrm{Im}\,\Pi(s) = \sum_{n\in 1^{--}} f_n^2\,\delta(s-m_n^2)$$



$$\rho^{(\gamma 2\pi)}(s) \equiv \mathrm{Im}\,F(s) \propto \sum_{n\in 1^{--}} f_n g_{\pi\pi n}\delta(s-m_n^2)$$



$$\rho_J^{(4\pi)}(s) \propto \sum_{n\in\left\{\substack{\text{even}^{++}\\ \text{odd}^{--}}\right.} g_{\pi\pi n}^2\delta(s-m_n^2)$$



These observables yield the simplest positive semidefinite system

$$\begin{pmatrix} \rho^{(\gamma\gamma)}(s) & \rho^{(\gamma 2\pi)}(s) \\ \rho^{(\gamma 2\pi)}(s) & \rho_1^{(4\pi)}(s) \end{pmatrix} \succeq 0$$

Low energy

At low energies ($E \ll M = m_\rho$), we can expand

$$\Pi_{\text{low}}(s) = a_0 + a_1 s + a_2 s^2 + \dots$$

$$F_{\text{low}}(s) = b_0 + b_1 s + b_2 s^2 + \dots \, .$$

Important: $b_0 \equiv 1$ (pions in the adjoint)

$$M_{\text{low}}(s,t) \quad = g_{1,0}\,(s+t) + g_{2,0}\,(s^2+t^2) + 2g_{2,1}\,(st) + \dots$$

Low-energy constants ↔ Wilson coefficients of chiral Lagrangian coupled to an off-shell $U(N_f)_V$ gauge field. Meaningful because J^μ_{aV} is an RG invariant.

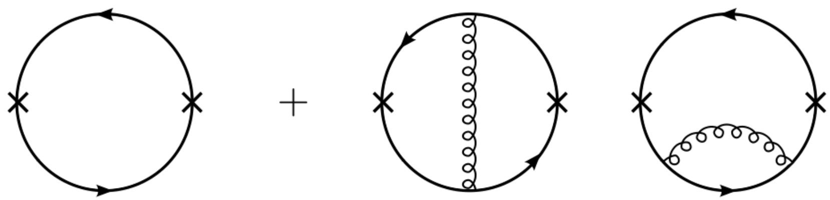
High energy

- For $\Pi(s)$, determined by the OPE:

$$\Pi(s) \sim \tilde{c}_{\mathbb{1}}(\alpha_s, s) + \frac{\tilde{c}_{\psi\psi}(\alpha_s, s)}{s^{3/2}} \langle \bar{\psi}_i \psi^i \rangle + \frac{\tilde{c}_{GG}(\alpha_s, s)}{s^2} \langle \mathrm{Tr}(G_{\rho\sigma} G^{\rho\sigma}) \rangle + \cdots$$

The identity contribution can be computed in pQCD, with systematic *log* corrections

$$\tilde{c}_{\mathbb{1}}(\alpha_s, s) = -\frac{N}{24\pi^2}\left(1 + \frac{12\pi\alpha_s(\mu)N}{32\pi^2}\right)\log(-s/\mu^2) + \cdots$$



Non-perturbative condensates *power*-suppressed (in fact $\tilde{c}_{\psi\psi} = 0$ in chiral limit)

- For $F(s)$, Brodsky-Farrar counting rules:

$$F(s) = O\left(\frac{1}{s}\right)$$

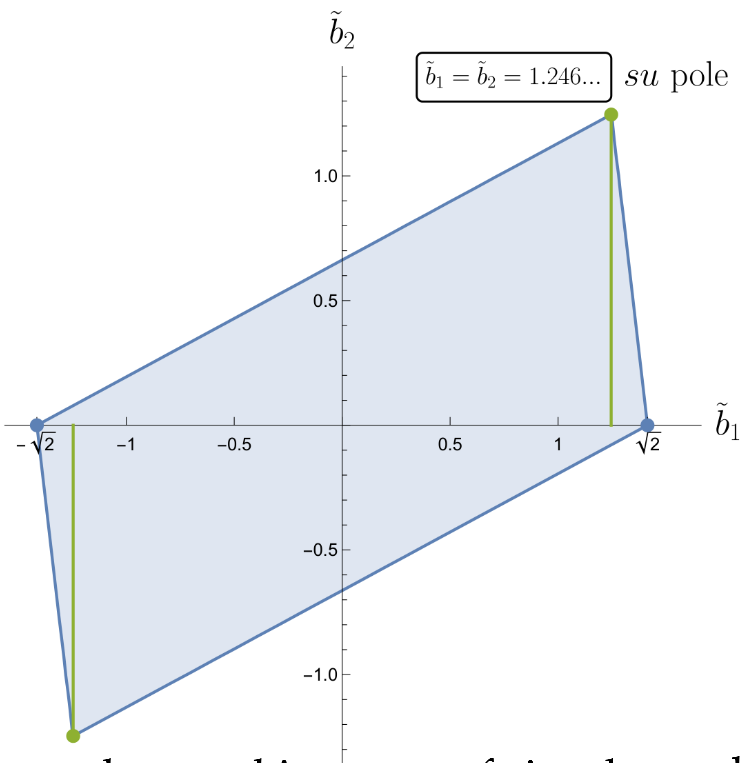
See also Efremov Radyushkin Brodsky Lepage

Rigorous bounds

Use $s \to \infty$ asymptotics just to fix the # of subtractions that kill arcs at ∞.

Standard machinery yields bounds on (doubly) homogeneous ratios,

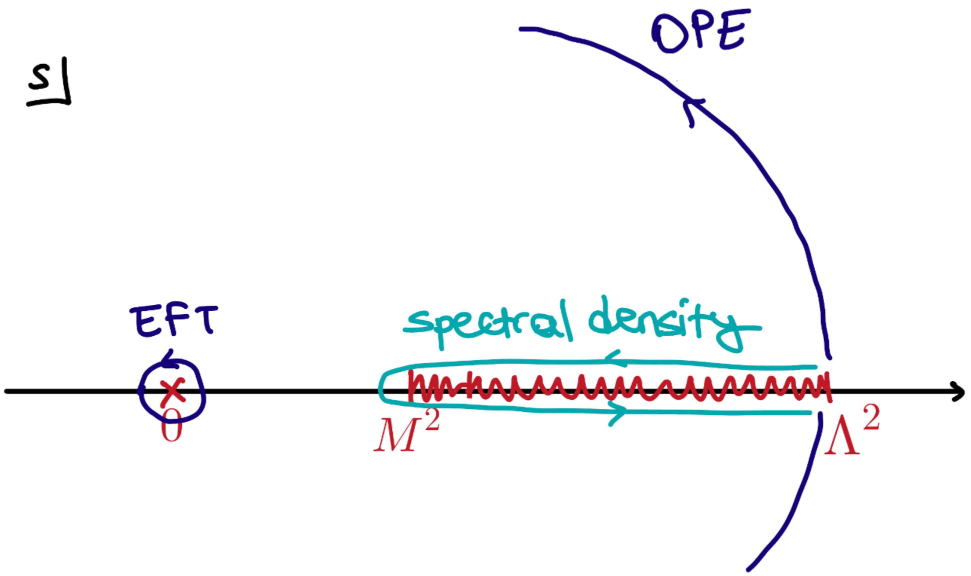
in units of the EFT cutoff, e.g. $\tilde{b}_k = \dfrac{b_k M^{2(k-1)}}{\sqrt{a_1 g_{1,0}}}$



Convex exclusion region understood in terms of simple unphysical solutions.

