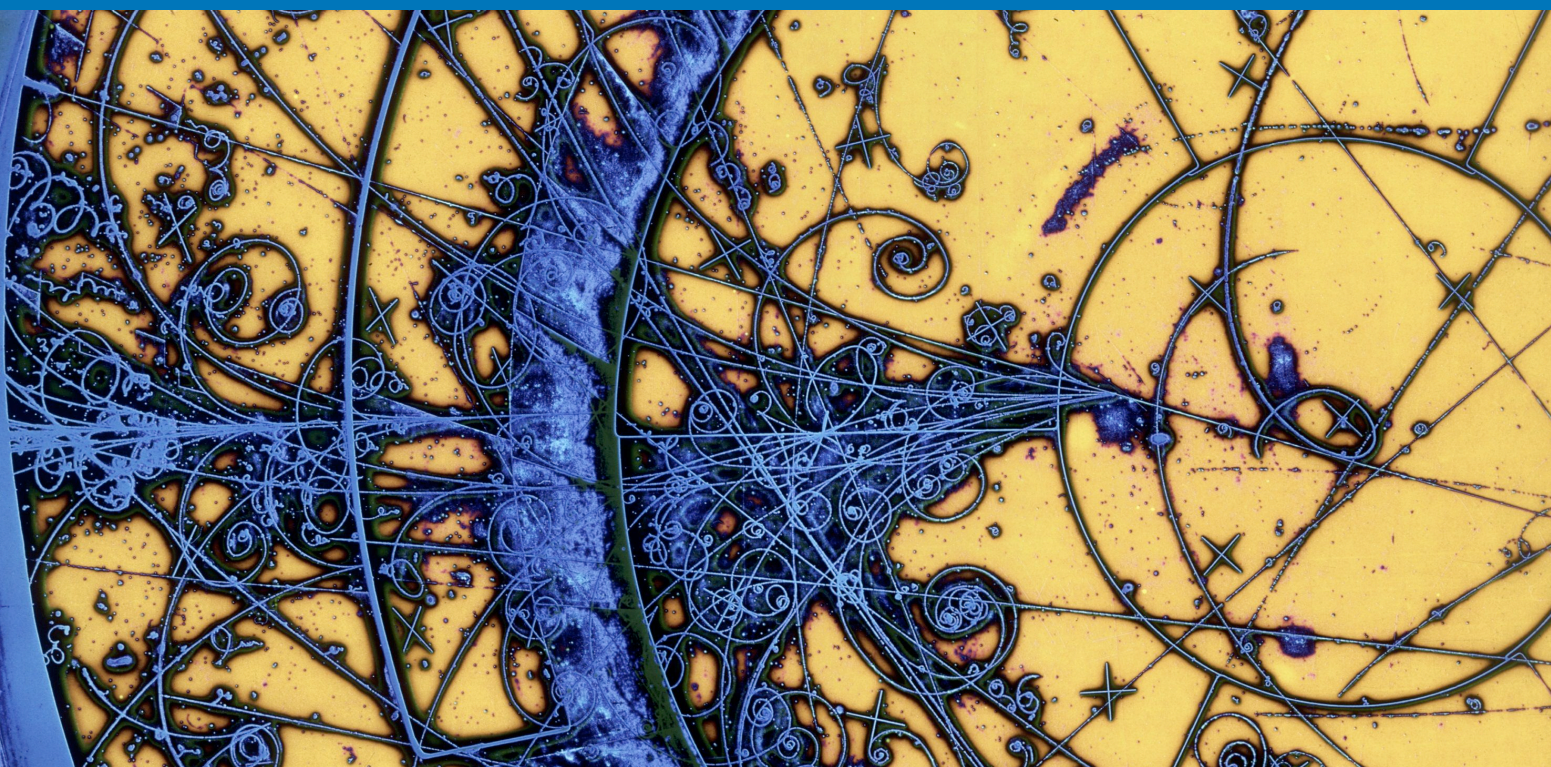




AMPLITUDES

Lecture 5

Pierpaolo Mastrolia (UNIPD, INFN)
Vatican Observatory School
June 2026

Scattering Amplitudes in Yang-Mills and Gravity

1. Photon decoupling and KK relations
2. Spinor helicity formalism and BCFW Recurrence Relation.

Kinematic Relations for Yang-Mills Amplitudes

3. Field-theory proof and Open-string monodromy derivation of BCJ relations

Kinematic Relations for Gravity Amplitudes

4. Color Kinematics Duality and Field-theory KLT relations

Amplitudes and Geometry

5. Intersection Theory for twisted DeRham co-homology.
6. KLT from twisted period/Riemann bilinear relations.

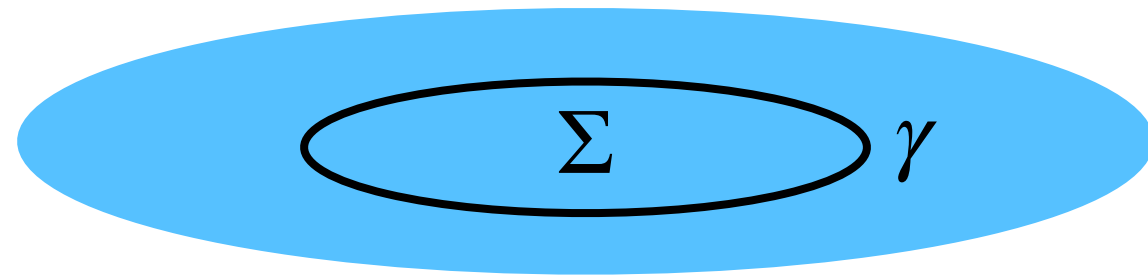
From Holes to de Rham Cohomology

An introductory route through homology

Homology detects holes by closed chains that are not boundaries.

Cohomology detects them by closed forms that are not exact.

The basic intuition: what is a hole?

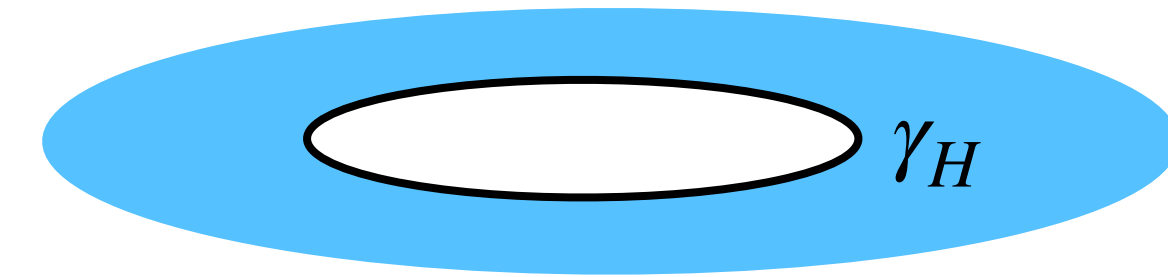


No hole

In a filled plane or disk, a closed loop usually bounds an inner region:

$$\gamma = \partial\Sigma.$$

The loop is closed, but it is also a boundary.



