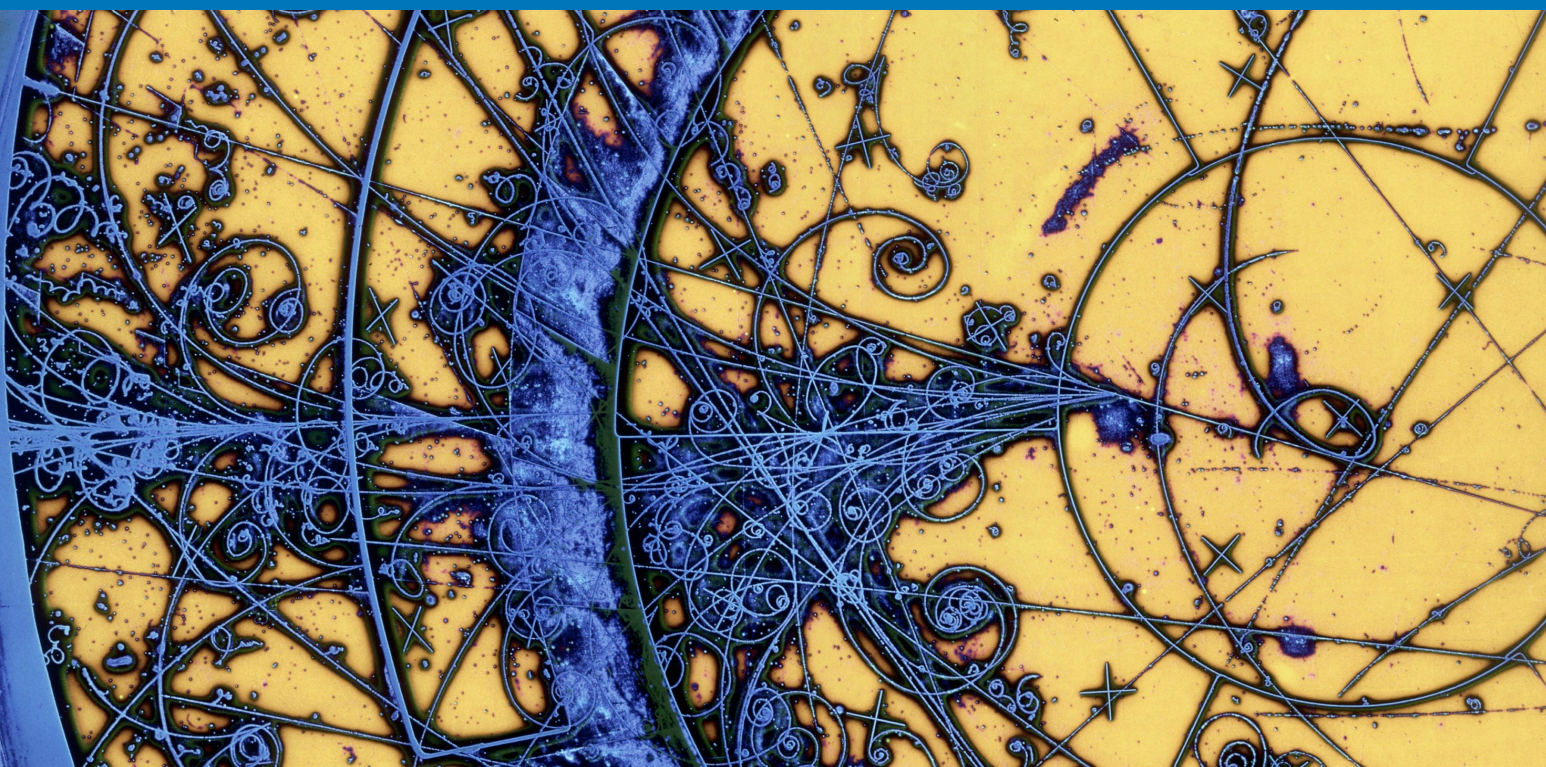




AMPLITUDES

Lecture 4

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June 2026

Scattering Amplitudes in Yang-Mills and Gravity

1. Photon decoupling and KK relations
2. Spinor helicity formalism and BCFW Recurrence Relation.

Kinematic Relations for Yang-Mills Amplitudes

3. Field-theory proof and Open-string monodromy derivation of BCJ relations

Kinematic Relations for Gravity Amplitudes

4. Color Kinematics Duality and Field-theory KLT relations

Amplitudes and Geometry

5. Intersection Theory for twisted DeRham co-homology.
6. KLT from twisted period/Riemann bilinear relations.

Scope and main idea

Goal

Explain the field-theory BCJ representation as the counterpart of monodromy-derived BCJ amplitude relations.

cubic graph representation $\implies$ color Jacobi identities
 $\implies$ kinematic Jacobi identities $\implies$ BCJ relations

Color-kinematics duality

Gauge-theory tree amplitudes can be written so that their kinematic numerators obey the same algebraic Jacobi identities as their color factors.

Color-ordered amplitudes

At tree level, Yang-Mills amplitudes may be decomposed into color factors and color-stripped partial amplitudes:

$$\mathcal{A}_n^{\text{tree}} = (ig)^{n-2} \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma(2)} x_1} \tilde{f}^{x_1 a_{\sigma(3)} x_2} \dots \tilde{f}^{x_{n-3} a_{\sigma(n-1)} a_n} A_n(1, \sigma(2), \dots, \sigma(n-1), n),$$

where

$$\tilde{f}^{abc} = \sqrt{2} f^{abc}$$

are antisymmetric structure constants.

Cubic graph representation

Instead of using both cubic and quartic Yang-Mills vertices, one can rewrite any tree amplitude using only cubic graphs. Quartic contact terms are absorbed into numerators by multiplying and dividing by propagators.

$$\mathcal{A}_n^{\text{tree}} = (ig)^{n-2} \sum_{i \in \text{cubic graphs}} \frac{c_i n_i}{D_i}$$

- c_i : color factor of graph i , built from cubic vertices $\tilde{f}^{abc}$.
- n_i : corresponding kinematic numerator.
- D_i : product of scalar propagators of graph i .

Four-point Amplitude

$$\mathcal{A}_4^{\text{tree}} =$$

At four points:

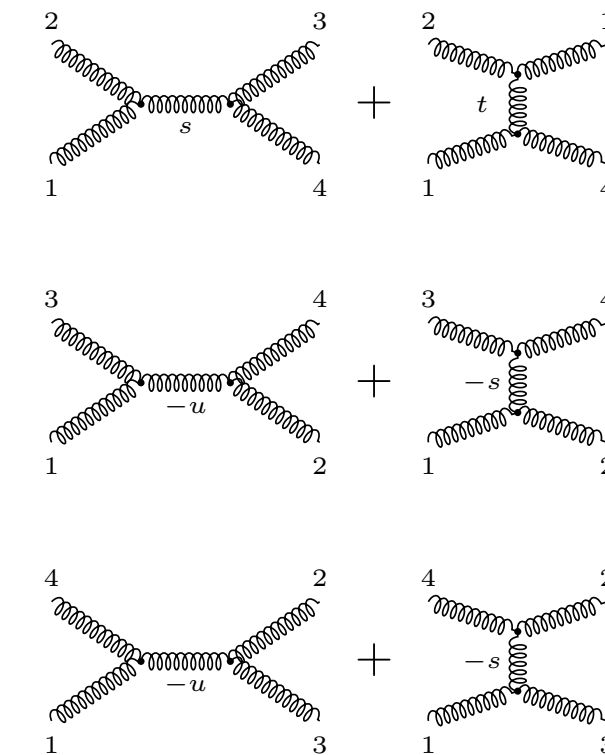
$$D_s = s, \quad D_t = t, \quad D_u = u, \quad s + t + u = 0.$$

The three independent color-ordered amplitudes can be written in terms of cubic diagrams as

$$A_4(1, 2, 3, 4) = \frac{n_s}{s} + \frac{n_t}{t},$$

$$A_4(1, 2, 4, 3) = -\frac{n_u}{u} - \frac{n_s}{s},$$

$$A_4(1, 3, 2, 4) = -\frac{n_t}{t} + \frac{n_u}{u}.$$



The signs reflect antisymmetry of the cubic vertices. The full four-gluon amplitude is

$$\mathcal{A}_4^{\text{tree}} = (ig)^2 \left(\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \right).$$