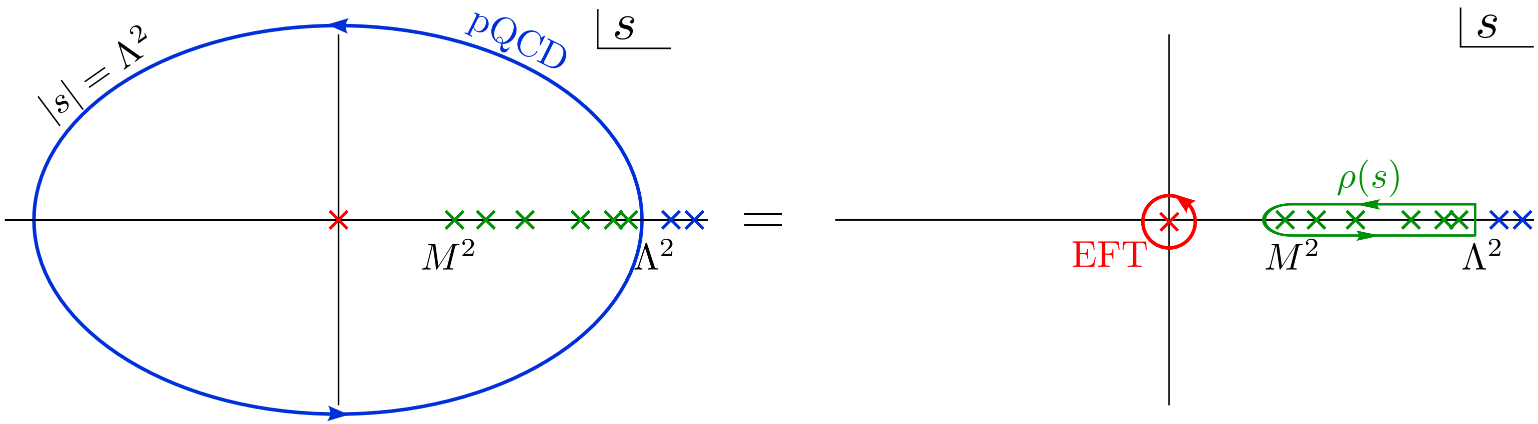
SVZ-like sum rules, bootstrapped

Evaluate high-energy arc at finite Λ, using pQCD. Inspired by Shifman-Vainshtein-Zakharov



Unlike SVZ we make no ansatz for the spectral density below Λ, beyond positivity.

SVZ-like sum rules, bootstrapped



$$\frac{1}{2\pi i}\oint_{\Lambda^2} ds\, \frac{\Pi_{\rm pQCD}(s)}{s^{k+1}} = \frac{1}{2\pi i}\oint_{0} ds\, \frac{\Pi_{\rm low}(s)}{s^{k+1}} \quad -\frac{1}{\pi}\int_{M^2}^{\Lambda^2} ds\, \frac{\mathrm{Im}\,\Pi(s)}{s^{k+1}}\,.$$

For $k \geq 1$ we smoothly recover previous sum rules as $\Lambda \to \infty$,

$$a_k = \int_{M^2}^{\Lambda^2} \frac{dm^2}{m^2}\, \rho^{(\gamma\gamma)}(m^2)\frac{1}{m^{2k}} + \frac{N}{24\pi^2}\frac{1}{k\Lambda^{2k}} + O\left(\frac{\Lambda^{-2k}}{\log\Lambda^2}\right), \qquad k = 1,2,\ldots\,.$$

But crucially, we can now also take $k \leq 0$, e.g. for $k = -1$:

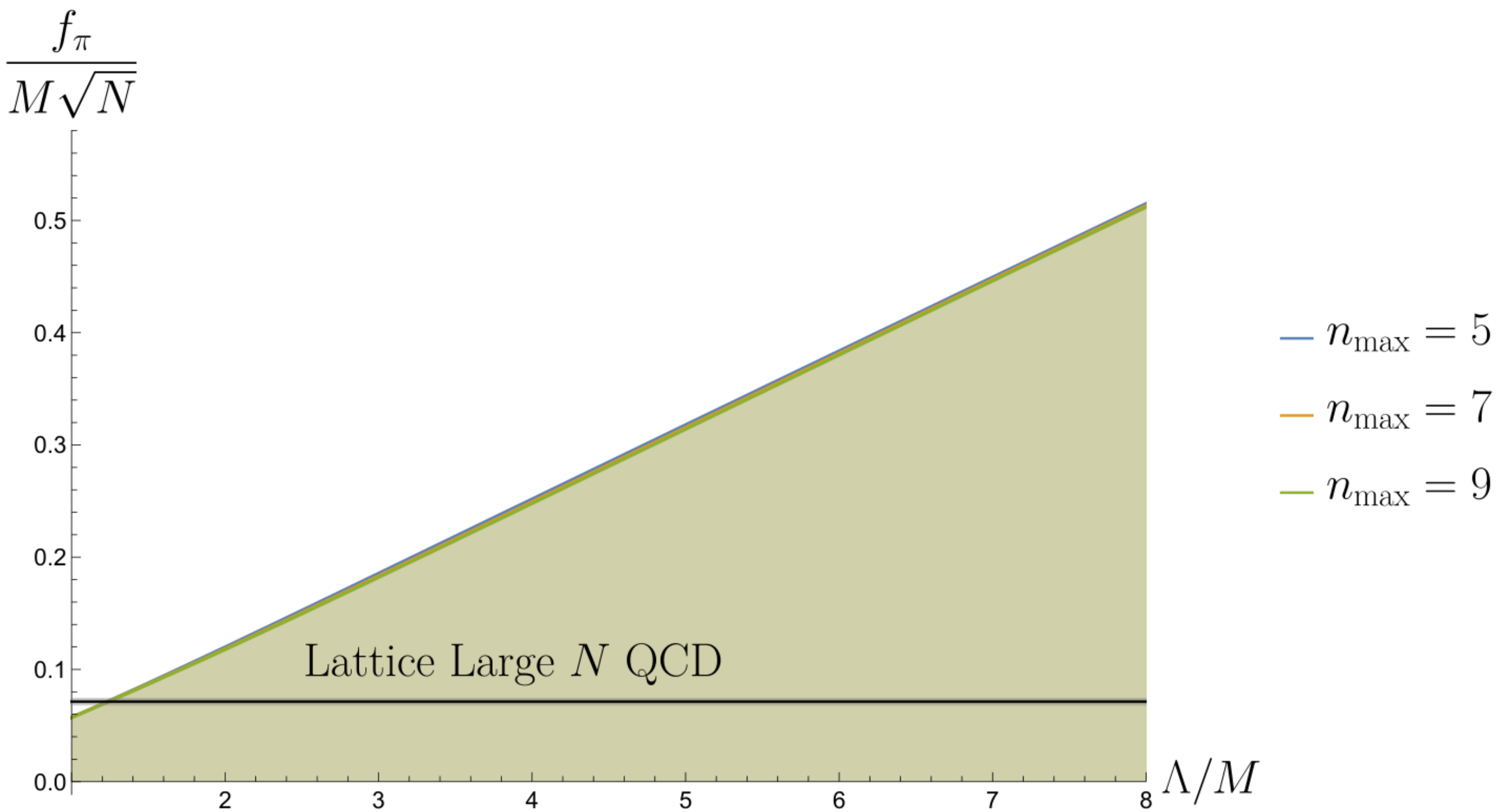
$$\alpha_{-1} \equiv \frac{N}{24\pi^2}\Lambda^2 = \int_{M^2}^{\Lambda^2} \frac{dm^2}{m^2}\, \rho^{(\gamma\gamma)}(m^2) m^2 + O\left(\frac{\Lambda^2}{\log\Lambda^2}\right)$$

An upper bound on f_π !

Trick: put an upper bound on $\tilde{b}_0^{\mathrm{SVZ}} \equiv \dfrac{b_0}{\sqrt{\alpha_{-1} g_{1,0}}}$.