A hole

In a punctured plane, a loop around the missing point satisfies

$$\partial\gamma_H = 0,$$

but there is no allowed surface Σ with

$$\gamma_H = \partial\Sigma.$$

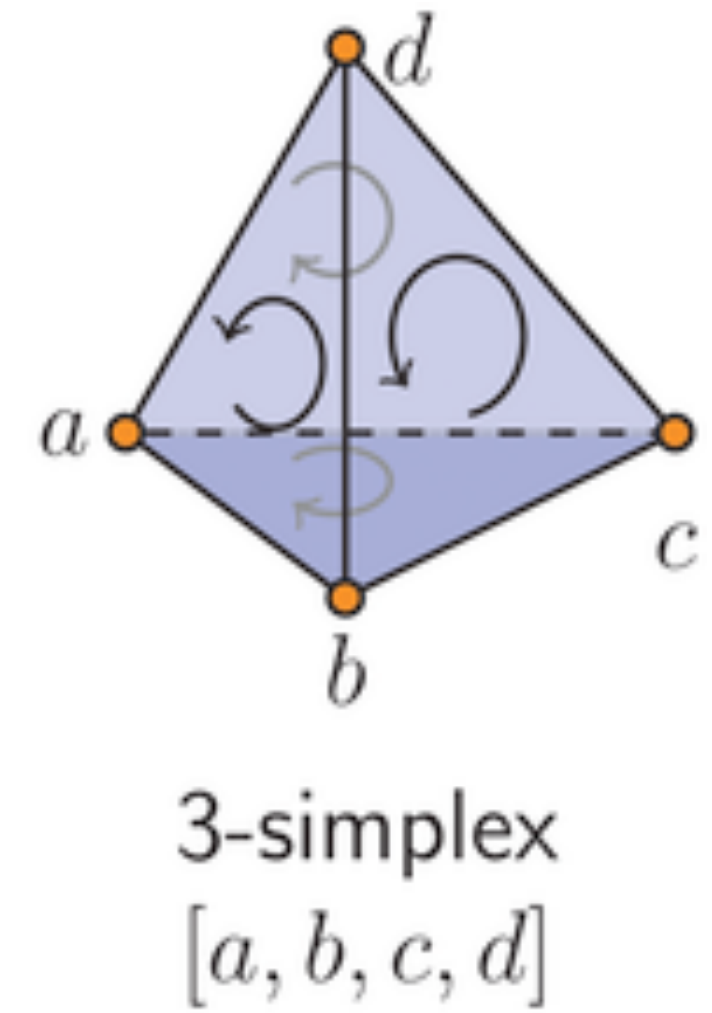
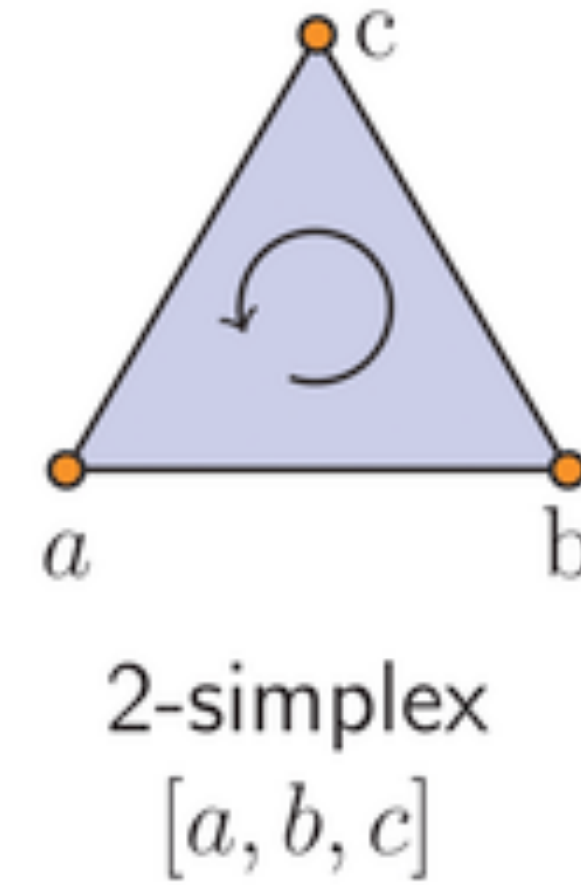
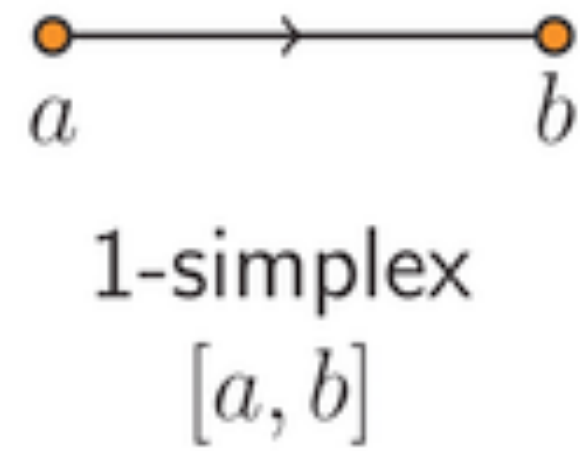
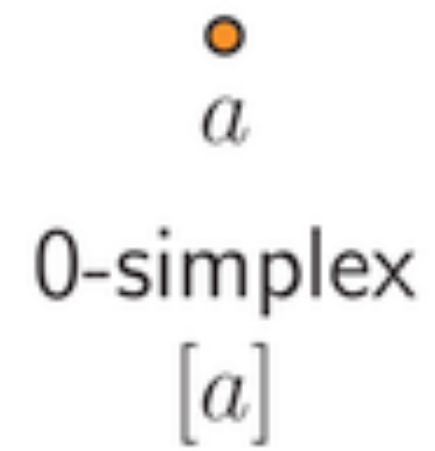
A hole is detected by a closed object that is not the boundary of anything.

Homology

Chains, boundaries, cycles, and holes

k-simplex

The simplest possible k-dimensional object with orientation



Chains: algebraic combinations of geometric pieces

A k -chain is a formal linear combination of oriented k -dimensional pieces.

Low-dimensional chains

$$C_0(X) : \quad c_0 = \sum_i n_i p_i$$

points with integer weights

$$C_1(X) : \quad c_1 = \sum_i n_i \gamma_i$$

oriented curves

Higher-dimensional chains

$$C_2(X) : \quad c_2 = \sum_i n_i \Sigma_i$$

oriented surfaces

$$C_3(X) : \quad c_3 = \sum_i n_i V_i$$

oriented volumes

The boundary operator ∂_k

The boundary map lowers dimension by one:

$$\partial_k : C_k(X) \longrightarrow C_{k-1}(X).$$

Examples

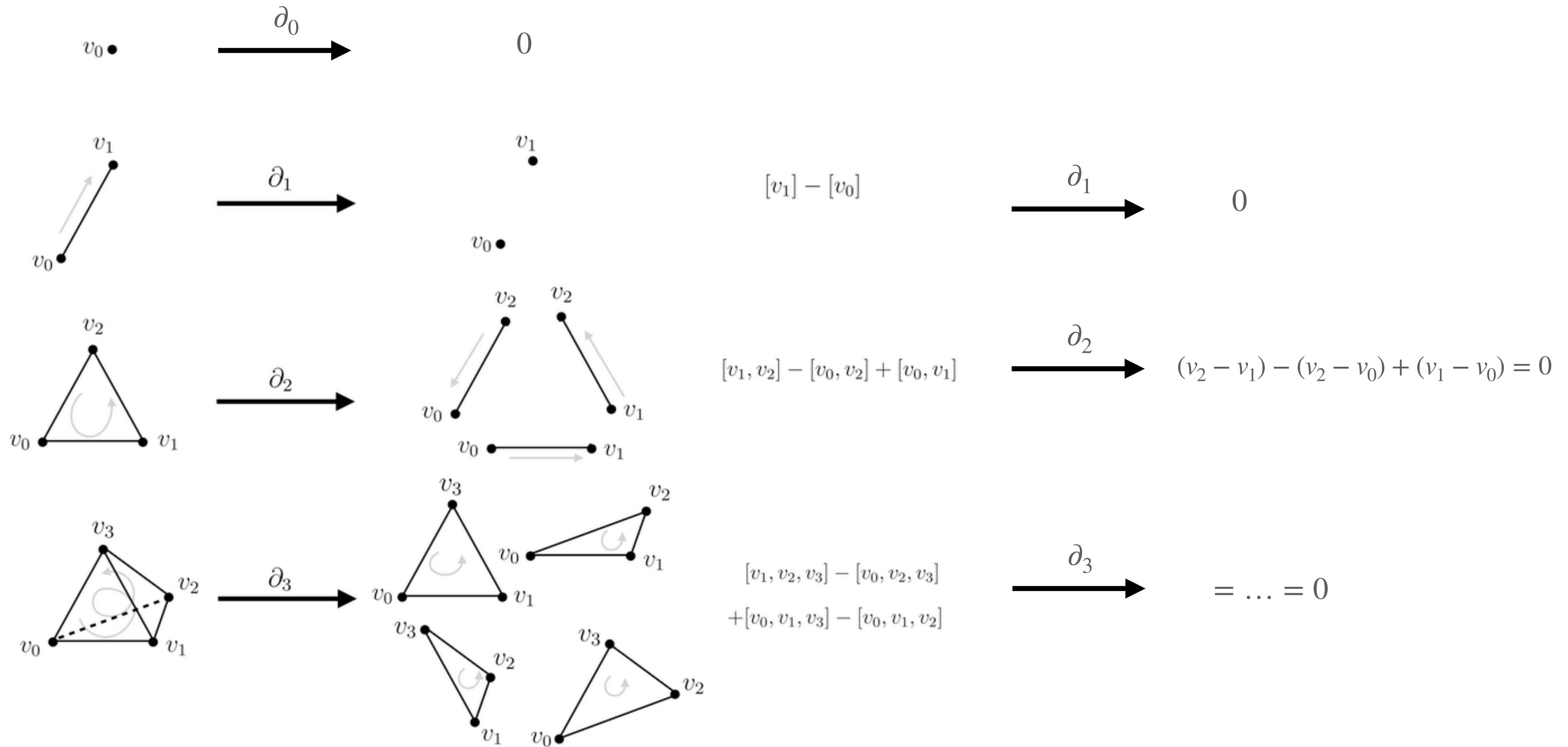
$$\partial p = 0$$

$$\partial[p, q] = q - p$$

$\partial\Sigma$ = oriented boundary curve of Σ

∂V = oriented surface enclosing V

The boundary operator ∂_k



The boundary of a boundary is zero

The fundamental identity is

$$\partial^2 = 0.$$

In words:

The boundary of a boundary has no boundary.

For example, if Σ is a surface and $\gamma = \partial\Sigma$, then

$$\partial\gamma = \partial(\partial\Sigma) = 0.$$

This is the algebraic reason why every boundary is automatically closed.

Cycles and boundaries

Cycles

A k -cycle is a k -chain with no boundary:

$$Z_k(X) = \ker \partial_k.$$

Thus

$$c \in Z_k(X) \iff \partial c = 0.$$

Boundaries

A k -boundary is the boundary of a $(k + 1)$ -chain:

$$B_k(X) = \operatorname{im} \partial_{k+1}.$$

Thus

$$c \in B_k(X) \iff c = \partial b.$$

Since $\partial^2 = 0$, every boundary is a cycle:

$$B_k(X) \subseteq Z_k(X).$$

Homology groups

The k -th homology group is

$$H_k(X) = \frac{Z_k(X)}{B_k(X)} = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

This means:

$$H_k(X) = \frac{\text{closed } k\text{-chains}}{\text{closed } k\text{-chains that bound}}.$$

A nonzero class $[c] \in H_k(X)$ is represented by

$$\partial c = 0, \quad c \neq \partial b \quad \text{for every } b \in C_{k+1}(X).$$

A hole is detected by a closed object that is not the boundary of anything.