Four-point color and kinematic Jacobi

A convenient convention for the color factors is

$$c_s = \tilde{f}^{a_1 a_2 \times} \tilde{f}^{\times a_3 a_4},$$

$$c_u = \tilde{f}^{a_1 a_3 \times} \tilde{f}^{\times a_2 a_4},$$

$$c_t = \tilde{f}^{a_2 a_3 \times} \tilde{f}^{\times a_4 a_1}.$$

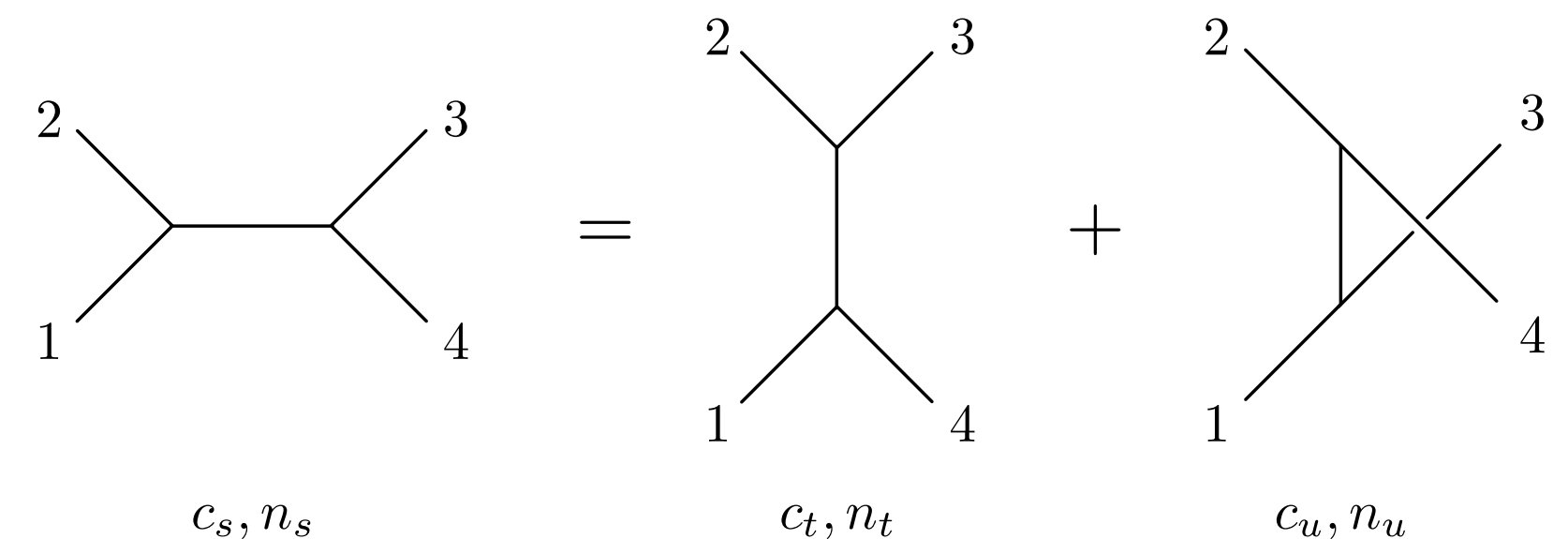
The Jacobi identity for the structure constants gives

$$c_s - c_u = c_t.$$

Color-kinematics duality says one can choose the kinematic numerators so that

$$n_s - n_u = n_t.$$

This is the simplest example of the BCJ representation.



Generalized gauge freedom

The numerators are not unique. The shift

$$n_s \rightarrow n_s + \rho s, \quad n_t \rightarrow n_t - \rho t, \quad n_u \rightarrow n_u - \rho u$$

leaves the color-ordered amplitudes unchanged, for example

$$\frac{n_s + \rho s}{s} + \frac{n_t - \rho t}{t} = \frac{n_s}{s} + \frac{n_t}{t}.$$

If the kinematic Jacobi relation holds before the shift, it still holds after the shift:

$$(n_s + \rho s) - (n_u - \rho u) - (n_t - \rho t) = (n_s - n_u - n_t) + \rho(s + u + t) = 0.$$

Thus at four points color-kinematics duality is essentially automatic once the amplitude is put in cubic form.

Four-point BCJ relation from kinematic Jacobi

The fundamental BCJ relation may be written as

$$0 = s_{12}A_4(1, 2, 3, 4) + (s_{12} + s_{23})A_4(1, 3, 2, 4).$$

Using

$$s = s_{12}, \quad u = s_{13}, \quad t = s_{14}, \quad s_{12} + s_{23} = -u,$$

we obtain

$$\begin{aligned} sA_4(1, 2, 3, 4) - uA_4(1, 3, 2, 4) &= s \left(\frac{n_s}{s} + \frac{n_t}{t} \right) - u \left(-\frac{n_t}{t} + \frac{n_u}{u} \right) \\ &= n_s - n_u + \frac{s + u}{t} n_t \\ &= n_s - n_u - n_t. \end{aligned}$$

Therefore the BCJ relation is equivalent to the kinematic Jacobi identity.

Counting labelled cubic tree graphs

For a cubic tree graph with n labelled external legs,

$$\underset{\text{[vertices]}}{V} = n - 2, \quad \underset{\text{[internal lines]}}{I} = n - 3, \quad \underset{\text{[external + internal lines]}}{E} = n + (n - 3) = 2n - 3.$$

To construct an $(n + 1)$ -point graph, insert the new leg on one of the $2n - 3$ edges:

$$N_{n+1} = (2n - 3)N_n, \quad N_3 = 1.$$

$$N_n = (2(n - 1) - 3)N_{n-1} = (2n - 5)N_{n-1}$$

Therefore

$$N_n = 1 \cdot 3 \cdot 5 \cdots (2n - 5) = (2n - 5)!!.$$

Examples:

$$n = 4 : 3!! = 3, \quad n = 5 : 5!! = 15.$$

Counting labelled cubic tree graphs

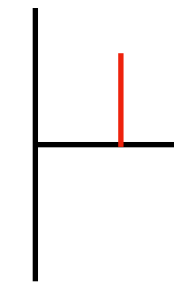
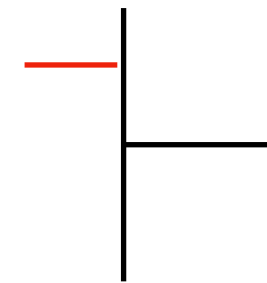
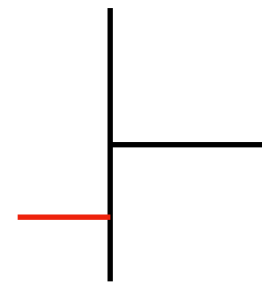
$$n = 3$$

$$N_3 = (2n - 5) = 1$$



$$n = 4$$

$$N_4 = (2n - 5)N_3 = 3$$



Counting labelled cubic tree graphs

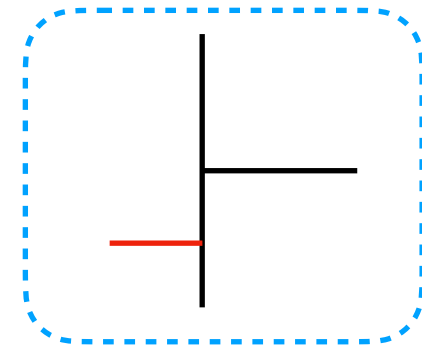
$$n = 3$$

$$N_3 = (2n - 5) = 1$$

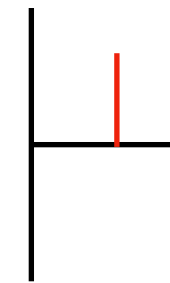
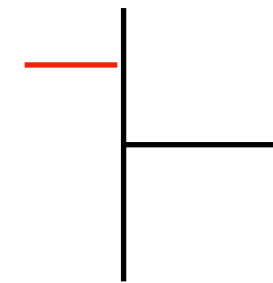
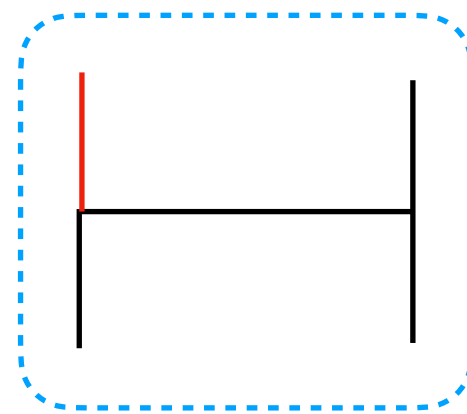


$$n = 4$$

$$N_4 = (2n - 5)N_3 = 3$$



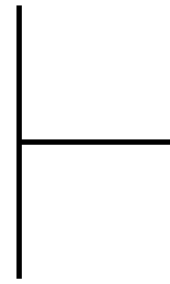
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Counting labelled cubic tree graphs

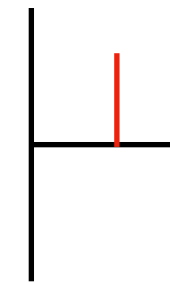
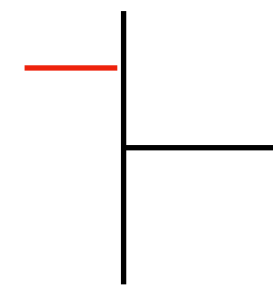
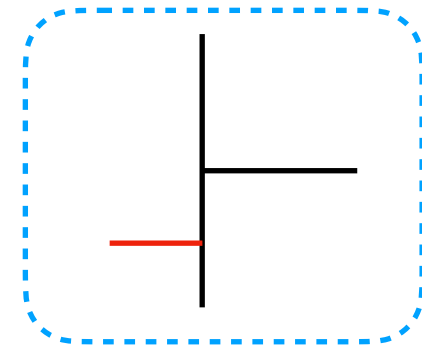
$$n = 3$$

$$N_3 = (2n - 5) = 1$$



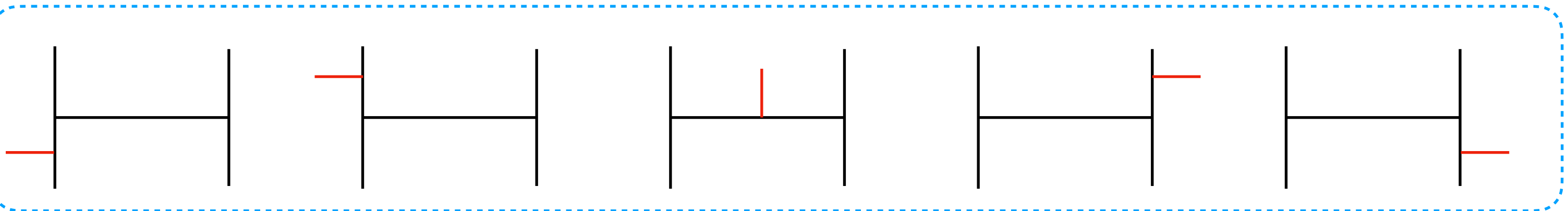
$$n = 4$$

$$N_4 = (2n - 5)N_3 = 3$$



$$n = 5$$

$$N_5 = (2n - 5)N_4 = 5 \cdot 3 = 15$$



+ other 10 diagrams

Five-point cubic representation

At five points, the number of distinct cubic diagrams is

$$(2n - 5)!! \big|_{n=5} = 15.$$

Hence

$$\mathcal{A}_5^{\text{tree}} = (ig)^3 \sum_{i=1}^{15} \frac{c_i n_i}{D_i}.$$

The denominators D_i are products of two propagators, such as $s_{12}s_{45}$ or $s_{23}s_{51}$. A

KK-independent set fixes legs 1 and 5:

$$A_5(1, \sigma(2, 3, 4), 5), \quad \sigma \in S_3.$$

Example five-point partial amplitudes

One possible cubic representation is

$$\begin{aligned} A_5(1, 2, 3, 4, 5) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}, \\ A_5(1, 3, 2, 4, 5) &= \frac{n_{15}}{s_{13}s_{45}} - \frac{n_2}{s_{32}s_{51}} - \frac{n_{10}}{s_{24}s_{13}} - \frac{n_4}{s_{45}s_{32}} - \frac{n_{11}}{s_{51}s_{24}}, \\ A_5(1, 3, 4, 2, 5) &= \frac{n_9}{s_{13}s_{25}} - \frac{n_5}{s_{34}s_{51}} + \frac{n_{10}}{s_{42}s_{13}} - \frac{n_8}{s_{25}s_{34}} + \frac{n_{11}}{s_{51}s_{42}}. \end{aligned}$$

The numerator labels refer to the fifteen possible cubic graphs. The precise labelling is conventional; the important point is that each numerator is paired with the color factor of the same graph.

