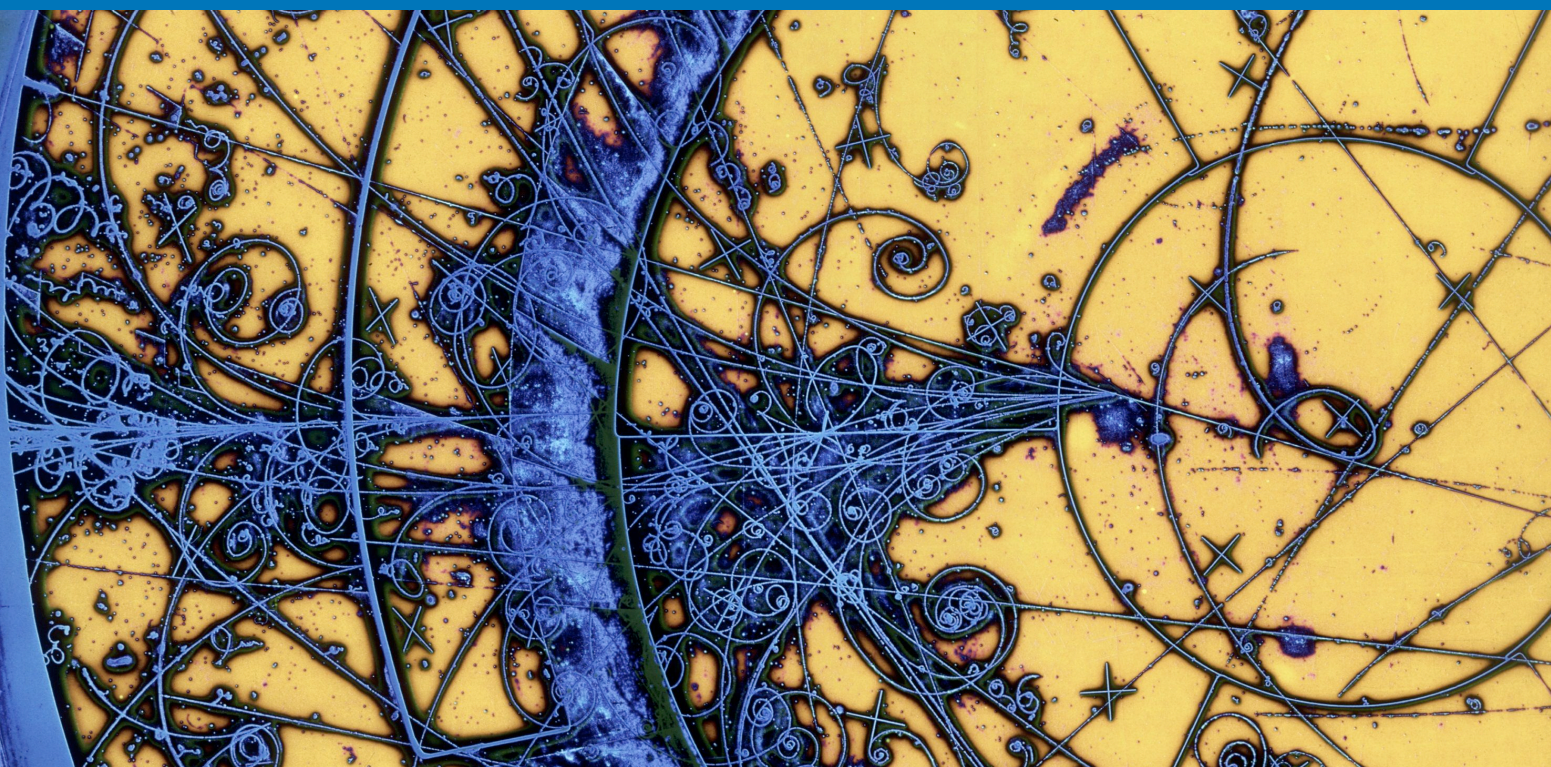




AMPLITUDES

Lecture 3

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Vatican Observatory School
June 2026

Scattering Amplitudes in Yang-Mills and Gravity

1. Photon decoupling and KK relations
2. Spinor helicity formalism and BCFW Recurrence Relation.

Kinematic Relations for Yang-Mills Amplitudes

3. Field-theory proof and Open-string monodromy derivation of BCJ relations

Kinematic Relations for Gravity Amplitudes

4. Color Kinematics Duality and Field-theory KLT relations

Amplitudes and Geometry

5. Intersection Theory for twisted DeRham co-homology.
6. KLT from twisted period/Riemann bilinear relations.

Field-theory proof of BCJ relation

Statement of the fundamental BCJ relation

The fundamental BCJ relation for primitive QCD tree amplitudes is

$$\sum_{i=2}^{n-1} \left(\sum_{j=i+1}^n 2p_2 \cdot p_j \right) A_n(1, 3, \dots, i, 2_g, i+1, \dots, n-1, n) = 0.$$

$$\sum_{j=1}^n p_j = 0, \quad p_2^2 = 0,$$

Equivalently, by momentum conservation,

$$\sum_{j=i+1}^n 2p_2 \cdot p_j = - \sum_{j=1}^i 2p_2 \cdot p_j.$$

$$\sum_{i=2}^{n-1} \left(\sum_{j=1}^i 2p_2 \cdot p_j \right) A_n(1, 3, \dots, i, 2_g, i+1, \dots, n-1, n) = 0.$$