Definition of a hole

Homological definition

A k -dimensional hole is detected by a k -chain c such that

$$\partial c = 0,$$

but

$$c \neq \partial b$$

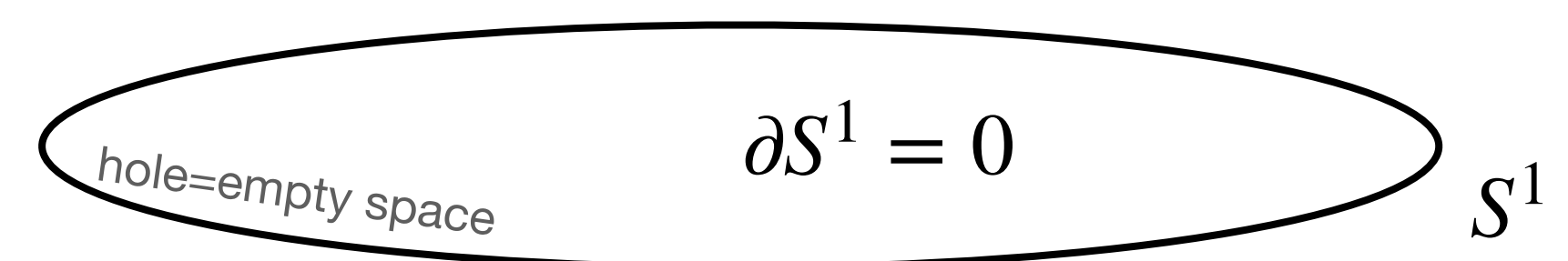
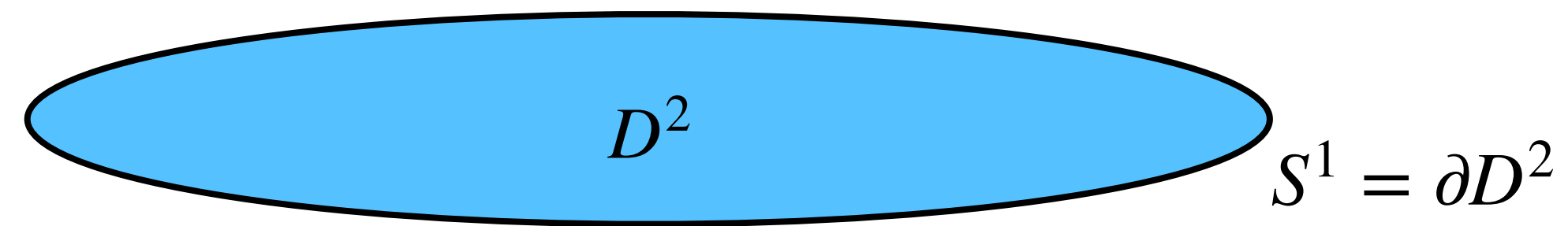
for every $(k + 1)$ -chain b in the space.

Equivalently,

$$[c] \neq 0 \in H_k(X).$$

So homology does not identify the hole itself as a point or region. It identifies a closed chain that cannot be filled inside the space.

Examples of homology classes



Disk D^2

The boundary circle satisfies

$$S^1 = \partial D^2.$$

So it is a cycle, but also a boundary:

$$[S^1] = 0 \in H_1(D^2).$$

No 1-dimensional hole.

Circle S^1

The whole circle satisfies

$$\partial S^1 = 0.$$

But no disk is contained in S^1 , so

$$[S^1] \neq 0 \in H_1(S^1).$$

One 1-dimensional hole.

de Rham Cohomology

Differential forms as detectors of holes

From chains to differential forms

Homology uses chains and the boundary operator:

$$C_k(X) \xrightarrow{\partial} C_{k-1}(X).$$

De Rham cohomology uses differential forms and the exterior derivative:

$$\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M).$$

The key parallel is

$$\boxed{\partial^2 = 0} \quad \Longleftrightarrow \quad \boxed{d^2 = 0.}$$

Closed and exact forms

Let $\omega \in \Omega^k(M)$.

Closed form

$$d\omega = 0.$$

The form has zero exterior derivative.

Exact form

$$\omega = d\eta$$

for some $(k - 1)$ -form η .

Since $d^2 = 0$, every exact form is closed:

$$\omega = d\eta \implies d\omega = d^2\eta = 0.$$

The de Rham cohomology group

The k -th de Rham cohomology group is

$$H_{\text{dR}}^k(M) = \frac{\ker(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\text{im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

In words:

$$H_{\text{dR}}^k(M) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

Thus cohomology measures closed forms that cannot be written as derivatives.

Example: a closed form which is exact

On $\mathbb{R}^2$, define the function

$$f(x, y) = x^2 + y^2.$$

Then

$$\omega = df = 2x \, dx + 2y \, dy.$$

So ω is exact by construction. It is also closed:

$$d\omega = d(df) = d^2f = 0.$$

Therefore

$\omega = 2x \, dx + 2y \, dy \text{ is closed and exact on } \mathbb{R}^2.$
--

Example: a closed form which is not exact

On the punctured plane

$$M = \mathbb{R}^2 \setminus \{0\},$$

consider the 1-form

$$\omega = \frac{-y \, dx + x \, dy}{x^2 + y^2}.$$

One checks that

$$d\omega = 0$$

on M , so ω is closed. But it is not exact, because on the unit circle S^1 ,

$$\int_{S^1} \omega = 2\pi \neq 0.$$

If $\omega = df$, then $\int_{S^1} df = 0$. Contradiction!

Why the non-exact form detects the hole

The form

$$\omega = \frac{-y \, dx + x \, dy}{x^2 + y^2}$$

measures angular change around the origin. For a loop γ going once around the missing point,

$$\int_{\gamma} \omega = 2\pi.$$

For a loop that does not wind around the hole,

$$\int_{\gamma} \omega = 0.$$

Thus the closed non-exact form detects the nontrivial homology class of loops around the puncture.

The homology/cohomology dictionary

Homology	de Rham cohomology
k -chains $C_k(X)$	k -forms $\Omega^k(M)$
$\partial : C_k \rightarrow C_{k-1}$	$d : \Omega^k \rightarrow \Omega^{k+1}$
$\partial^2 = 0$	$d^2 = 0$
cycles: $\partial c = 0$	closed forms: $d\omega = 0$
boundaries: $c = \partial b$	exact forms: $\omega = d\eta$
$H_k = Z_k / B_k$	$H_{\text{dR}}^k = \ker d / \text{im } d$

The bridge: integration and Stokes' theorem

A k -form can be integrated over a k -chain:

$$\langle \omega | c \rangle = \int_c \omega$$

The bridge between boundaries and derivatives is Stokes' theorem:

$$\langle \omega | \partial c \rangle = \boxed{\int_{\partial c} \omega = \int_c d\omega.} = \langle d\omega | c \rangle$$

Consequences:

- closed forms give invariants of homology classes;
- exact forms integrate to zero on cycles;
- non-exact closed forms can detect holes.

Takeaway

Homology

A hole is detected by a cycle that is not a boundary.

de Rham cohomology

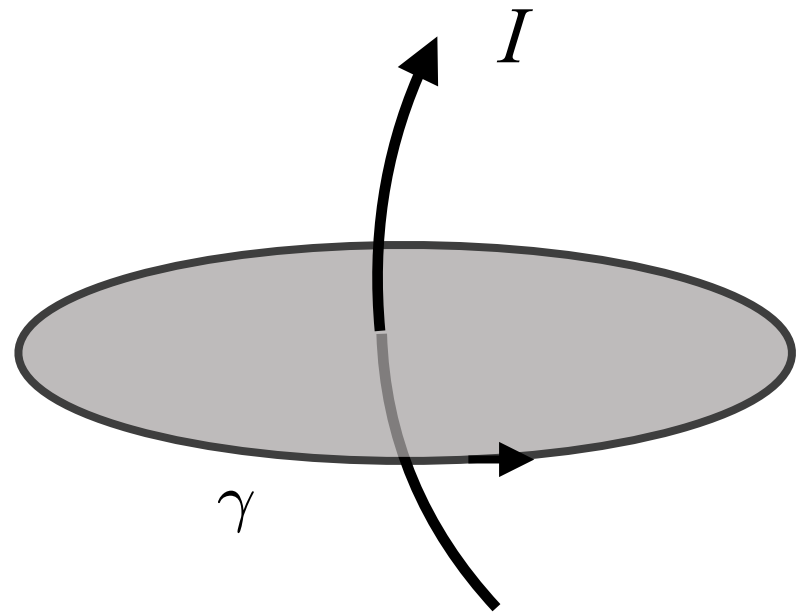
A hole is detected by a closed form that is not exact.

The two viewpoints are linked by integration and Stokes' theorem.