Color Jacobi and kinematic Jacobi at five points

The fifteen color factors are not independent. Whenever three cubic graphs differ only by the routing of one internal edge, their color factors obey

$$c_\alpha + c_\beta - c_\gamma = 0.$$

Color-kinematics duality states that the numerators can be chosen to obey the same identities:

$$n_\alpha + n_\beta - n_\gamma = 0.$$

At five points this is no longer automatic. Writing the amplitude in cubic form is easy; finding numerators satisfying all kinematic Jacobi identities is nontrivial. The BCJ conjecture is that such a representation exists for all tree amplitudes.

Five-point BCJ relation

The five-point fundamental BCJ relation is

$$0 = s_{12}A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23})A_5(1, 3, 2, 4, 5) \\ + (s_{12} + s_{23} + s_{24})A_5(1, 3, 4, 2, 5).$$

Substituting the cubic representations and simplifying gives

$$0 = \frac{n_4 - n_1 + n_{15}}{s_{45}} - \frac{n_{10} - n_9 + n_{15}}{s_{13}} - \frac{n_5 - n_2 + n_{11}}{s_{51}} - \frac{n_3 - n_5 + n_8}{s_{34}}.$$

Each numerator combination is a kinematic Jacobi identity, for example

$$n_4 - n_1 + n_{15} = 0, \quad n_{10} - n_9 + n_{15} = 0.$$

General BCJ representation

The general tree-level BCJ representation is

$$\mathcal{A}_n^{\text{tree}} = (ig)^{n-2} \sum_{i \in \text{cubic graphs}} \frac{c_i n_i}{\prod_{\alpha_i} s_{\alpha_i}}$$

Color factors satisfy antisymmetry and Jacobi relations:

$$c_i \rightarrow -c_i \quad \text{under a flip of one cubic vertex,} \quad c_i + c_j + c_k = 0.$$

The BCJ representation requires the numerators to satisfy the same algebraic relations:

$$n_i \rightarrow -n_i \quad \text{under the same flip,} \quad n_i + n_j + n_k = 0 \quad \text{whenever} \quad c_i + c_j + c_k = 0.$$

Conceptual interpretation: **Color-Kinematic Duality**

The color factors are built from the Lie algebra of the gauge group. Their Jacobi identities are ordinary Lie-algebra identities.

Surprising statement

The kinematic numerators can be arranged to behave as if they were color factors of a second algebra:

color algebra	$\longleftrightarrow$	kinematic algebra.
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This is not manifest in the Yang-Mills Lagrangian. It becomes visible only after rewriting the amplitude in a suitable cubic-graph form.

Relation to BCJ amplitude relations

BCJ amplitude relations reduce the number of independent color-ordered tree amplitudes:

$$(n-2)! \longrightarrow (n-3)!.$$

They have two complementary explanations:

Open-string picture	Monodromy of integration contours, followed by $\alpha' \rightarrow 0$
Field-theory picture	Cubic graphs with color-dual kinematic numerators

Thus kinematic Jacobi identities create linear dependencies among color-ordered amplitudes.

Bridge to gravity: the double copy

If a Yang-Mills tree amplitude is written as

$$\mathcal{A}_n^{\text{tree}} = (ig)^{n-2} \sum_i \frac{c_i n_i}{D_i}$$

with numerators satisfying color-kinematics duality, then a corresponding gravity amplitude is obtained by replacing color factors by a second copy of kinematic numerators:

$$\mathcal{M}_n^{\text{tree}} = i \left(\frac{\kappa}{2} \right)^{n-2} \sum_i \frac{n_i \tilde{n}_i}{D_i}.$$

In the simplest case $\tilde{n}_i = n_i$. More generally, the two numerator sets may come from two different gauge theories.

$$\text{gravity} = (\text{gauge theory}) \times (\text{gauge theory}).$$

This is closely related to KLT relations.

Momentum Kernel: the bridge from BCJ to KLT

The purpose of this section is to introduce the *momentum kernel*

$$\mathcal{S}[\alpha|\beta]_p$$

and to explain why it is the natural object connecting BCJ relations to KLT relations.

In the previous section, BCJ relations were obtained from open-string monodromy in the field-theory limit. Here we reorganize those relations into a compact matrix form. The same matrix of kinematic invariants that encodes BCJ relations will later appear in the KLT formula, where it pairs two color-ordered Yang–Mills amplitudes into a gravity amplitude.

Schematically,

$$\boxed{\text{BCJ: } \mathcal{S} \cdot A = 0}$$

in an overcomplete gauge-theory basis, while

$$\boxed{\text{KLT: } M_n = A^T \mathcal{S} \tilde{A}}.$$

Thus the momentum kernel is the bridge between amplitude relations in gauge theory and the gravity/gauge-theory relation.

Guiding idea: gravity as a bilinear of gauge theory

Schematic KLT message

$$M_n = A_n^T S \tilde{A}_n.$$

- A_n and $\tilde{A}_n$ are color-ordered Yang–Mills tree amplitudes.
- M_n is the corresponding gravity tree amplitude.
- S is the momentum kernel: a matrix of Mandelstam products.
- The same kernel also packages BCJ amplitude relations.

Field-theory KLT relation

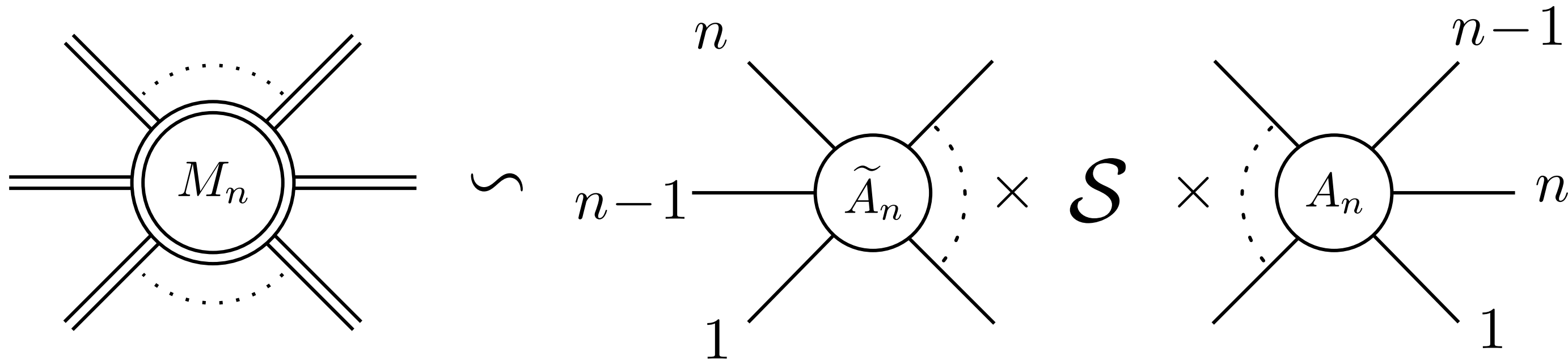
$$\begin{aligned} M_n = (-1)^{n+1} \sum_{\sigma} \sum_{\gamma, \beta} & \mathcal{S}[\gamma(\sigma(2), \dots, \sigma(j-1)) | \sigma(2, \dots, j-1)]_{k_1} \\ & \times \mathcal{S}[\sigma(j), \dots, \sigma(n-2) | \beta(\sigma(j), \dots, n-2))]_{k_{n-1}} \\ & \times A_n(1, \sigma(2, \dots, n-2), n-1, n) \\ & \times \tilde{A}_n(\gamma(\sigma(2), \dots, \sigma(j-1)), 1, n-1, \beta(\sigma(j), \dots, \sigma(n-2)), n) \end{aligned}$$

with the freedom of choosing $2 \leq j \leq n-1$.

The general KLT formula admits different-looking representations, labelled by a parameter j .