Recalling that $b_0 \equiv 1$ and $g_{1,0} \sim 1/f_\pi^2$,

we get an upper bound on $f_\pi/\sqrt{N}$ in units of $M = m_\rho$

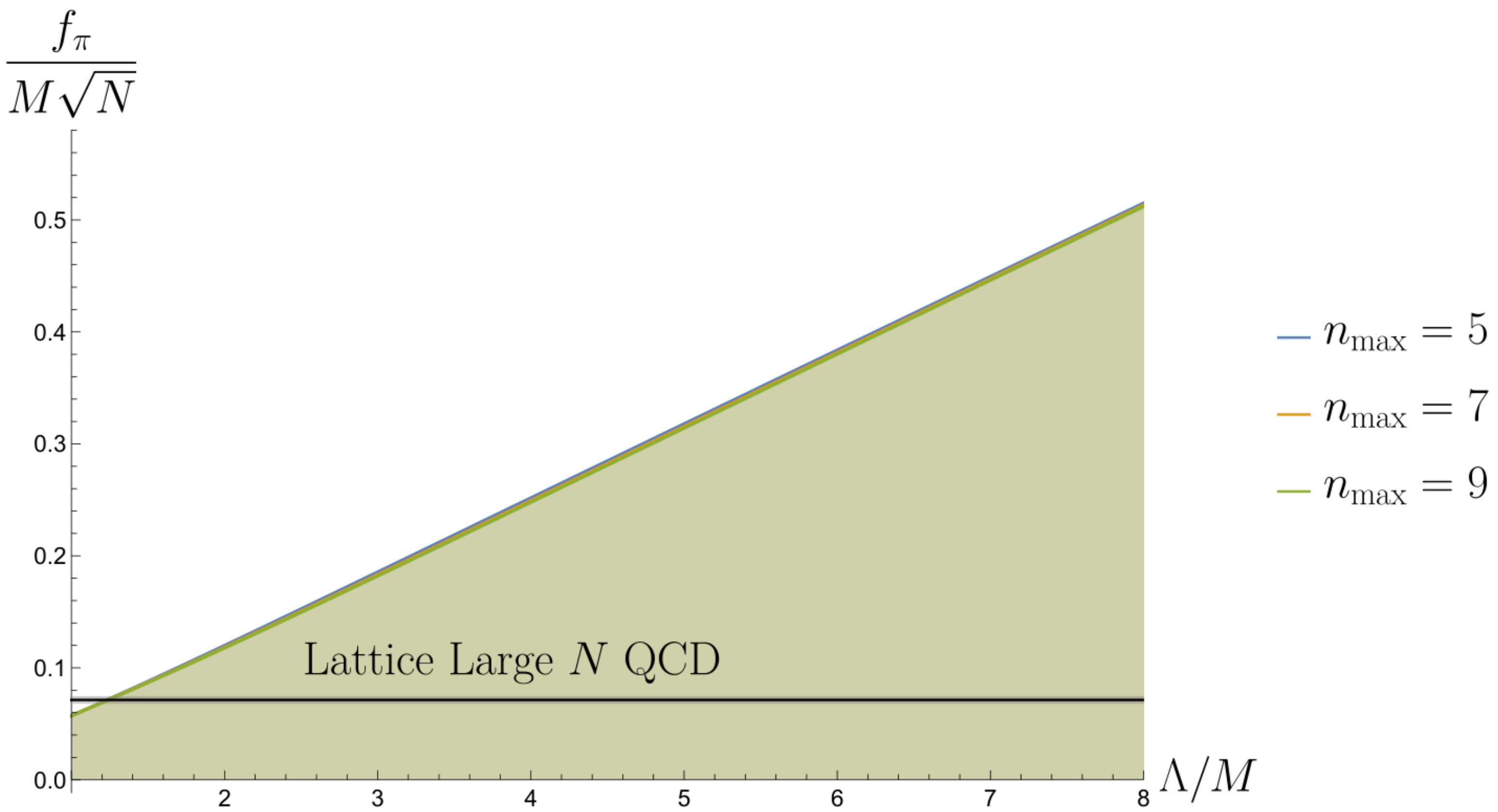


…or a lower bound on Λ

Using recent large lattice result $\dfrac{f_\pi}{m_\rho\sqrt{N}} = 0.0713(34)$ [TEK in chiral limit, Bonanno et al]