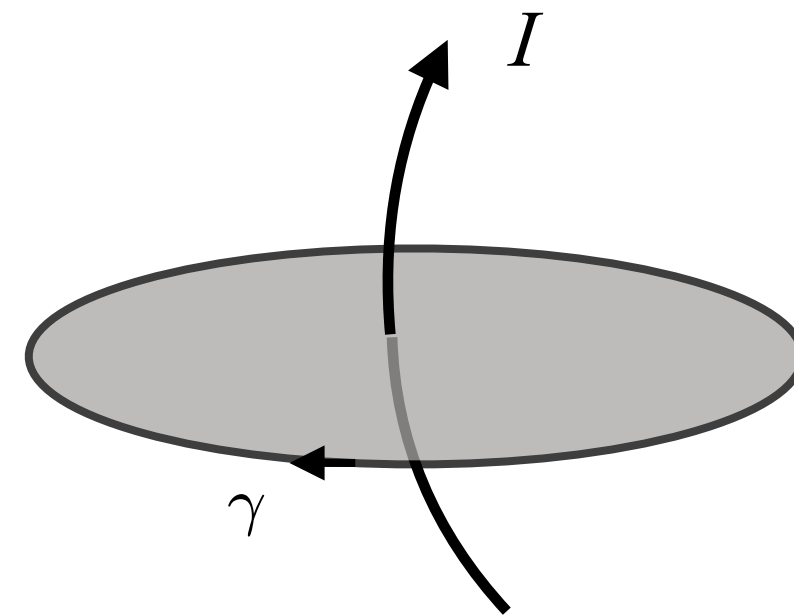
Modern Calculus

Algebraic structure of (regulated) Integrals

Ampere's Law

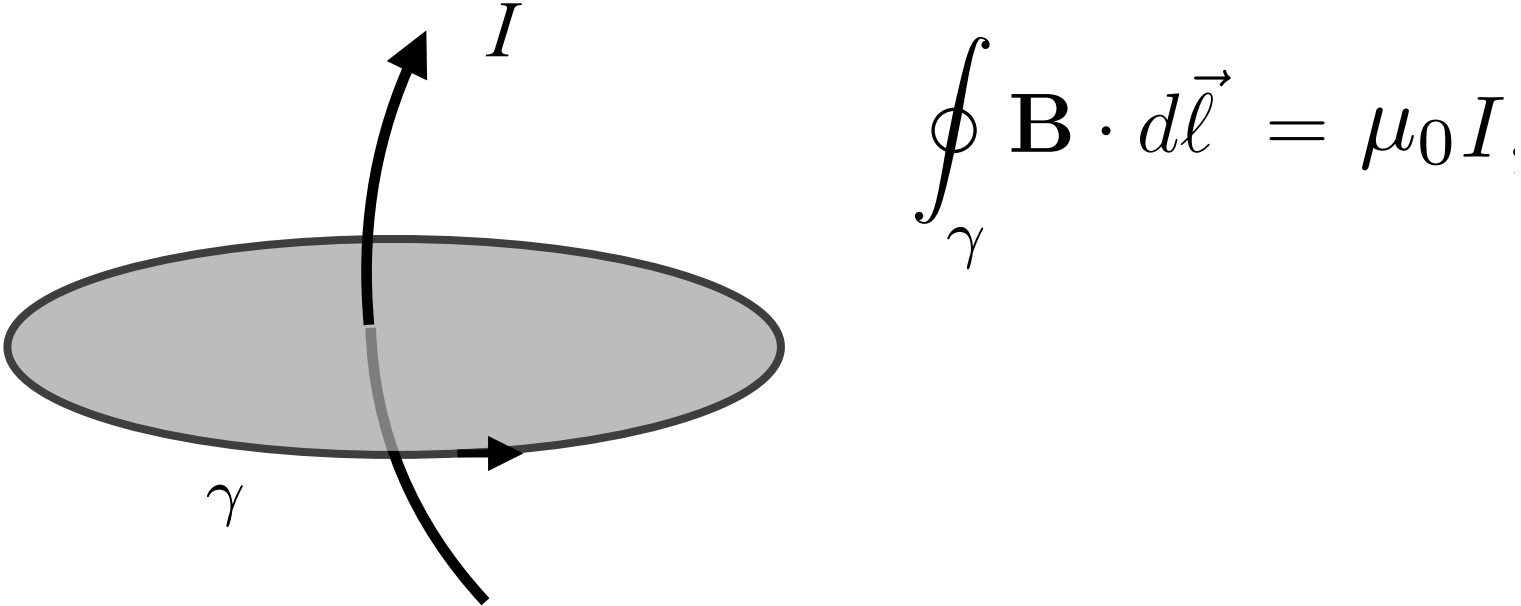


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

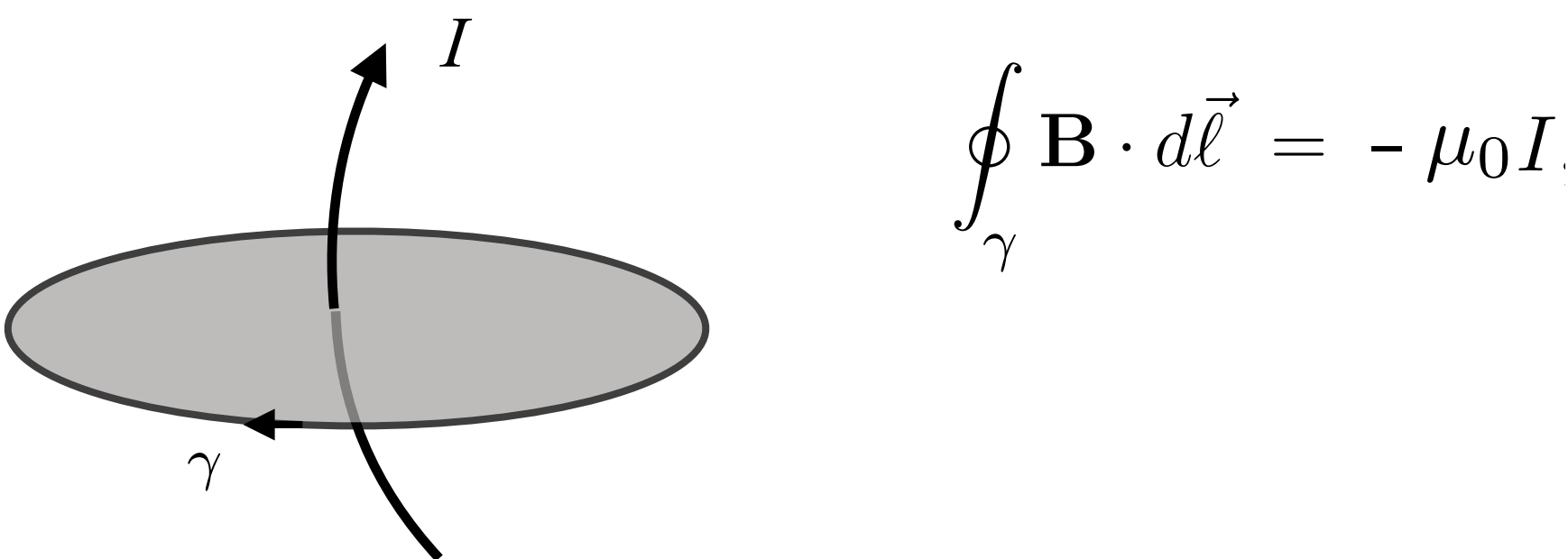


$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$

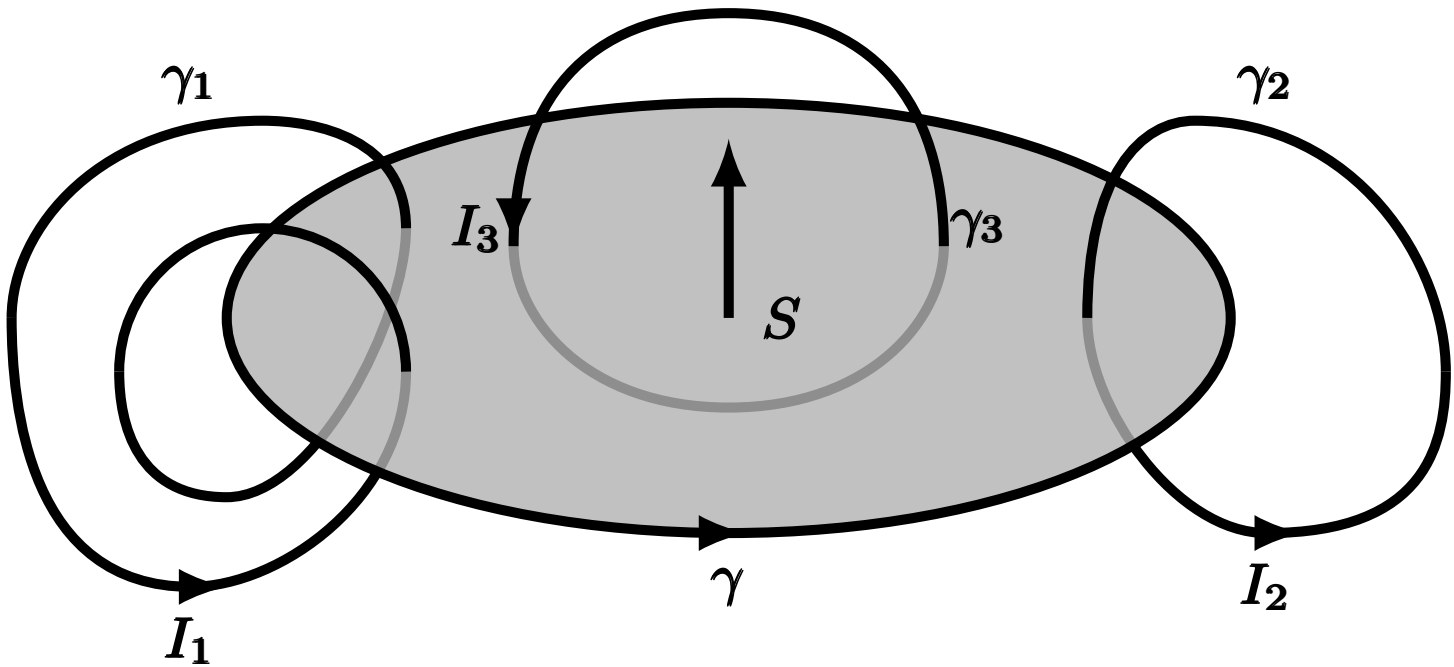
Ampere's Law



$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$



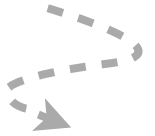
$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$



● Integral decomposition by geometry

$$\oint_{\gamma} \mathbf{B} \cdot d\vec{\ell} = \mu_0 \sum_k (\pm n_k) I_k$$

$$n_k = \text{Link}(\gamma_k, \gamma)$$

 Master Contributions

 Gauss' Linking Number

$$\text{Link}(\gamma_1, \gamma) = +2, \text{Link}(\gamma_2, \gamma) = -1, \text{and } \text{Link}(\gamma_3, \gamma) = 0$$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,..., Cacciatori, **P.M.**

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_C u(\mathbf{z}) \varphi_n(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function