Field-theory KLT relation

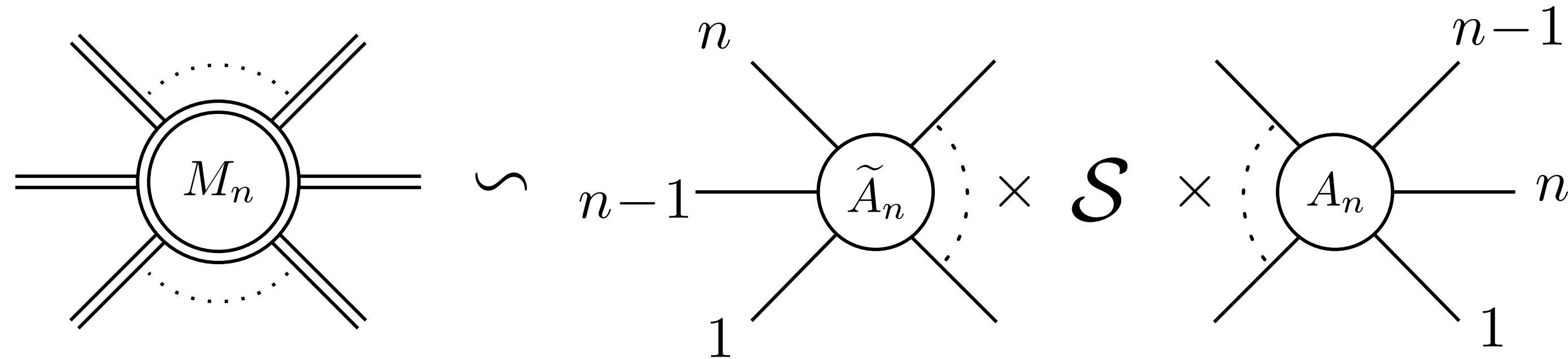
For $j = n - 1$



$$M_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} \tilde{A}_n(n-1, n, \tilde{\sigma}_{2,n-2}, 1) \mathcal{S}[\tilde{\sigma}_{2,n-2} | \sigma_{2,n-2}]_{k_1} A_n(1, \sigma_{2,n-2}, n-1, n)$$

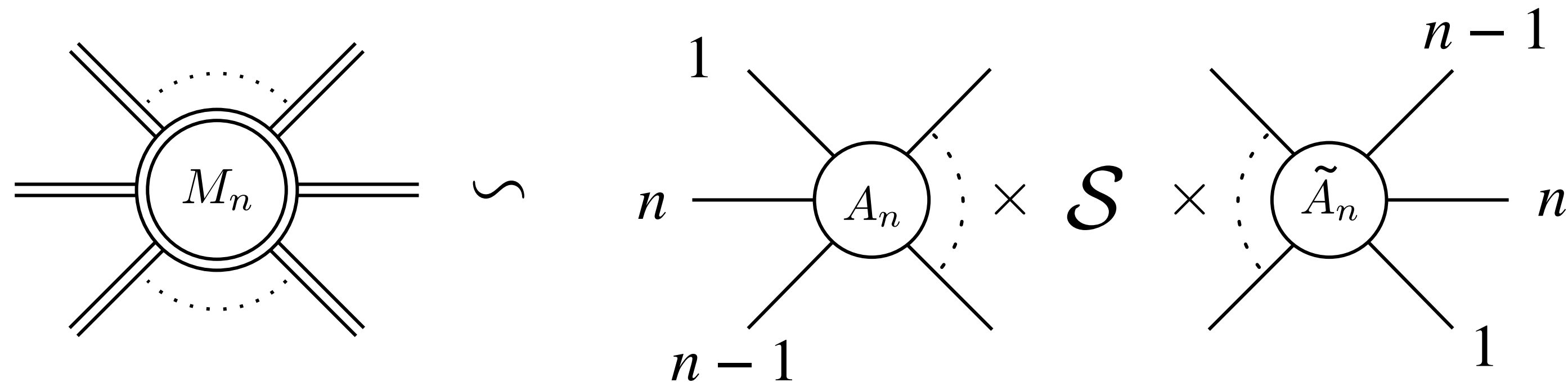
Field-theory KLT relation

For $j = n - 1$



$$M_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} \tilde{A}_n(n-1, n, \tilde{\sigma}_{2,n-2}, 1) \mathcal{S}[\tilde{\sigma}_{2,n-2} | \sigma_{2,n-2}]_{k_1} A_n(1, \sigma_{2,n-2}, n-1, n)$$

and for $j = 2$



$$M_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma} \in S_{n-3}} A_n(1, \sigma_{2,n-2}, n-1, n) S[\tilde{\sigma}_{2,n-2} | \sigma_{2,n-2}]_{p_1} \tilde{A}_n(n-1, n, \tilde{\sigma}_{2,n-2}, 1).$$

so the sum contains $(n-3)! \times (n-3)!$ terms.

$$\sigma_{2,n-2} \equiv \sigma(2, \dots, n-2).$$

Definition of the momentum kernel

Let

$$I = (i_1, i_2, \dots, i_k), \quad J = (j_1, j_2, \dots, j_k)$$

be two orderings of the same label set. Then

$$S[i_1, \dots, i_k \mid j_1, \dots, j_k]_p = \prod_{t=1}^k \left(s_{pi_t} + \sum_{q>t} \theta(i_t, i_q) s_{i_t i_q} \right).$$

$$s_{ab} = 2p_a \cdot p_b = (p_a + p_b)^2.$$

θ rule

$\theta(i_t, i_q) = 1$ if i_t and i_q appear in opposite order in I and J ; otherwise it is zero.

$$\mathcal{S}[i_1, \dots, i_k \mid j_1, \dots, j_k]_p = \mathcal{S}[j_k, \dots, j_1 \mid i_k, \dots, i_1]_p$$

Algorithm for computing $\mathcal{S}[I \mid J]_p$

$$\mathcal{S}[i_1, \dots, i_k \mid j_1, \dots, j_k]_p = \prod_{t=1}^k \left(s_{pi_t} + \sum_{q>t} \theta(i_t, i_q) s_{i_t i_q} \right)$$

- 1 Move through $I = (i_1, \dots, i_k)$ from left to right.
- 2 For each i_t , start with a factor s_{pi_t} .
- 3 Look at labels i_q to the right of i_t in I .
- 4 If i_t and i_q are reversed in J , add $s_{i_t i_q}$.
- 5 Multiply the resulting factors.

Simple kernel examples

No label

$$\mathcal{S}[\emptyset \mid \emptyset]_p = 1$$

For one label,

$$\boxed{\mathcal{S}[2 \mid 2]_{p_1} = s_{12}.$$

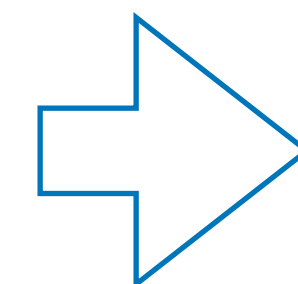
For the two orderings (2, 3) and (3, 2)

$$\mathcal{S}[2, 3 \mid 2, 3]_{p_1} = s_{12}s_{13},$$

$$\mathcal{S}[2, 3 \mid 3, 2]_{p_1} = (s_{12} + s_{23})s_{13},$$

$$\mathcal{S}[3, 2 \mid 2, 3]_{p_1} = (s_{13} + s_{23})s_{12},$$

$$\mathcal{S}[3, 2 \mid 3, 2]_{p_1} = s_{13}s_{12}.$$



$$\mathcal{S}_{p_1} = \begin{pmatrix} s_{12}s_{13} & (s_{12} + s_{23})s_{13} \\ (s_{13} + s_{23})s_{12} & s_{13}s_{12} \end{pmatrix}$$

Low-point KLT relations

At three points,

$$M_3(1, 2, 3) = A_3(1, 2, 3) \tilde{A}_3(1, 2, 3).$$

At four points,

$$M_4(1, 2, 3, 4) = -s_{12} A_4(1, 2, 3, 4) \tilde{A}_4(1, 2, 4, 3).$$

At five points, one compact representation is

$$\begin{aligned} M_5 = & s_{12}s_{34}A_5(1, 2, 3, 4, 5)\tilde{A}_5(2, 1, 4, 3, 5) \\ & + s_{13}s_{24}A_5(1, 3, 2, 4, 5)\tilde{A}_5(3, 1, 4, 2, 5). \end{aligned}$$

How the kernel encodes BCJ relations

BCJ relations can be written as

$$0 = \sum_{\sigma \in S_{n-2}} S[\beta(2, \dots, n-1) \mid \sigma(2, \dots, n-1)]_{p_1} A_n(1, \sigma(2, \dots, n-1), n).$$

where β is just some arbitrary permutation of $2, 3, \dots, n-1$.

Symbolically,

$$S \cdot A = 0$$

Interpretation

The same object S is both the KLT gluing matrix and a compact generator of BCJ identities.

How the kernel encodes BCJ relations

the five-point case with $\beta(2, 3, 4) = (2, 3, 4)$,

$$\begin{aligned} & \sum_{\sigma \in S_3} \mathcal{S}[2, 3, 4 | \sigma(2, 3, 4)]_{k_1} A_5(1, \sigma(2, 3, 4), 5) \\ &= \sum_{\sigma \in OP(\{2\} \cup \{3, 4\})} \mathcal{S}[2, 3, 4 | \sigma(2, 3, 4)]_{k_1} A_5(1, \sigma(2, 3, 4), 5) \quad + \quad \sum_{\sigma \in OP(\{2\} \cup \{4, 3\})} \mathcal{S}[2, 3, 4 | \sigma(2, 3, 4)]_{k_1} A_5(1, \sigma(2, 3, 4), 5) . \\ &= s_{13}s_{14} \left[s_{12} A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23}) A_5(1, 3, 2, 4, 5) + (s_{12} + s_{23} + s_{24}) A_5(1, 3, 4, 2, 5) \right] \\ &\quad + s_{14}(s_{13} + s_{34}) \left[s_{12} A_5(1, 2, 4, 3, 5) + (s_{12} + s_{24}) A_5(1, 4, 2, 3, 5) + (s_{12} + s_{23} + s_{24}) A_5(1, 4, 3, 2, 5) \right] \\ &= 0 \quad + \quad 0 \end{aligned}$$

Field-theory proof of KLT

Define the right-hand side of the KLT relation as

$$R_n = (-1)^{n+1} \sum_{\sigma, \tilde{\sigma}} A_n(1, \sigma, n-1, n) \mathcal{S}[\tilde{\sigma} \mid \sigma]_{p_1} \tilde{A}_n(n-1, n, \tilde{\sigma}, 1).$$

The goal is to show

$$R_n = M_n.$$

The proof proceeds by induction in n .