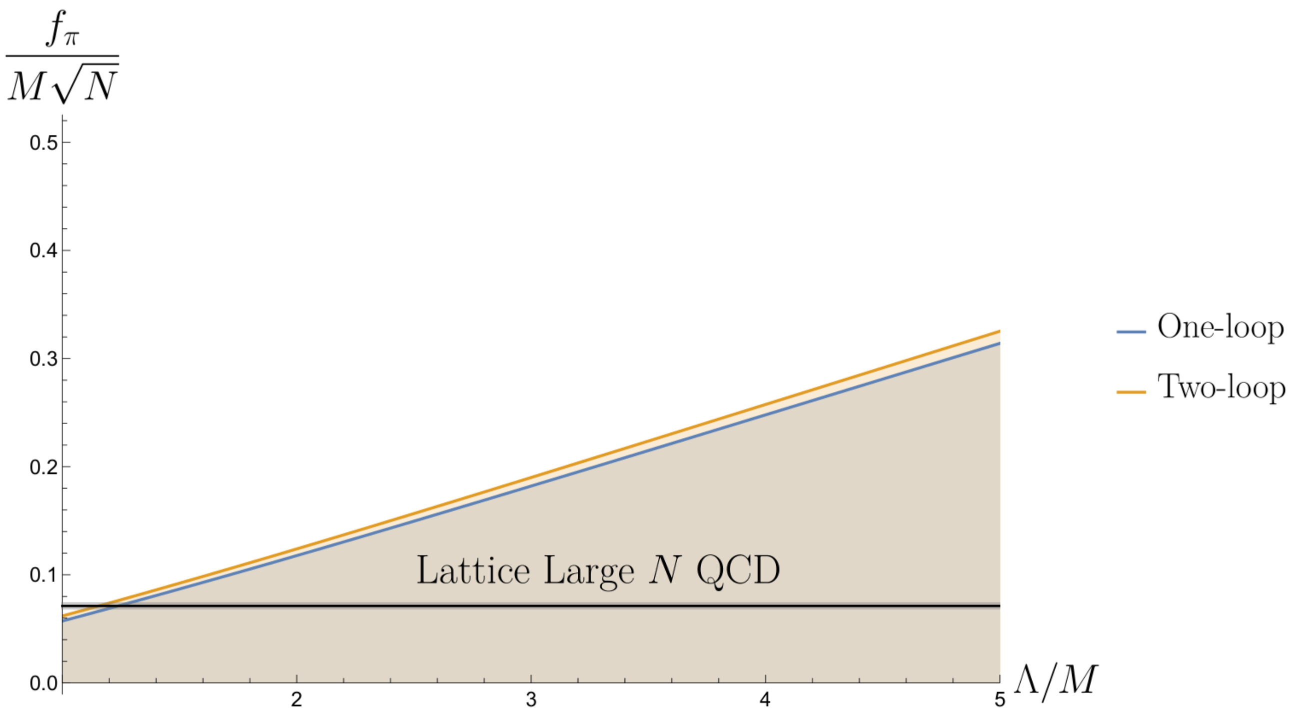
we can view this as a lower bound on the perturbative scale,

$\Lambda \geq (1.24 \pm 0.06)m_\rho \approx 960\,\mathrm{MeV}$. Fairly low. SVZ literature takes 1-1.5 GeV.