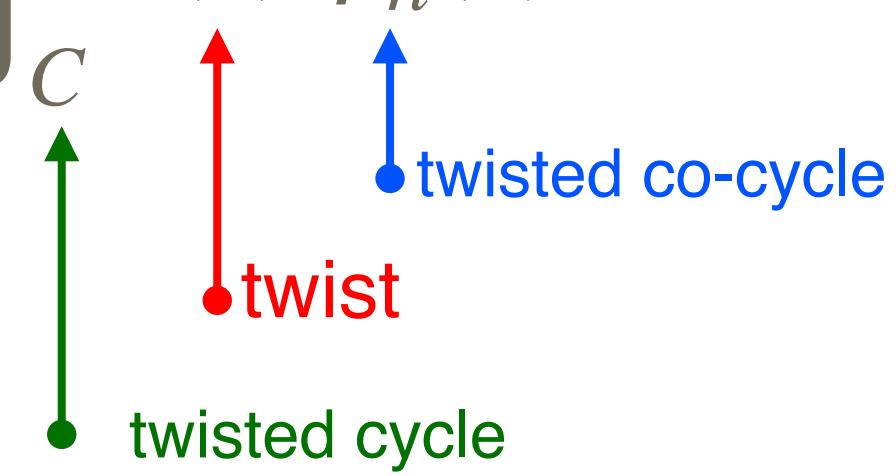
$$u(\partial\mathcal{C}) = 0$$

$\varphi_n(\mathbf{z})$ is a differential n -form

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,..., Cacciatori, **P.M.**

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_C u(\mathbf{z}) \varphi_n(\mathbf{z})$$


twisted cycle

twist

twisted co-cycle

$u(\mathbf{z})$ is a multivalued function

$$u(\partial\mathcal{C}) = 0$$

$\varphi_n(\mathbf{z})$ is a differential n -form

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,..., Cacciatori, **P.M.**

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_C u(\mathbf{z}) \varphi_n(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function

$$u(\partial\mathcal{C}) = 0$$

$\varphi_n(\mathbf{z})$ is a differential n -form

● Gamma function

$$u = z^{s-1} e^{-z}, \quad \mathcal{C} = (0, \infty)$$

$$\varphi_1 = \frac{dz}{z^{n_1}}$$

● Euler Beta function

$$u = z^a (z - 1)^b, \quad \mathcal{C} = (0, 1)$$

$$\varphi_1 = \frac{dz}{z^{n_1} (z - 1)^{n_2}}$$

● ${}_2F_1$ Hypergeometric

$$u = z^a (z - 1)^b (z - 1/x)^c, \quad \mathcal{C} = (0, 1)$$

$$\varphi_1 = \frac{dz}{z^{n_1} (z - 1)^{n_2} (z - 1/x)^{n_3}}$$

● Feynman Integrals

.....

.....

.....

● Quantum Mechanics and QFT

● ...and many more

$$n_i \in \mathbb{Z}$$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,..., Cacciatori, **P.M.**

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \dots, z_m)$

$$I = \int_C u(\mathbf{z}) \varphi_n(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function

$$u(\partial\mathcal{C}) = 0$$

$\varphi_n(\mathbf{z})$ is a differential n -form

- **The dawn of Relations among Integrals:**

- **Equivalence Classes of DIFFERENTIAL FORMS**

There could exist many forms φ_m that upon integration give the same result I

- **Equivalence Classes of INTEGRATION CONTOURS**

There could exist many contours $\mathcal{C}$ that do not alter the the result of I

$$n_i \in \mathbb{Z}$$

Vector Space Structure of Twisted Period Integrals

Feynman [- *Euler-Mellin* - *GKZ* - *A-hypergeometric*] Integrals

● **Basis** E_i , $i = 1, \dots, \nu$ ν = dimension of the vector space

● **Orthogonality** $E_i \bullet E_j = \delta_{ij}$ ● **Completeness** $\sum_{i=1}^{\nu} E_i E_i = \mathbb{I}_{\nu \times \nu}$

● **Vector decomposition** $I = \sum_{i=1}^{\nu} c_i E_i$

● **Projections** $c_i = I \bullet E_i$



Vector Space Structure of Twisted Period Integrals

Feynman [- *Euler-Mellin* - *GKZ* - *A-hypergeometric*] Integrals

● **Basis** E_i , $i = 1, \dots, \nu$ $\downarrow$
 ν = dimension of the vector space

● **Orthogonality** $E_i \bullet E_j = \delta_{ij}$ ● **Completeness** $\sum_{i=1}^{\nu} E_i E_i = \mathbb{I}_{\nu \times \nu}$

● **Vector decomposition** $I = \sum_{i=1}^{\nu} c_i E_i$

● **Projections** $c_i = I \bullet E_i$

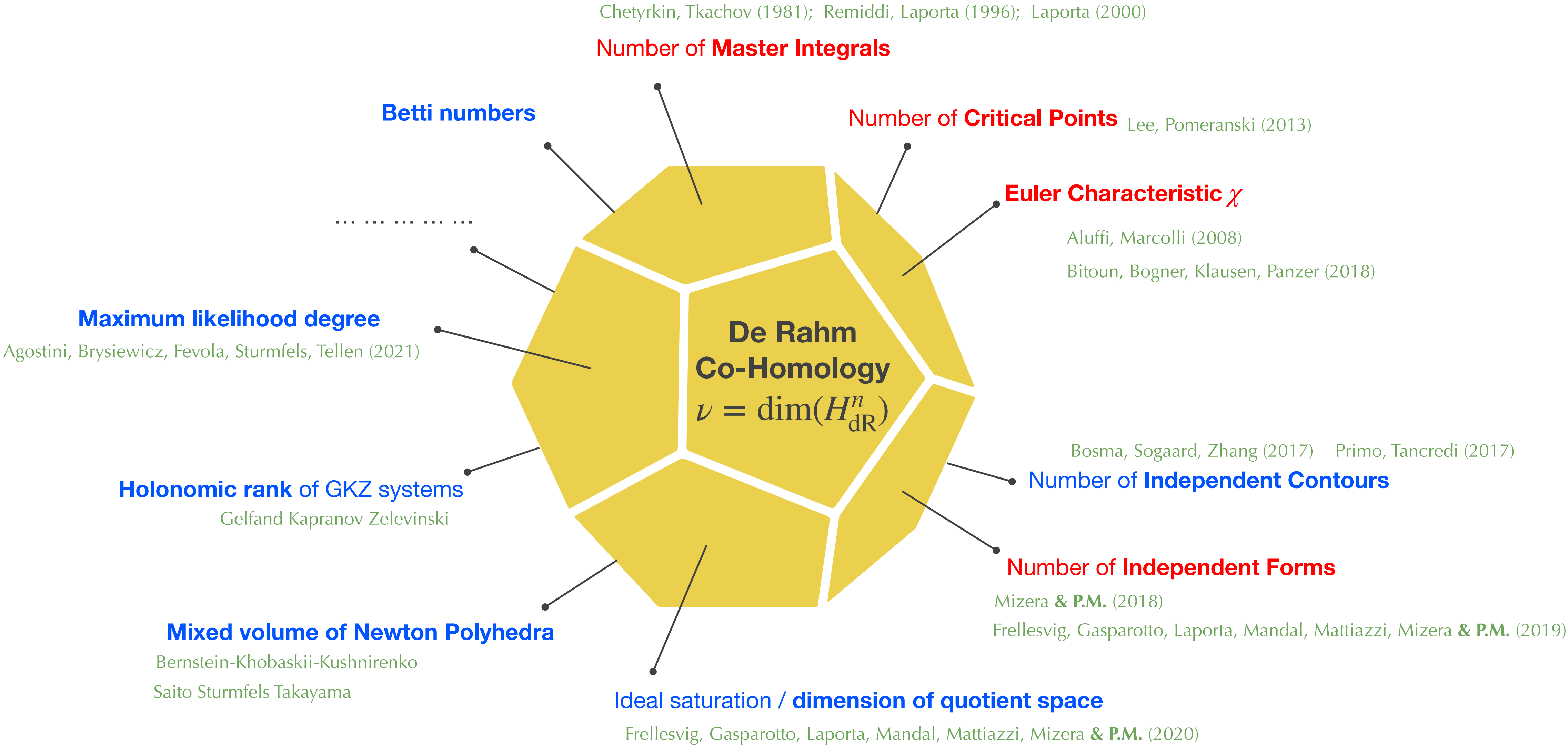
Questions:

Q1: what is ν ?

Q2: what is the inner product $\bullet$?

Vector Space Dimensions / counting “holes”

A1: answering Q1



Vector Space Dimensions / counting “holes”

A1: answering Q1

$$I = \int_C u(\mathbf{z}) \varphi_n(\mathbf{z})$$

● Dimension of De Rham twisted co-homology group

$$\nu = \dim(H_{\text{dR}}^n) = \left\{ \# \text{ sol's of } \frac{\partial}{\partial z_i} \log(u(\mathbf{z})) = 0 \right\}$$

Number of **Critical Points**

$$\begin{aligned} \text{if } u(\mathbf{z}) = \mathbf{z}^\rho f(\mathbf{z})^\gamma \quad \nu &= (-1)^n (n + 1 - \chi(V_E)) \\ &= (-1)^n (\chi(\mathbb{CP}^n) - \chi(V_E)) \end{aligned}$$