8.1 Induction hypothesis

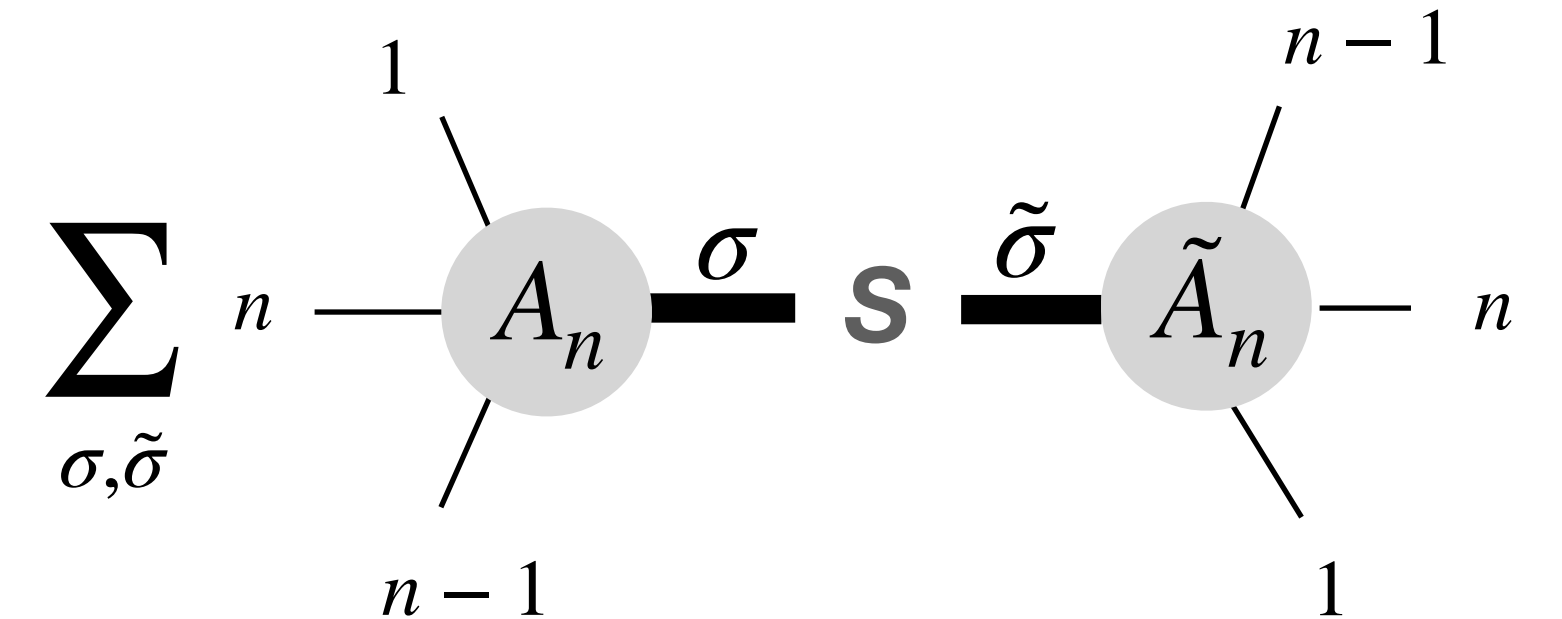
Assume that KLT holds for all amplitudes with fewer than n external legs. $R_k = M_k$, $3 \leq k \leq n$

8.2 Complex deformation

Perform a BCFW deformation on two external legs and consider $R_n(z)$. If the boundary term at infinity vanishes, Cauchy's theorem gives

$$R_n(0) = - \sum_{z_* \neq 0} \text{Res}_{z=z_*} \frac{R_n(z)}{z}.$$

The finite poles correspond to physical factorisation channels.



Field-theory proof of KLT

Type I residues: poles in only one Yang–Mills factor

Suppose a pole appears in A_n but not in $\tilde{A}_n$.

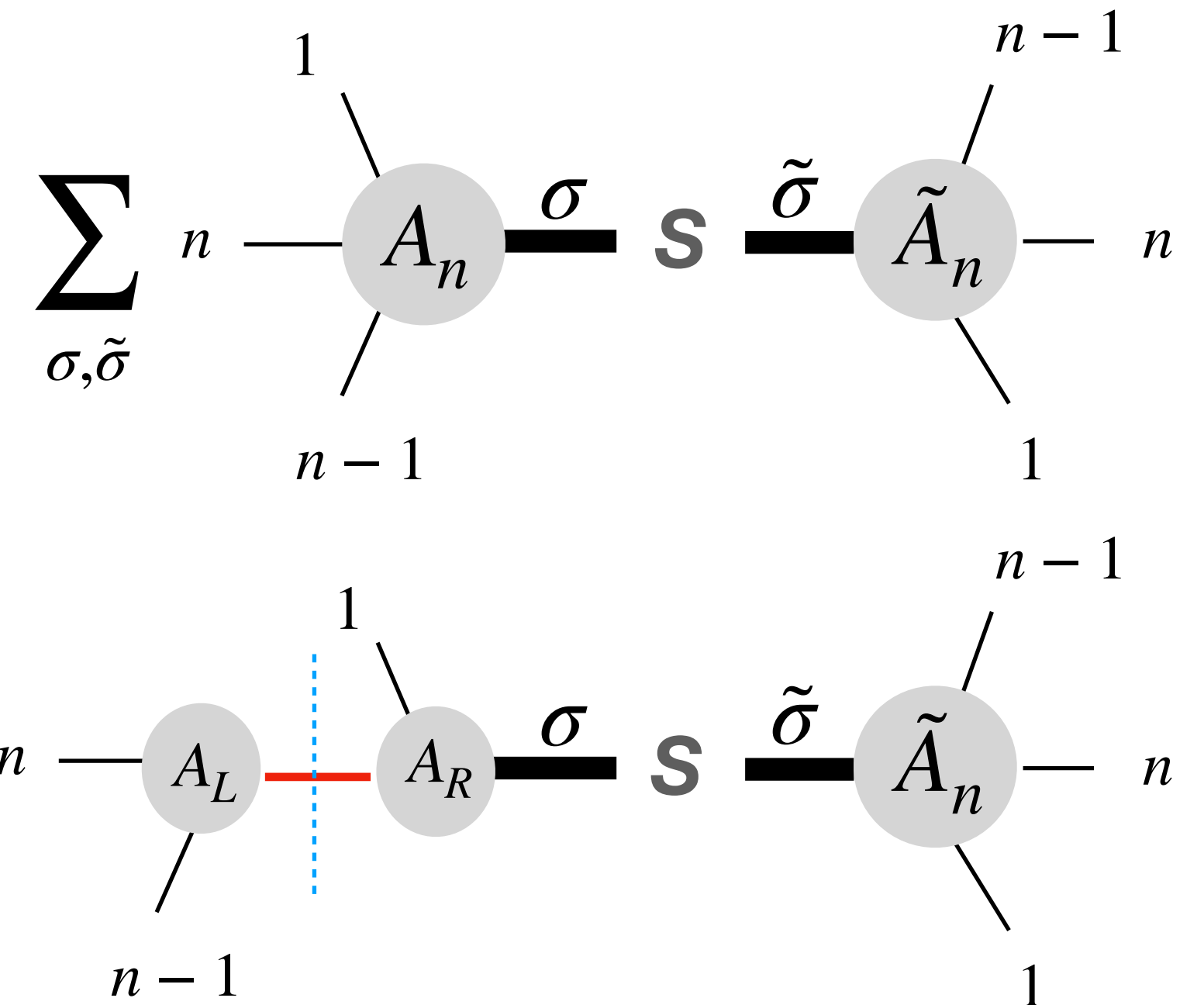
$$A_n \longrightarrow A_L \frac{1}{P^2} A_R$$

Then the residue contains a sum of the schematic form

$$\sum_{\sigma} \mathcal{S}[\beta \mid \sigma] A_R(\dots, \sigma, \dots).$$

This is precisely a BCJ relation written in momentum-kernel form. Therefore

Type I residues vanish by BCJ.



Field-theory proof of KLT

Type II residues: common poles in both Yang–Mills factors

If the same pole appears in both Yang–Mills factors, both amplitudes factorise:

$$A_n \longrightarrow A_L \frac{1}{P^2} A_R, \quad \tilde{A}_n \longrightarrow \tilde{A}_L \frac{1}{P^2} \tilde{A}_R.$$

The momentum kernel factorises compatibly:

$$\mathcal{S}_n \longrightarrow \mathcal{S}_L \mathcal{S}_R.$$

Thus the residue becomes

$$\left(A_L^T \mathcal{S}_L \tilde{A}_L \right) \frac{1}{P^2} \left(A_R^T \mathcal{S}_R \tilde{A}_R \right).$$

By the induction hypothesis,

$$A_L^T \mathcal{S}_L \tilde{A}_L = M_L, \quad A_R^T \mathcal{S}_R \tilde{A}_R = M_R.$$