…or a lower bound on Λ

Two-loop correction shifts the bound to $\Lambda \gtrsim 900\,\mathrm{MeV}$

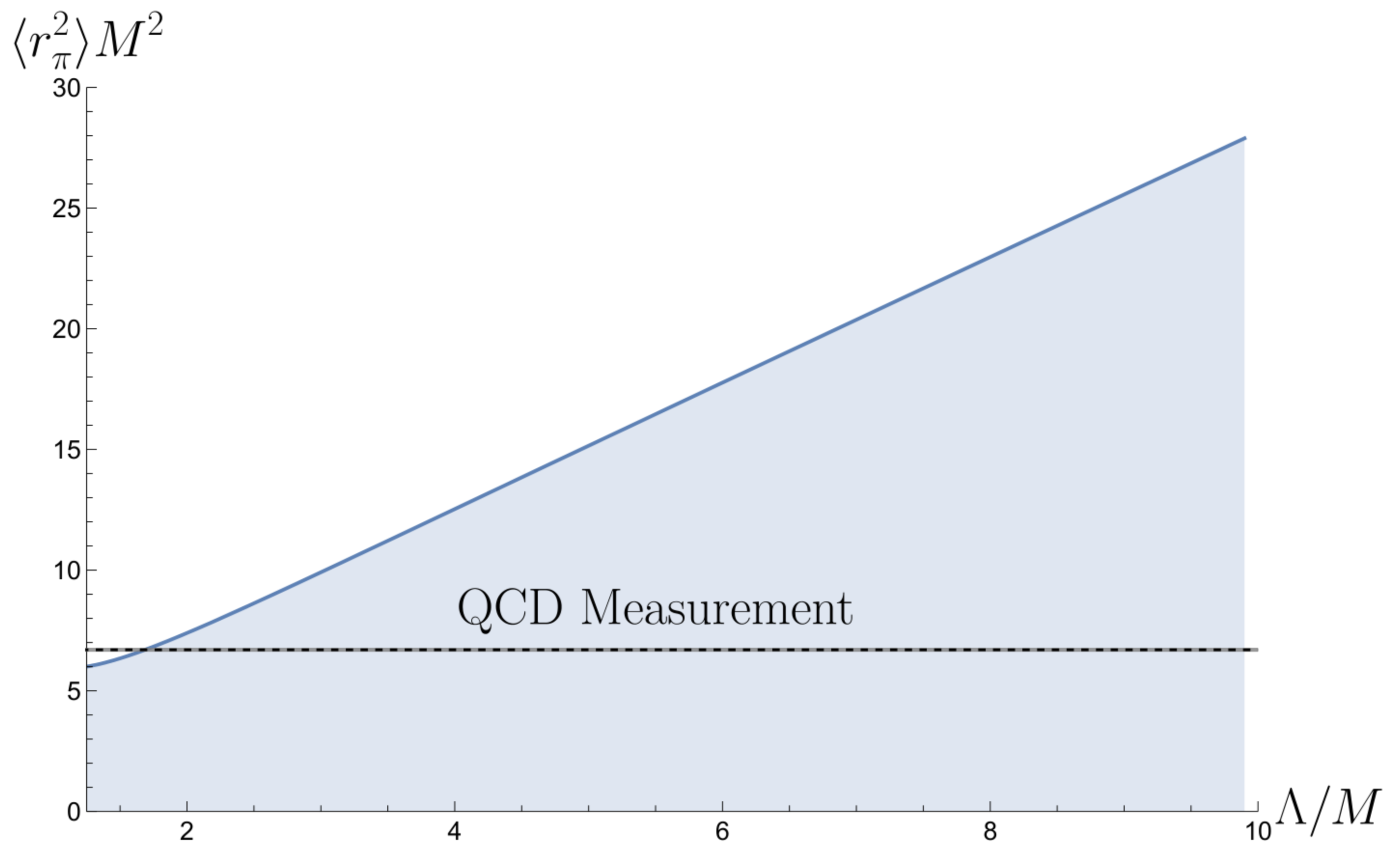


A bound on the pion charge radius

Same template leads to a bound on $b_1 = \dfrac{1}{6}\langle r_\pi^2 \rangle$

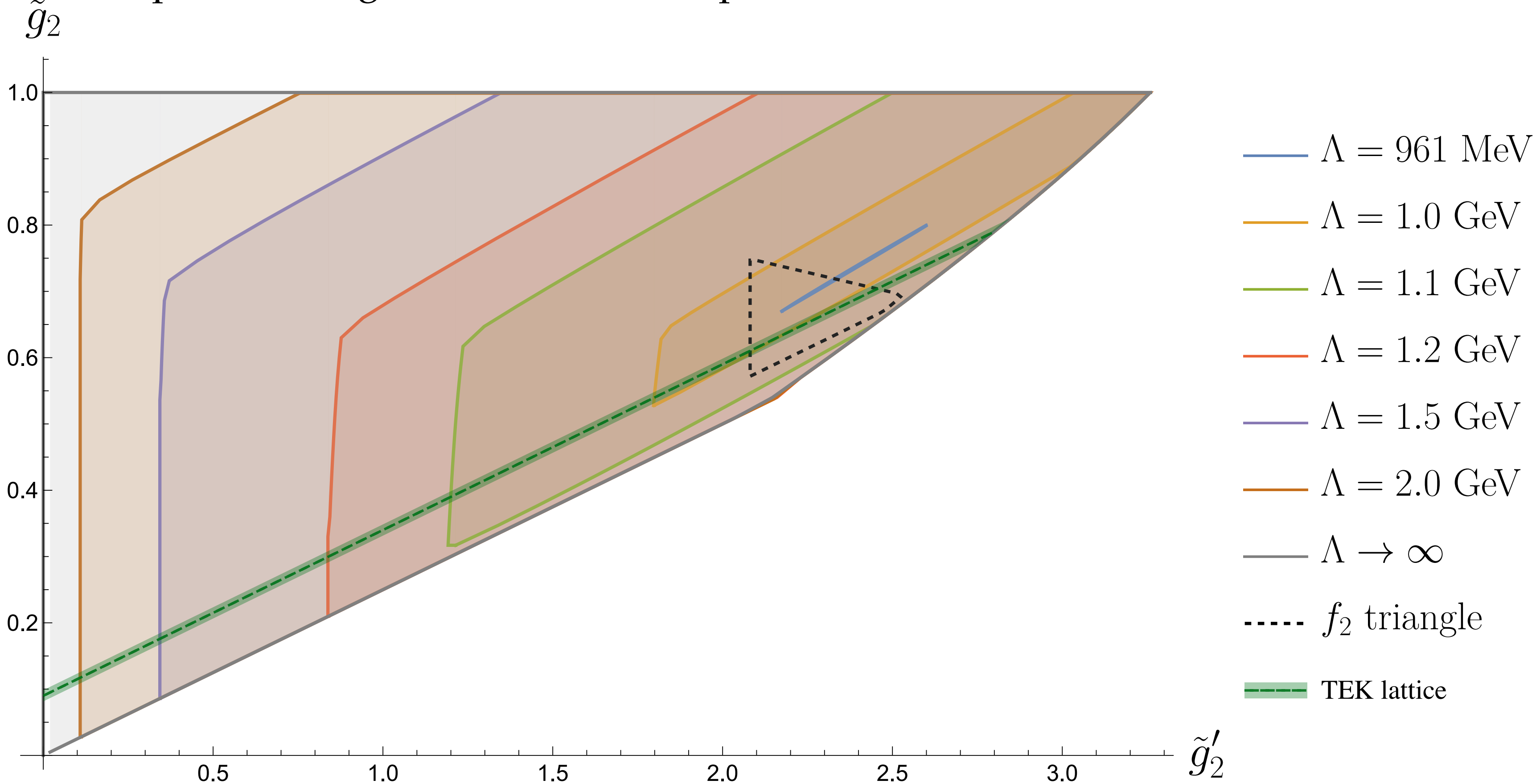
Comparing for fun with real world data (as no large N lattice results exist),

one finds $\Lambda \gtrsim 1.7 m_\rho \approx 1.32\,\mathrm{GeV}$.



UV input on the EFT

Fixing $f_\pi/\sqrt{N}$ to the large N lattice value, we force a backreaction of the form factor bootstrap on the original EFT exclusion plot.



Remarkably, as $\Lambda \to \Lambda_{\text{min}}$, allowed region migrates to where one expects QCD to sit, based on purely IR information (masses and couplings of ρ and f_2) or the TEK lattice

What is the minimal physical input
to corner large N QCD?