Number of **Independent Forms**
Euler Characteristic χ

V_E = vanishing locus of $\{\mathbf{z} f(\mathbf{z}) = 0\}$

Example: Beta and 2F1

$$\int_C u \, \varphi$$

● Euler Beta integral

$$\beta(p,q) \equiv \int_0^1 z^p (1-z)^q \frac{dz}{z(1-z)} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\omega = d \log(u) = \frac{\partial_z u}{u} = \frac{p}{z} - \frac{q}{(1-z)}$$

$$u = z^p (1-z)^q \, , \quad \varphi = \frac{dz}{z(1-z)} \, ,$$

$$\omega = 0 \quad \Rightarrow \quad \nu = \dim H^1 = 1$$

Example: Beta and 2F1

$$\int_C u \varphi$$

●Euler Beta integral

$$\beta(p,q) \equiv \int_0^1 z^p (1-z)^q \frac{dz}{z(1-z)} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$u = z^p (1-z)^q \ , \quad \varphi = \frac{dz}{z(1-z)} \ ,$$

$$\omega = d \log(u) = \frac{\partial_z u}{u} = \frac{p}{z} - \frac{q}{(1-z)}$$

$$\omega = 0 \quad \Rightarrow \quad \nu = \dim H^1 = 1$$

● ${}_2F_1$ Gauss hypergeometric integral

$$\Gamma(b)\Gamma(c-b) {}_2F_1(a,b,c;x) \equiv \int_0^1 z^b (1-z)^{c-b} (1-xz)^{-a} \frac{dz}{z(1-z)}$$

$$u = z^b (1-z)^{c-b} (1-xz)^{-a} \ , \quad \varphi = \frac{dz}{z(1-z)}$$

$$\omega = d \log(u) = \frac{\partial_z u}{u} = \frac{b}{z} - \frac{c-b}{(1-z)} + \frac{ax}{(1-zx)}$$

$$\omega = 0 \quad \Rightarrow \quad \nu = \dim H^1 = 2$$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

0 =

$\int_C d(u \varphi_{m-1})$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

0 =

$$\int_{\mathcal{C}} d\left(u \, \varphi_{m-1}\right) = \int_{\mathcal{C}} \left(u \, d\varphi_{m-1} + du \wedge \varphi_{m-1}\right)$$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

0 =

$$\int_{\mathcal{C}} d\left(u \, \varphi_{m-1}\right) = \int_{\mathcal{C}} \left(u \, d\varphi_{m-1} + du \wedge \varphi_{m-1}\right) = \int_{\mathcal{C}} u\left(d + \omega \wedge \right)\varphi_{m-1}$$

●

Covariant Derivative

$\omega \equiv d \log u$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d\left(u \varphi_{m-1}\right) = \int_{\mathcal{C}} \left(u \, d\varphi_{m-1} + du \wedge \varphi_{m-1}\right) = \int_{\mathcal{C}} u \left(d + \omega \wedge \right) \varphi_{m-1} = \int_{\mathcal{C}} u \, \nabla_{\omega} \varphi_{m-1}$$

● **Covariant Derivative** $\omega \equiv d \log u$ $\nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

0 =

$$\int_{\mathcal{C}} d\left(u \varphi_{m-1}\right) = \int_{\mathcal{C}} \left(u d\varphi_{m-1} + du \wedge \varphi_{m-1}\right) = \int_{\mathcal{C}} u\left(d + \omega \wedge\right) \varphi_{m-1} = \int_{\mathcal{C}} u \nabla_{\omega} \varphi_{m-1}$$

● Covariant Derivative

$$\omega \equiv d \log u \qquad \nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$$

● Integrals

$$I \quad = \quad \int_{\mathcal{C}} u \varphi_m \quad = \quad \int_{\mathcal{C}} u \left(\varphi_m + \nabla_{\omega} \varphi_{m-1} \right) \quad = \quad \int_{\mathcal{C} + \partial \Gamma} u \varphi_m$$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

0 =

$$\int_{\mathcal{C}} d\left(u \varphi_{m-1}\right)=\int_{\mathcal{C}}\left(u d \varphi_{m-1}+d u \wedge \varphi_{m-1}\right)=\int_{\mathcal{C}} u\left(d+\omega \wedge\right) \varphi_{m-1}=\int_{\mathcal{C}} u \nabla_{\omega} \varphi_{m-1}$$

● Covariant Derivative

$$\omega \equiv d \log u \qquad \nabla_{\omega} \equiv d+\omega \wedge \equiv u^{-1} \cdot d \cdot u$$

● Integrals

$$I=\int_{\mathcal{C}} u \varphi_m=\int_{\mathcal{C}} u\left(\varphi_m+\nabla_{\omega} \varphi_{m-1}\right)=\int_{\mathcal{C}+\partial \Gamma} u \varphi_m$$

● Twisted Cohomology Group

$$H_{\omega}^m(X)=\frac{\text { Ker }\left(\nabla_{\omega}: \varphi_m \rightarrow \varphi_{m+1}\right)}{\text { Im }\left(\nabla_{\omega}: \varphi_{m-1} \rightarrow \varphi_m\right)}$$

Basics of Intersection Theory for deRham (co)-Homology

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$0 = \int_{\mathcal{C}} d\left(u \varphi_{m-1}\right) = \int_{\mathcal{C}} \left(u d\varphi_{m-1} + du \wedge \varphi_{m-1}\right) = \int_{\mathcal{C}} u\left(d + \omega \wedge \right) \varphi_{m-1} = \int_{\mathcal{C}} u \nabla_{\omega} \varphi_{m-1}$

● Covariant Derivative

$\omega \equiv d \log u \qquad \nabla_{\omega} \equiv d + \omega \wedge \equiv u^{-1} \cdot d \cdot u$

● Integrals

$I = \int_{\mathcal{C}} u \varphi_m = \int_{\mathcal{C}} u \left(\varphi_m + \nabla_{\omega} \varphi_{m-1} \right) = \int_{\mathcal{C} + \partial \Gamma} u \varphi_m$

● Twisted Cohomology Group

$H_{\omega}^m(X) = \frac{\text{Ker}(\nabla_{\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$

● Twisted Homology Group

$H_m^{\omega}(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$

Basics of Intersection Theory for deRham (co)-Homology

$$u \rightarrow u^{-1} \qquad \Longleftrightarrow \qquad \omega \rightarrow -\omega$$

Consider the $(m - 1)$ -differential form φ_{m-1} ,

$$0 = \int_{\mathcal{C}} d\Big(u^{-1} \, \varphi_{m-1}\Big) = \int_{\mathcal{C}} \Big(u^{-1} \, d\varphi_{m-1} - u^{-2} du \wedge \varphi_{m-1}\Big) = \int_{\mathcal{C}} u^{-1} \Big(d - \omega \wedge \Big) \varphi_{m-1} = \int_{\mathcal{C}} u^{-1} \, \nabla_{-\omega} \varphi_{m-1}$$

- Dual Covariant Derivative

$$\nabla_{-\omega} \equiv d - \omega \wedge \equiv u \cdot d \cdot u^{-1}$$

- Dual Integrals

$$\tilde{I} \; = \; \int_{\mathcal{C}} u^{-1} \, \varphi_m \; = \; \int_{\mathcal{C}} u^{-1} \, \Big(\varphi_m + \nabla_{-\omega} \varphi_{m-1}\Big) = \int_{\mathcal{C} + \partial \Gamma} u^{-1} \, \varphi_m$$

- Dual Twisted Cohomology Group

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

- Dual Twisted Homology Group

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

Pairings of Cycles and Co-cycles

- Twisted and Dual-Twisted Co-Homology Groups

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

Pairings of Cycles and Co-cycles

● Twisted and Dual-Twisted Co-Homology Groups

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

[a deep identity]

Pairings of Cycles and Co-cycles

- Twisted and Dual-Twisted Co-Homology Groups

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

- Basic building blocks

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

Pairings of Cycles and Co-cycles

● **Twisted and Dual-Twisted Co-Homology Groups**

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

● **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

● **Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L \mid \mathcal{C}_R \rangle \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

Pairings of Cycles and Co-cycles

● **Twisted and Dual-Twisted Co-Homology Groups**

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

● **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

● **Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L \mid \mathcal{C}_R \rangle \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

● **Dual Integrals ::** pairings of cycles and co-cycles

$$[\mathcal{C}_L \mid \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

Pairings of Cycles and Co-cycles

- **Twisted and Dual-Twisted Co-Homology Groups**

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)}$$

$$H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega$$

$$| \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$| \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m$$

$$[\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

- **Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

- **Dual Integrals ::** pairings of cycles and co-cycles

$$[\mathcal{C}_L | \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

- **Intersection numbers for cycles ::** pairings of cycles

$$[\mathcal{C}_L | \mathcal{C}_R] \equiv \text{intersection number}$$

Pairings of Cycles and Co-cycles

- **Twisted and Dual-Twisted Co-Homology Groups**

$$H^m_\omega(X) = \frac{\text{Ker}(\nabla_\omega : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_\omega : \varphi_{m-1} \rightarrow \varphi_m)} \qquad H^m_{-\omega}(X) = \frac{\text{Ker}(\nabla_{-\omega} : \varphi_m \rightarrow \varphi_{m+1})}{\text{Im}(\nabla_{-\omega} : \varphi_{m-1} \rightarrow \varphi_m)} \qquad H^\omega_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})} \qquad H^{-\omega}_m(X) = \frac{\text{Ker}(\partial : \mathcal{C}_{m+1} \rightarrow \mathcal{C}_m)}{\text{Im}(\partial : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1})}$$

$$\dim H^m_{\pm\omega} = \dim H^{\pm\omega}_m \equiv \nu$$

- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad | \varphi_R \rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega} \qquad | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H^\omega_m \qquad [\mathcal{C}_L | \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H^{-\omega}_m$$

- **Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L | \mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = I$$

- **Dual Integrals ::** pairings of cycles and co-cycles

$$[\mathcal{C}_L | \varphi_R \rangle \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = \tilde{I}$$

- **Intersection numbers for cycles ::** pairings of cycles

$$[\mathcal{C}_L | \mathcal{C}_R] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles ::** pairings of co-cycles

$$\langle \varphi_L | \varphi_R \rangle \equiv \int_{\mathcal{C}} \iota(\varphi_L) \wedge \varphi_R$$