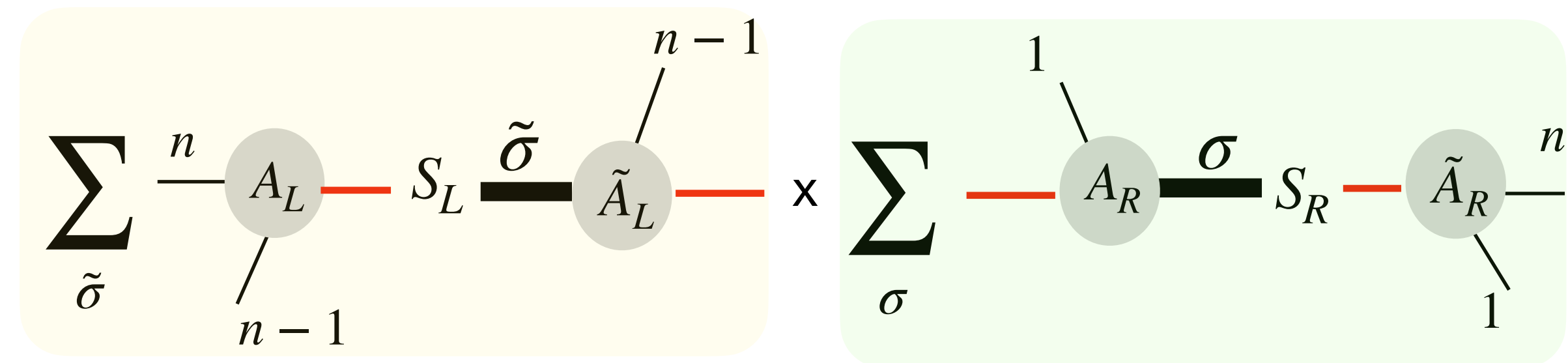
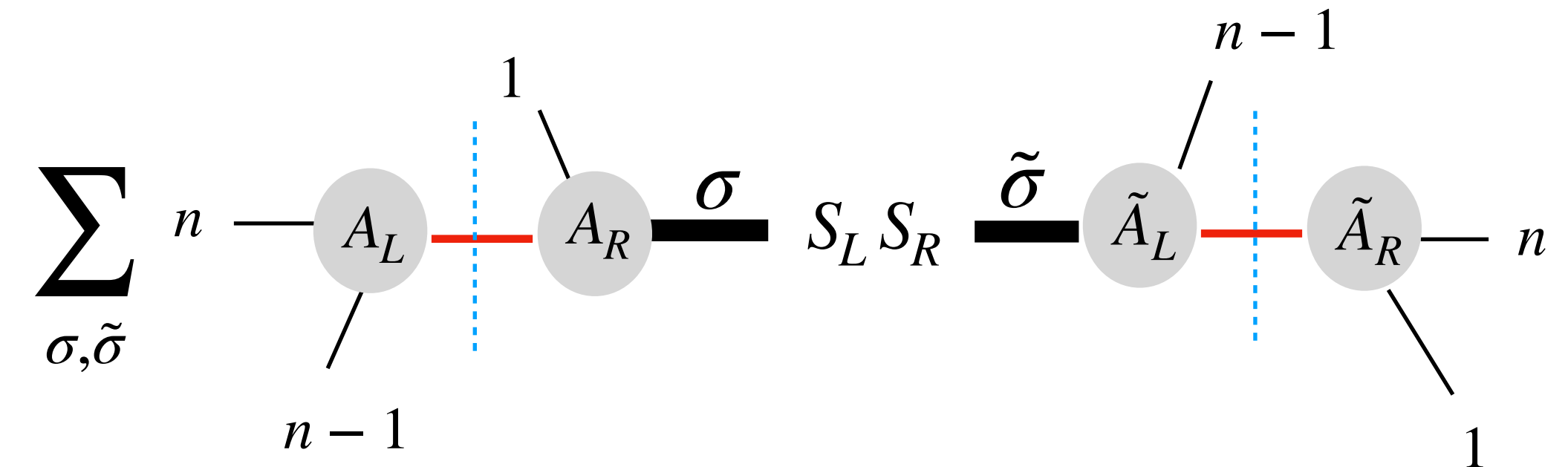
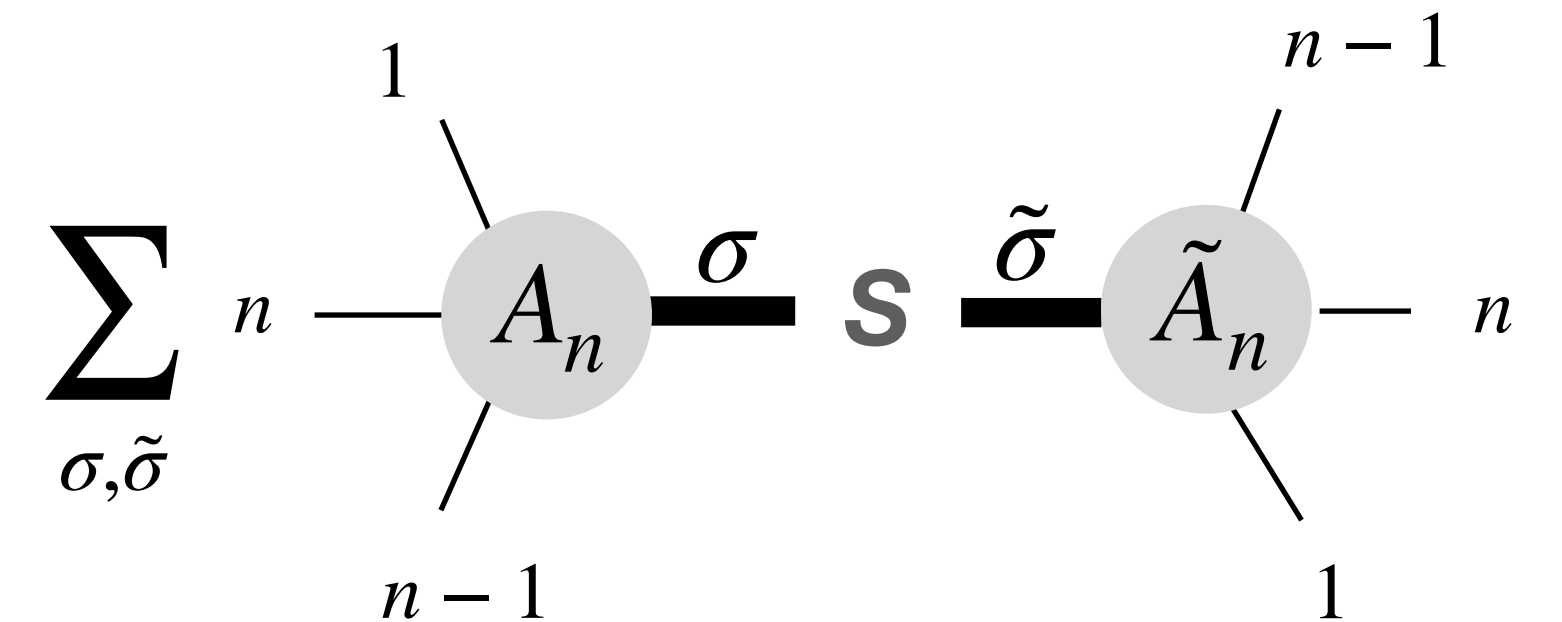
Therefore the residue is

$$\boxed{M_L \frac{1}{P^2} M_R},$$

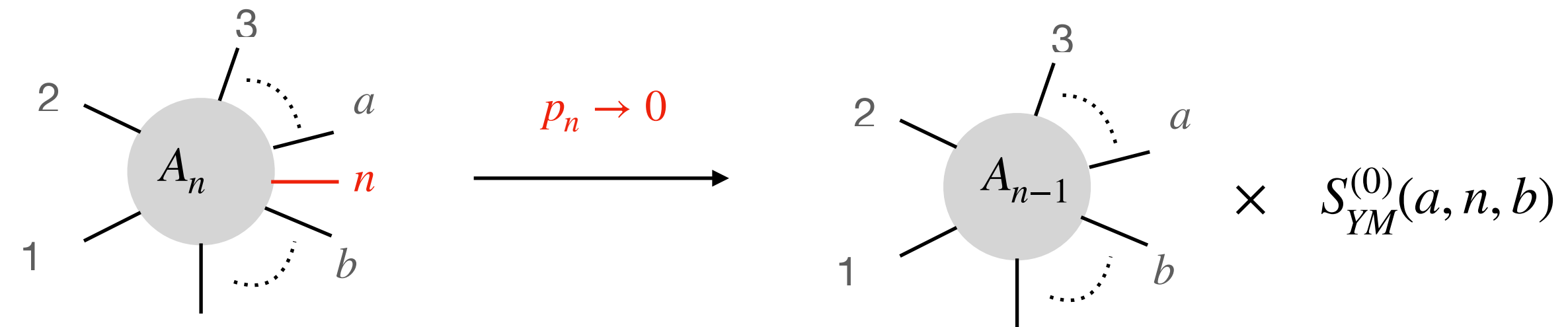
which is exactly the gravity factorisation residue.

Summing over all residues reproduces the BCFW recursion for the gravity amplitude. Hence

$$\boxed{R_n = M_n.}$$



Soft gluon theorem



Let a positive-helicity gluon n^+ become soft. If it is inserted between adjacent color-neighbours a and b ,

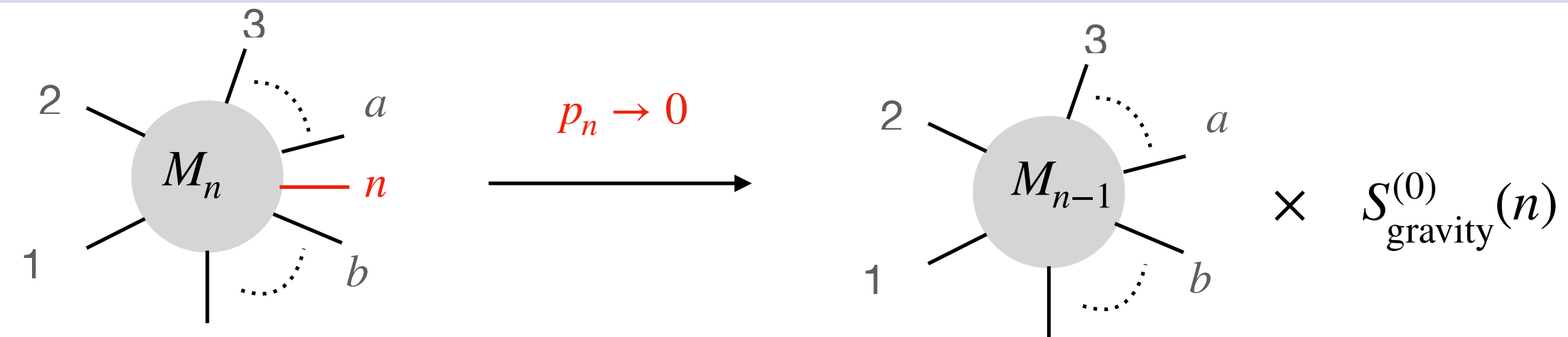
$$A_n(\dots, a, n^+, b, \dots) \xrightarrow{p_n \rightarrow 0} S^{\text{YM}}(a, n^+, b) A_{n-1}(\dots, a, b, \dots),$$

where

$$S^{\text{YM}}(a, n^+, b) = \frac{\langle ab \rangle}{\langle an \rangle \langle nb \rangle}.$$

The soft factor depends on the two color-neighbours of the soft leg.