Linear Algebra

- Cohomology Basis :: bra

$\langle e_i | \quad i = 1, 2, \dots, \nu$
- generic form

$\langle \varphi |$
- Cohomology Dual-Basis :: ket

$| h_j \rangle \quad j = 1, 2, \dots, \nu$
- generic form

$| \psi \rangle$

Linear Algebra

- Cohomology Basis :: bra

$\langle e_i|$

$i = 1, 2, \dots, \nu$
- generic form

$\langle \varphi|$
- Cohomology Dual-Basis :: ket

$|h_j\rangle$

$j = 1, 2, \dots, \nu$
- generic form

$|\psi\rangle$

$(\nu+1) \times (\nu+1)$ matrix **M**

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \dots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \dots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \dots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix}$$

Linear Algebra

● Cohomology Basis :: bra

$\langle e_i | \quad i = 1, 2, \dots, \nu$

● generic form

$\langle \varphi |$

● Cohomology Dual-Basis :: ket

$| h_j \rangle \quad j = 1, 2, \dots, \nu$

● generic form

$|\psi\rangle$

$(\nu+1) \times (\nu+1)$ matrix **M**

$$\mathbf{M} = \begin{pmatrix} \langle \varphi | \psi \rangle & \langle \varphi | h_1 \rangle & \langle \varphi | h_2 \rangle & \dots & \langle \varphi | h_\nu \rangle \\ \langle e_1 | \psi \rangle & \langle e_1 | h_1 \rangle & \langle e_1 | h_2 \rangle & \dots & \langle e_1 | h_\nu \rangle \\ \langle e_2 | \psi \rangle & \langle e_2 | h_1 \rangle & \langle e_2 | h_2 \rangle & \dots & \langle e_2 | h_\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu | \psi \rangle & \langle e_\nu | h_1 \rangle & \langle e_\nu | h_2 \rangle & \dots & \langle e_\nu | h_\nu \rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi | \psi \rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi | \psi \rangle - \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} \right)$$

Linear Algebra

- Cohomology Basis :: bra

$\langle e_i|$

$i = 1, 2, \dots, \nu$
- generic form

$\langle \varphi|$
- Cohomology Dual-Basis :: ket

$|h_j\rangle$

$j = 1, 2, \dots, \nu$
- generic form

$|\psi\rangle$

$(\nu+1) \times (\nu+1)$ matrix **M**

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \dots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \dots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \dots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi|\psi\rangle - \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} \right) = 0$$

$$\mathbf{C}_{ij} = \langle e_i|h_j\rangle$$

$$\det \mathbf{C} \neq 0 \quad \Rightarrow \quad \langle \varphi|\psi\rangle = \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B}$$

Linear Algebra

● Cohomology Basis :: bra

$\langle e_i|$

$i = 1, 2, \dots, \nu$

● generic form

$\langle \varphi|$

● Cohomology Dual-Basis :: ket

$|h_j\rangle$

$j = 1, 2, \dots, \nu$

● generic form

$|\psi\rangle$

$(\nu+1) \times (\nu+1)$ matrix $\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \dots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \dots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \dots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi|\psi\rangle - \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} \right) = 0$$

$\mathbf{C}_{ij} = \langle e_i|h_j\rangle$

$\det \mathbf{C} \neq 0 \quad \Rightarrow \quad \langle \varphi|\psi\rangle = \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} = \sum_{i,j=1}^\nu \langle \varphi|h_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i|\psi\rangle$

Linear Algebra

- Cohomology Basis :: bra

$\langle e_i|$

$i = 1, 2, \dots, \nu$
- generic form

$\langle \varphi|$
- Cohomology Dual-Basis :: ket

$|h_j\rangle$

$j = 1, 2, \dots, \nu$
- generic form

$|\psi\rangle$

$(\nu+1) \times (\nu+1)$ matrix $\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \dots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \dots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \dots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi|\psi\rangle - \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} \right) = 0$$

$\mathbf{C}_{ij} = \langle e_i|h_j\rangle$

$\det \mathbf{C} \neq 0 \quad \Rightarrow \quad \langle \varphi|\psi\rangle = \mathbf{A}^\top \mathbf{C}^{-1} \mathbf{B} = \sum_{i,j=1}^\nu \langle \varphi|h_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i|\psi\rangle$



$$\sum_{i,j=1}^\nu |h_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c$$

Identity Resolution

$$\dim H_{\pm\omega}^m = \dim H_m^{\pm\omega} \equiv \nu$$

● **Forms**

● **Cohomology Basis**

$$\langle e_i | \qquad i = 1, 2, \dots, \nu$$

● **Cohomology Dual-Basis**

$$| h_j \rangle \qquad j = 1, 2, \dots, \nu$$

$$\sum_{i,j=1}^{\nu} | h_j \rangle (\mathbf{C}^{-1})_{ji} \langle e_i | = \mathbb{I}_c$$

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

Metric Matrix for Forms

Identity Resolution

$$\dim H_{\pm\omega}^m = \dim H_m^{\pm\omega} \equiv \nu$$

● **Forms**

● Cohomology Basis $\langle e_i| \quad i = 1, 2, \dots, \nu$

● Cohomology Dual-Basis $|h_j\rangle \quad j = 1, 2, \dots, \nu$

$$\sum_{i,j=1}^{\nu} |h_j\rangle (\mathbf{C}^{-1})_{ji} \langle e_i| = \mathbb{I}_c$$

$\mathbf{C}_{ij} \equiv \langle e_i|h_j\rangle$
Metric Matrix for Forms

● **Contours**

● Homology Basis $[\mathcal{C}_{L,i}| \quad i = 1, 2, \dots, \nu$

● Homology Dual-Basis $|\mathcal{C}_{R,j}] \quad j = 1, 2, \dots, \nu$

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_{R,j}] (\mathbf{H}^{-1})_{ji} [\mathcal{C}_{L,i}| = \mathbb{I}_h$$

$\mathbf{H}_{ji} \equiv [\mathcal{C}_{L,i}|\mathcal{C}_{R,j}]$
Metric Matrix for Contours

Linear Relations

Flux Decomposition

$$\int_{\gamma} \varphi = \text{[Diagram of a diamond-shaped surface with a central dot, an arrow labeled } \gamma \text{ along the bottom edge, and an arrow labeled } \varphi \text{ pointing upwards from the center]} = \langle \varphi | \gamma \rangle$$

Flux Decomposition

$$\int_{\gamma} \varphi = \text{[Diagram: A gray parallelogram representing a surface with a central point. A curved arrow labeled γ points along the bottom edge, and another curved arrow labeled φ points upwards from the center.]} = \langle \varphi | \gamma \rangle$$

● Contour decomposition

$$\text{[Diagram: A gray parallelogram representing a surface. A blue contour γ_1 enters from the top left, goes down, then right, then up, and exits to the top right. A red contour γ_2 enters from the bottom left, goes up, then right, then down, and exits to the bottom right. Dotted lines indicate the continuation of these contours. A central point has a curved arrow labeled φ pointing upwards. The entire surface is labeled γ at the bottom.]} = \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

$$|\gamma] = \sum_i a_i |\gamma_i]$$

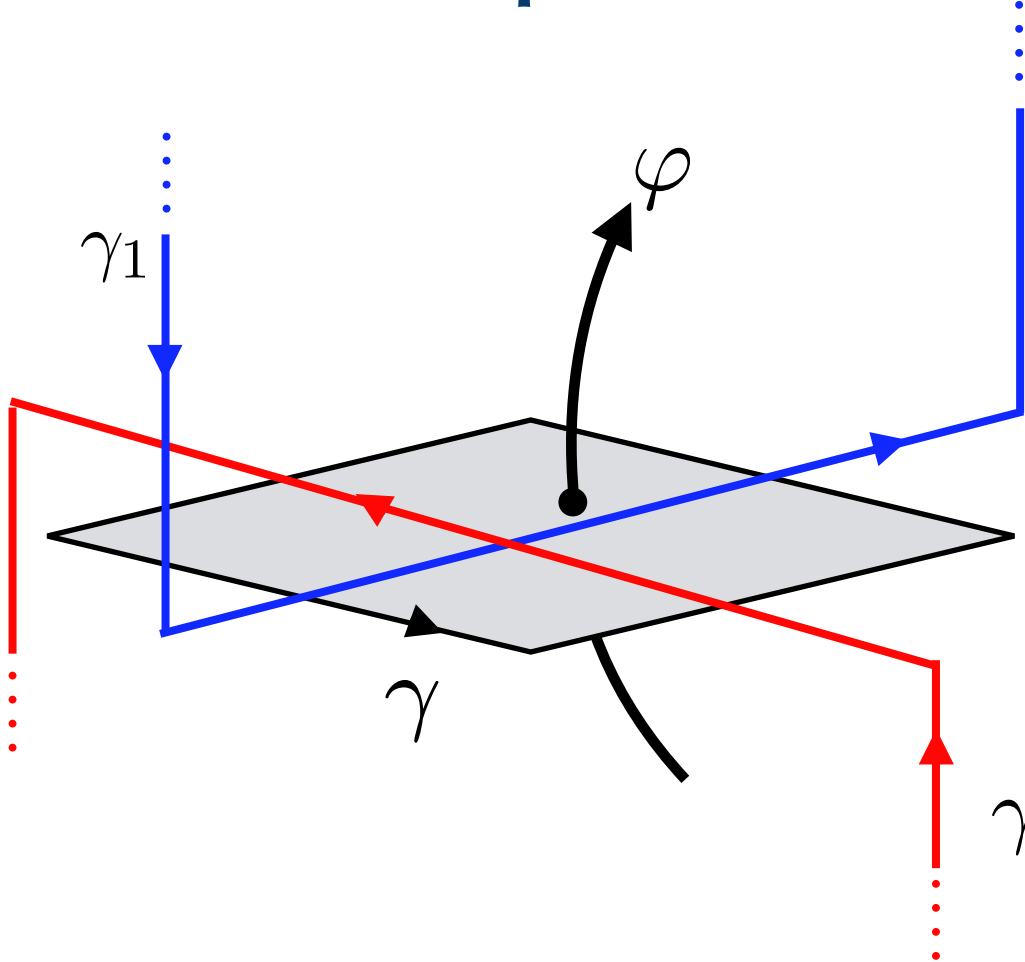
● Coefficients are **Intersection Numbers (contours)**

$$a_i = [\gamma_i | \gamma] , \quad [\gamma_i | \gamma_j] = \delta_{ij}$$

Flux Decomposition

$$\int_{\gamma} \varphi = \text{[Diagram: A diamond-shaped surface with a vector γ tangent to its edge and a vector φ normal to its surface]} = \langle \varphi | \gamma \rangle$$

● **Contour decomposition**

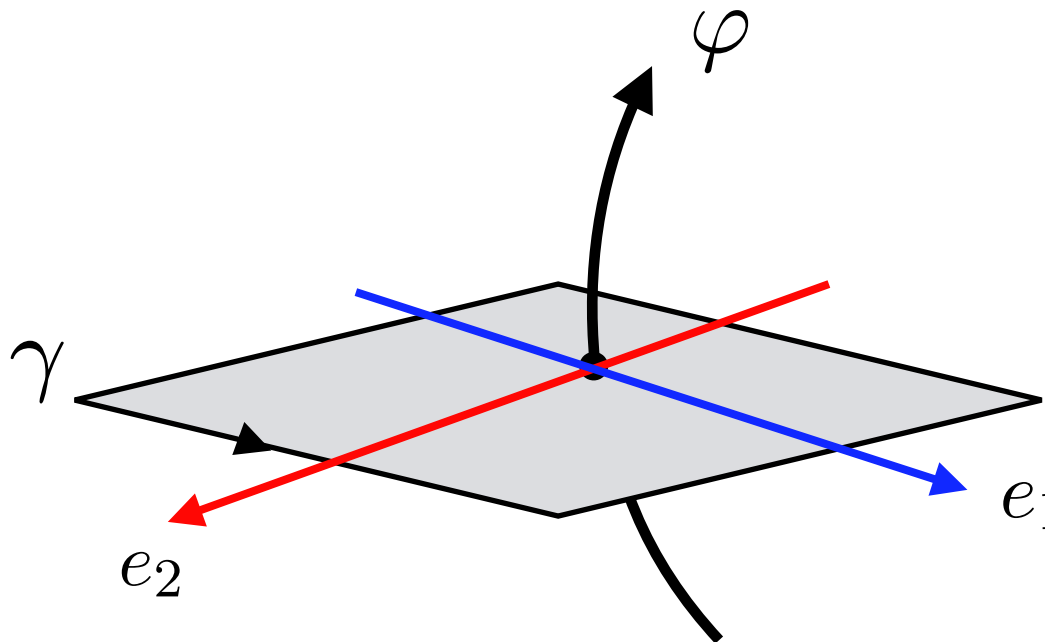

$$= \sum_{i=1} a_i \int_{\gamma_i} \varphi$$

$$|\gamma\rangle = \sum_i a_i |\gamma_i\rangle$$

● **Coefficients are Intersection Numbers (contours)**

$$a_i = [\gamma_i | \gamma] , \quad [\gamma_i | \gamma_j] = \delta_{ij}$$

● **Form decomposition**


$$= \sum_i c_i \int_{\gamma} e_i$$

$$\langle \varphi | = \sum_i c_i \langle e_i |$$

● **Coefficients are Intersection Numbers (forms)**

$$c_i = \langle \varphi | e_i \rangle , \quad \langle e_i | e_j \rangle = \delta_{ij}$$

Twisted Period Integrals and Intersection Theory

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019-20)

$$I = \langle \varphi | \mathcal{C}]$$

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z}) \, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}] \, , \qquad i = 1, \dots, \nu \, ,$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

Twisted Period Integrals and Intersection Theory

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019-20)

$$I = \langle \varphi | \mathcal{C}]$$

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z}) e_i(\mathbf{z}) = \langle e_i | \mathcal{C}] , \quad i = 1, \dots, \nu ,$$

$$I = \sum_{i=1}^{\nu} c_i J_i \quad \Longleftrightarrow \quad \langle \varphi | = \sum_{i=1}^{\nu} c_i \langle e_i |$$

● (Left) Form decomposition

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \underbrace{\langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}}_{=1} \langle e_i | = \sum_{i=1}^{\nu} c_i \langle e_i |$$



$$c_i = \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}$$

[independent of $|h_j\rangle$]

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

● Twisted Riemann Period Relations Cho, Matsumoto (1995)

$$\langle \varphi_i | \varphi_j \rangle = \sum_{k, \ell} \underbrace{\langle \varphi_i | \gamma_k \rangle [\gamma_\ell | \gamma_k]^{-1} [\gamma_\ell | \varphi_j]}_{=1} = (\mathbf{P}_\omega \cdot \mathbf{H}^{-1} \cdot \mathbf{P}_{-\omega})_{ij}$$

$$\mathbf{P}_\omega = \begin{pmatrix} \int_{\gamma_1} u \varphi_1 & \int_{\gamma_2} u \varphi_1 & \cdots & \int_{\gamma_\nu} u \varphi_1 \\ \int_{\gamma_1} u \varphi_2 & \int_{\gamma_2} u \varphi_2 & \cdots & \int_{\gamma_\nu} u \varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1} u \varphi_\nu & \int_{\gamma_2} u \varphi_\nu & \cdots & \int_{\gamma_\nu} u \varphi_\nu \end{pmatrix} \quad \mathbf{H} = \begin{pmatrix} [\gamma_1 | \gamma_1] & [\gamma_1 | \gamma_2] & \cdots & [\gamma_1 | \gamma_\nu] \\ [\gamma_2 | \gamma_1] & [\gamma_2 | \gamma_2] & \cdots & [\gamma_2 | \gamma_\nu] \\ \vdots & \vdots & \ddots & \vdots \\ [\gamma_\nu | \gamma_1] & [\gamma_\nu | \gamma_2] & \cdots & [\gamma_\nu | \gamma_\nu] \end{pmatrix} \quad \mathbf{P}_{-\omega} = \begin{pmatrix} \int_{\gamma_1} u^{-1} \varphi_1 & \int_{\gamma_2} u^{-1} \varphi_1 & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_1 \\ \int_{\gamma_1} u^{-1} \varphi_2 & \int_{\gamma_2} u^{-1} \varphi_2 & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1} u^{-1} \varphi_\nu & \int_{\gamma_2} u^{-1} \varphi_\nu & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_\nu \end{pmatrix}$$

Twisted Riemann Periods Relations (TRPR)

● Twisted Riemann Period Relations Cho, Matsumoto (1995)

$$[\gamma_i | \gamma_j] = \sum_{k, \ell} [\gamma_i | \varphi_k] \underbrace{\langle \varphi_\ell | \varphi_k \rangle^{-1}}_{=1} \langle \varphi_\ell | \gamma_j \rangle = (\mathbf{P}_{-\omega} \cdot \mathbf{C}^{-1} \cdot \mathbf{P}_\omega)_{ij}$$

$$\mathbf{P}_\omega = \begin{pmatrix} \int_{\gamma_1} u \varphi_1 & \int_{\gamma_2} u \varphi_1 & \cdots & \int_{\gamma_\nu} u \varphi_1 \\ \int_{\gamma_1} u \varphi_2 & \int_{\gamma_2} u \varphi_2 & \cdots & \int_{\gamma_\nu} u \varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1} u \varphi_\nu & \int_{\gamma_2} u \varphi_\nu & \cdots & \int_{\gamma_\nu} u \varphi_\nu \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} \langle \varphi_1 | \varphi_1 \rangle & \langle \varphi_1 | \varphi_2 \rangle & \cdots & \langle \varphi_1 | \varphi_\nu \rangle \\ \langle \varphi_2 | \varphi_1 \rangle & \langle \varphi_2 | \varphi_2 \rangle & \cdots & \langle \varphi_2 | \varphi_\nu \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \varphi_\nu | \varphi_1 \rangle & \langle \varphi_\nu | \varphi_2 \rangle & \cdots & \langle \varphi_\nu | \varphi_\nu \rangle \end{pmatrix} \quad \mathbf{P}_{-\omega} = \begin{pmatrix} \int_{\gamma_1} u^{-1} \varphi_1 & \int_{\gamma_2} u^{-1} \varphi_1 & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_1 \\ \int_{\gamma_1} u^{-1} \varphi_2 & \int_{\gamma_2} u^{-1} \varphi_2 & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1} u^{-1} \varphi_\nu & \int_{\gamma_2} u^{-1} \varphi_\nu & \cdots & \int_{\gamma_\nu} u^{-1} \varphi_\nu \end{pmatrix}$$

EXTRA

Master Decomposition Formulas

● (Left) Form decomposition

$$\langle \phi_L | = \sum_{i=1}^v c_i \langle e_i |$$

$$c_i = \sum_{j=1}^v \langle \phi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

● (Right) Form decomposition

$$| \phi_R \rangle = \sum_{i=1}^v \tilde{c}_i | h_i \rangle$$

$$\tilde{c}_i = \sum_{j=1}^v \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j | \phi_R \rangle$$

● (Left) Contour decomposition

$$[\mathcal{C}_L | = \sum_i \tilde{a}_i [\mathcal{C}_{L,i} |$$

$$\tilde{a}_i = \sum_{j=1}^v [\mathcal{C}_L | \mathcal{C}_{R,j}] \left(\mathbf{H}^{-1} \right)_{ji}$$

● (Right) Contour decomposition

$$| \mathcal{C}_R] = \sum_i a_i | \mathcal{C}_{R,i}]$$

$$a_i = \sum_{j=1}^v \left(\mathbf{H}^{-1} \right)_{ij} [\mathcal{C}_{L,j} | \mathcal{C}_R]$$