Soft graviton theorem



For gravity, there is no color ordering. A useful positive-helicity form is

$$M_n(1, 2, \dots, n^+) \xrightarrow{p_n \rightarrow 0} S^{\text{grav}}(n^+) M_{n-1}(1, 2, \dots, n-1),$$

with

$$S^{\text{grav}}(n^+) = - \sum_{i=2}^{n-2} \frac{\langle 1i \rangle \langle n-1, i \rangle [in]}{\langle 1n \rangle \langle n, n-1 \rangle \langle ni \rangle},$$

up to convention-dependent overall signs.

KLT soft-limit mechanism

Start with

$$M_n = (-1)^{n+1} \sum_{\sigma, \gamma \in S_{n-3}} A_n(1, \sigma, n-1, n^+) S[\gamma \mid \sigma]_{p_1} \tilde{A}_n(n-1, n^+, \gamma, 1).$$

The two Yang–Mills factors give soft factors:

$$A_n \rightarrow S^{\text{YM}}(n-1, n^+, 1) A_{n-1},$$

$$\tilde{A}_n \rightarrow S^{\text{YM}}(n-1, n^+, i) \tilde{A}_{n-1},$$

where i is the first label in γ .

Gravity soft factor from two gluon soft factors

The KLT soft limit contains products

$$S^{\text{YM}}(n-1, n^+, 1) S^{\text{YM}}(n-1, n^+, i).$$

Using

$$S^{\text{YM}}(n-1, n^+, 1) = \frac{\langle n-1, 1 \rangle}{\langle n-1, n \rangle \langle n1 \rangle},$$

$$S^{\text{YM}}(n-1, n^+, i) = \frac{\langle n-1, i \rangle}{\langle n-1, n \rangle \langle ni \rangle},$$

we obtain

$$S_L^{\text{YM}} S_R^{\text{YM}} = \frac{1}{s_{n,n-1}} \frac{\langle n-1, 1 \rangle [n, n-1]}{\langle n-1, n \rangle \langle n1 \rangle} \frac{\langle n-1, i \rangle}{\langle ni \rangle}.$$

Four-point MHV example: setup

Study

$$M_4(1^-, 2^-, 3^+, 4^+)$$

using

$$M_4(1, 2, 3, 4) = -s_{12} A_4(1, 2, 3, 4) \tilde{A}_4(1, 2, 4, 3).$$

Take leg 3 soft:

$$p_3 \rightarrow \varepsilon p_3, \quad \varepsilon \rightarrow 0,$$

with symmetric spinor scaling

$$|3\rangle \rightarrow \sqrt{\varepsilon} |3\rangle, \quad |3] \rightarrow \sqrt{\varepsilon} |3].$$

Thus $\langle i3 \rangle, [i3] = O(\sqrt{\varepsilon})$ and $s_{i3} = O(\varepsilon)$.

Four-point Yang–Mills soft factors

For the left amplitude,

$$A_4(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle},$$

so

$$A_4(1, 2, 3, 4) = \underbrace{\frac{\langle 24 \rangle}{\langle 23 \rangle \langle 34 \rangle}}_{S^{\text{YM}}(2,3,4)} A_3(1, 2, 4).$$

For the right amplitude,

$$\tilde{A}_4(1, 2, 4, 3) = \underbrace{\frac{\langle 41 \rangle}{\langle 43 \rangle \langle 31 \rangle}}_{\tilde{S}^{\text{YM}}(4,3,1)} \tilde{A}_3(1, 2, 4).$$

Each Yang–Mills soft factor scales as $O(1/\varepsilon)$.

Four-point KLT soft factor

Substitute the soft limits into KLT:

$$M_4(1^-, 2^-, 3^+, 4^+) \rightarrow -s_{12} S_L(2, 3, 4) \tilde{S}_R(4, 3, 1) M_3(1^-, 2^-, 4^+).$$

Thus

$$S_{\text{grav}}^{(0)}(3^+) = -s_{12} \left(\frac{\langle 24 \rangle}{\langle 23 \rangle \langle 34 \rangle} \right)^{S^{\text{YM}}(2,3,4)} \left(\frac{\langle 41 \rangle}{\langle 43 \rangle \langle 31 \rangle} \right)^{\tilde{S}^{\text{YM}}(4,3,1)}.$$

Using $\langle 43 \rangle = -\langle 34 \rangle$,

$$S_{\text{grav}}^{(0)}(3^+) = s_{12} \frac{\langle 24 \rangle \langle 41 \rangle}{\langle 23 \rangle \langle 31 \rangle \langle 34 \rangle^2}.$$

$$s_{12} = s_{34} = \langle 34 \rangle [43] \quad = [43] \frac{\langle 24 \rangle \langle 41 \rangle}{\langle 23 \rangle \langle 31 \rangle \langle 34 \rangle} = \frac{[34] \langle 14 \rangle \langle 24 \rangle}{\langle 34 \rangle \langle 13 \rangle \langle 23 \rangle} \quad \text{KLT soft factor} \quad \text{[#]}$$

Comparison with Weinberg at four points

The leading positive-helicity soft graviton factor can be written as

$$S_{\text{grav}}^{(0)}(3^+) = \sum_{a \neq 3} \frac{[3a]}{\langle 3a \rangle} \frac{\langle xa \rangle}{\langle x3 \rangle} \frac{\langle ya \rangle}{\langle y3 \rangle}.$$

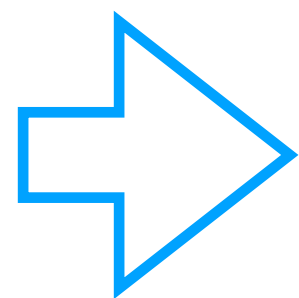
Weinberg soft factor

Choose $x = 1$, $y = 2$. Only $a = 4$ contributes:

$$S_{\text{grav}}^{(0)}(3^+) = \frac{[34] \langle 14 \rangle \langle 24 \rangle}{\langle 34 \rangle \langle 13 \rangle \langle 23 \rangle}.$$

[#]
KLT soft factor

the KLT expression agrees with the Weinberg soft factor.



$$S_{\text{grav}}^{(0)}(3^+) = -s_{12} S_{\text{YM}}(2, 3, 4) \tilde{S}_{\text{YM}}(4, 3, 1).$$

Four-point lesson

At four points,

$$M_4 = -s_{12}A_4\tilde{A}_4.$$

Under the soft scaling of leg 3,

$$A_4 \sim \frac{1}{\varepsilon}A_3, \quad \tilde{A}_4 \sim \frac{1}{\varepsilon}\tilde{A}_3, \quad s_{12} = s_{34} = O(\varepsilon).$$

Therefore

$$M_4 \sim \varepsilon \cdot \frac{1}{\varepsilon^2} A_3 \tilde{A}_3 \sim \frac{1}{\varepsilon} M_3.$$

Final takeaway

Across the four-point and five-point examples, the same mechanism appears:

$$A_{n+1} \sim \frac{1}{\varepsilon} A_n, \quad \tilde{A}_{n+1} \sim \frac{1}{\varepsilon} \tilde{A}_n, \quad S_{n+1} \sim \varepsilon S_n.$$

Thus

$$M_{n+1} = A_{n+1}^T S_{n+1} \tilde{A}_{n+1} \sim \frac{1}{\varepsilon} A_n^T S_n \tilde{A}_n = \frac{1}{\varepsilon} M_n.$$

Core lesson

The momentum kernel supplies exactly the soft power needed to turn two gauge-theory soft singularities into the universal gravity soft singularity.

Mechanism

Yang–Mills soft $\times$ Yang–Mills soft $\times$ KLT kernel = gravity soft.

EXTRA

Starting point: a KLT representation

Consider a KLT formula written schematically as

$$M_n = \sum_{\alpha, \beta} A_n(\hat{1}, \alpha, n-1, \hat{n}) \mathcal{S}[\alpha \mid \beta]_{p_1} \tilde{A}_n(n-1, \hat{n}, \beta, \hat{1}).$$

We use a BCFW shift on legs 1 and n :

$$\hat{p}_1(z) = p_1 + zq, \quad \hat{p}_n(z) = p_n - zq, \quad q^2 = q \cdot p_1 = q \cdot p_n = 0.$$

A residue occurs when some channel becomes on shell:

$$P_I^2(z_I) = 0, \quad P_I(z) = \hat{p}_1(z) + \sum_{i \in I} p_i.$$

Type 1: pole only in one Yang-Mills factor

Suppose the ordering α splits as

$$\alpha = (\sigma, \gamma),$$

so that

$$A_n(\hat{1}, \sigma, \gamma, n-1, \hat{n}) \longrightarrow \sum_h A_L(\hat{1}, \sigma, -\hat{P}^h) \frac{1}{P_I^2} A_R(\hat{P}^{-h}, \gamma, n-1, \hat{n}).$$

But the second amplitude does not split:

$$\tilde{A}_n(n-1, \hat{n}, \beta, \hat{1}) \quad \text{is still an } n\text{-point amplitude.}$$

Therefore the residue contains

$$\sum_{\sigma, \gamma, \beta} A_L \frac{1}{P_I^2} A_R \mathcal{S}[\sigma, \gamma \mid \beta]_{p_1} \tilde{A}_n(n-1, \hat{n}, \beta, \hat{1}).$$

The two possible residue types

Type 1: single-sided pole

Only one Yang-Mills amplitude factorises:

$$A_n \longrightarrow A_L \frac{1}{P_I^2} A_R, \quad \tilde{A}_n \text{ does not factorise} \implies \text{one KLT kernel remains.}$$

Type 2: double-sided pole

Both Yang-Mills amplitudes factorise in the same channel:

$$A_n \longrightarrow A_L \frac{1}{P_I^2} A_R, \quad \tilde{A}_n \longrightarrow \tilde{A}_L \frac{1}{P_I^2} \tilde{A}_R. \implies \text{the KLT kernel factorises.}$$

Kernel factorisation requires matching block decompositions on both sides.

Why only one kernel in Type 1?

There is only one kernel because only one side has split:

$$A_n \rightarrow A_L \frac{1}{P_I^2} A_R, \quad \tilde{A}_n \not\rightarrow \tilde{A}_L \frac{1}{P_I^2} \tilde{A}_R.$$

So there is no pair of lower-point amplitudes to combine into two lower KLT formulas.

The remaining sum has the schematic form

$$\sum_{\beta} \mathcal{S}[\sigma, \gamma \mid \beta]_{p_1} \tilde{A}_n(n-1, \hat{n}, \beta, \hat{1}).$$

This is a KLT-kernel/BCJ annihilation identity:

$$\sum_{\beta} \mathcal{S}[\sigma, \gamma \mid \beta]_{p_1} \tilde{A}_n(n-1, \hat{n}, \beta, \hat{1}) = 0$$

for the single-sided pole configuration. Thus these residues vanish.

Type 2: pole on both sides

Now both color orderings split into compatible blocks:

$$\alpha = (\sigma, \gamma), \quad \beta = (\rho, \tau),$$

with

$$\{\sigma\} = \{\rho\}, \quad \{\gamma\} = \{\tau\}.$$

Then

$$A_n \longrightarrow \sum_h A_L(\hat{1}, \sigma, -\hat{P}^h) \frac{1}{P_I^2} A_R(\hat{P}^{-h}, \gamma, n-1, \hat{n}),$$

$$\tilde{A}_n \longrightarrow \sum_{\tilde{h}} \tilde{A}_L(\hat{P}^{\tilde{h}}, \rho, \hat{1}) \frac{1}{P_I^2} \tilde{A}_R(n-1, \hat{n}, \tau, -\hat{P}^{-\tilde{h}}).$$

Now the kernel sees the same left/right split on both sides.

Momentum-kernel factorisation identity

When the two orderings have compatible blocks, the KLT kernel factorises:

$$\boxed{\mathcal{S}[\sigma, \gamma \mid \rho, \tau]_{p_1} = \mathcal{S}[\sigma \mid \rho]_{p_1} \mathcal{S}[\gamma \mid \tau]_{p_1 + p_\sigma}}$$

where

$$p_\sigma = \sum_{i \in \sigma} p_i.$$

The second kernel has a shifted reference momentum because the right block sees the total momentum flowing through the left block:

$$p_1 \longrightarrow p_1 + p_\sigma.$$

Why the kernel factorises: direct product argument

Recall the momentum kernel

$$\mathcal{S}[i_1, \dots, i_k \mid j_1, \dots, j_k]_p = \prod_{t=1}^k \left(s_{i_t p} + \sum_{q>t} \theta(i_t, i_q)_{\{j\}} s_{i_t i_q} \right).$$

For labels in the first block σ , the product gives

$$\mathcal{S}[\sigma \mid \rho]_{p_1}.$$

For a label $i \in \gamma$, the cross terms with the first block combine as

$$s_{ip_1} + \sum_{j \in \sigma} s_{ij} = 2p_i \cdot \left(p_1 + \sum_{j \in \sigma} p_j \right) = s_{i, p_1 + p_\sigma}.$$

Hence the remaining product is

$$\mathcal{S}[\gamma \mid \tau]_{p_1 + p_\sigma}.$$

The double-sided residue becomes lower-point KLT

Using kernel factorisation, the residue becomes

$$\begin{aligned} \text{Res}_{P_I^2=0} M_n &\sim \sum_{h, \tilde{h}} \left[\sum_{\sigma, \rho} A_L(\hat{1}, \sigma, -\hat{P}^h) \mathcal{S}[\sigma \mid \rho]_{p_1} \tilde{A}_L(\hat{P}^{\tilde{h}}, \rho, \hat{1}) \right] \frac{1}{P_I^2} \\ &\times \left[\sum_{\gamma, \tau} A_R(\hat{P}^{-h}, \gamma, n-1, \hat{n}) \mathcal{S}[\gamma \mid \tau]_{p_1 + p_\sigma} \tilde{A}_R(n-1, \hat{n}, \tau, -\hat{P}^{-\tilde{h}}) \right]. \end{aligned}$$

Each bracket is a lower-point KLT formula, so

$$\boxed{\text{Res}_{P_I^2=0} M_n = \sum_h M_L(\hat{1}, \dots, -\hat{P}^h) \frac{1}{P_I^2} M_R(\hat{P}^{-h}, \dots, \hat{n})}$$

which is the expected gravity factorisation.

Summary

Single-sided pole

A_n factorises, $\tilde{A}_n$ does not.

There is only one KLT kernel:

$$\mathcal{S}[\sigma, \gamma \mid \beta]_{p_1}.$$

The residue vanishes by BCJ/KLT-kernel annihilation.

Double-sided pole

A_n factorises, $\tilde{A}_n$ factorises.

The compatible split forces

$$\boxed{\mathcal{S}_n \longrightarrow \mathcal{S}_L \mathcal{S}_R}$$

and the residue becomes the product of two lower-point gravity amplitudes.

Soft limit of the KLT momentum kernel

Replacing τ by ϵ

Understand the leading behavior when one external momentum becomes soft:

$$p_s = \epsilon q, \quad \epsilon \rightarrow 0.$$

KLT momentum kernel

A common convention for the KLT momentum kernel is

$$\mathcal{S}[i_1, \dots, i_k \mid j_1, \dots, j_k]_p = \prod_{t=1}^k \left(s_{i_t p} + \sum_{q>t} \theta(i_t, i_q)_{\{j\}} s_{i_t i_q} \right).$$

Here

$$s_{ab} = 2p_a \cdot p_b.$$

Ordering function

$\theta(i_t, i_q)_{\{j\}} = 1$ if i_t and i_q have opposite relative order in the two orderings, and 0 otherwise.

Soft scaling of Mandelstam invariants

Take the soft leg to be

$$p_s = \epsilon q, \quad \epsilon \rightarrow 0.$$

Then every invariant involving the soft leg is linear in ϵ :

$$s_{sa} = 2p_s \cdot p_a = \epsilon 2q \cdot p_a.$$

Therefore any factor in the KLT kernel that contains s contributes one power of ϵ .

soft leg in the kernel $\implies \mathcal{S}_{\text{KLT}} \sim \mathcal{O}(\epsilon)$

General soft-leg placement

Suppose the soft label appears in both orderings:

$$\mathcal{S}[\alpha, s, \beta \mid \gamma, s, \delta]_p.$$

Then the leading soft behavior is

$$\mathcal{S}[\alpha, s, \beta \mid \gamma, s, \delta]_p = \epsilon F_s(\alpha, \beta; \gamma, \delta) \mathcal{S}[\alpha, \beta \mid \gamma, \delta]_p + \mathcal{O}(\epsilon^2).$$

The soft factor is

$$F_s(\alpha, \beta; \gamma, \delta) = 2q \cdot p + \sum_{\substack{i_q \text{ right of } s \\ \text{in the first ordering}}} \theta(s, i_q)_{\{\gamma, s, \delta\}} 2q \cdot p_{i_q}.$$

Equivalent form before substituting $p_s = \epsilon q$

Before writing the soft momentum explicitly as $p_s = \epsilon q$, the same statement is

$$\mathcal{S}[\alpha, s, \beta \mid \gamma, s, \delta]_p = \left(s_{sp} + \sum_{\substack{i_q \text{ right of } s \\ \text{in the first ordering}}} \theta(s, i_q) s_{si_q} \right) \mathcal{S}[\alpha, \beta \mid \gamma, \delta]_p + \mathcal{O}(p_s^2).$$

Since each s_{sa} is order ϵ , the prefactor is order ϵ .

Simple special case: soft leg last

If the soft leg is placed last in both orderings, then no later labels appear:

$$\mathcal{S}[\alpha, s \mid \beta, s]_p = s_{sp} \mathcal{S}[\alpha \mid \beta]_p + \mathcal{O}(\epsilon^2).$$

Using

$$s_{sp} = 2p_s \cdot p = \epsilon \, 2q \cdot p,$$

we get

$$\mathcal{S}[\alpha, s \mid \beta, s]_p = \epsilon (2q \cdot p) \mathcal{S}[\alpha \mid \beta]_p + \mathcal{O}(\epsilon^2).$$

Takeaway

Momentum kernel

When the soft leg is included in the KLT kernel,

$$\mathcal{S}_{\text{KLT}} \sim \mathcal{O}(\epsilon).$$

Inverse kernel

The inverse kernel has the reciprocal leading behavior:

$$\mathcal{S}_{\text{KLT}}^{-1} \sim \mathcal{O}(\epsilon^{-1}).$$

This inverse soft pole is the structure that can combine with gauge-theory amplitudes to reproduce the usual Weinberg soft behavior of gravity amplitudes.