Bern-Carrasco-Johannsson (BCJ) relations / 4- and 5-gluon amplitudes

$$s_{ij} \equiv 2p_i \cdot p_j$$

● 4-gluon case

$$(s_{23} + s_{24})A_4(1, 2, 3, 4) + s_{24}A_4(1, 3, 2, 4) = 0.$$

Using momentum conservation,

$$s_{21} + s_{23} + s_{24} = 0,$$

we have

$$s_{23} + s_{24} = -s_{12}.$$

Therefore the four-point BCJ relation may be written as

$$s_{12}A_4(1, 2, 3, 4) - s_{24}A_4(1, 3, 2, 4) = 0.$$

● 5-gluon case

$$(s_{23} + s_{24} + s_{25})A_5(1, 2, 3, 4, 5) + (s_{24} + s_{25})A_5(1, 3, 2, 4, 5) + s_{25}A_5(1, 3, 4, 2, 5) = 0.$$

Using momentum conservation,

$$s_{21} + s_{23} + s_{24} + s_{25} = 0,$$

we may rewrite this as

$$s_{12}A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23})A_5(1, 3, 2, 4, 5) - s_{25}A_5(1, 3, 4, 2, 5) = 0.$$

BCJ relations / the proof

The proof has two main ingredients:

1. A complex deformation of the three distinguished legs 1, 2_g and n .
2. Induction in the number of external particles.

● 3- particle deformation

We deform the momenta of legs 1, 2_g and n :

$$p_1 \longrightarrow \hat{p}_1(z), \quad p_2 \longrightarrow \hat{p}_2(z), \quad p_n \longrightarrow \hat{p}_n(z),$$

with

$$\hat{p}_1(0) = p_1, \quad \hat{p}_2(0) = p_2, \quad \hat{p}_n(0) = p_n.$$

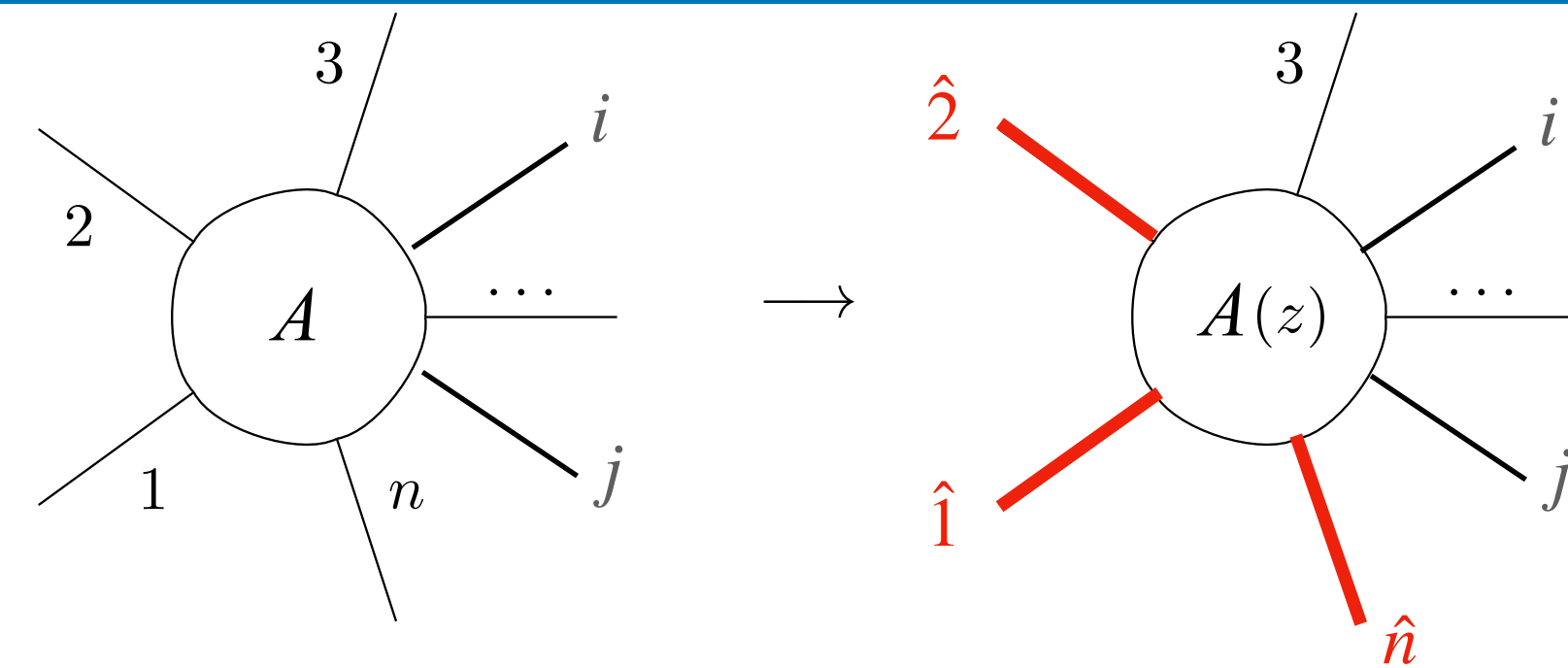
The deformation is chosen so that

$$\hat{p}_1(z) + \hat{p}_2(z) + \hat{p}_n(z) = p_1 + p_2 + p_n,$$

and the on-shell conditions are preserved:

$$\hat{p}_1(z)^2 = m_1^2, \quad \hat{p}_2(z)^2 = 0, \quad \hat{p}_n(z)^2 = m_n^2.$$

- (i) momentum conservation is preserved;
- (ii) all shifted external momenta remain on shell;
- (iii) the resulting BCJ sum has no pole at infinity.



$$|\hat{1}] = |1] - z y_1 |n], \quad |\hat{1}\rangle = |1\rangle,$$

$$|\hat{2}] = |2] - z y_2 |n], \quad |\hat{2}\rangle = |2\rangle,$$

$$|\hat{n}\rangle = |n\rangle + z y_1 |1\rangle + z y_2 |2\rangle, \quad |\hat{n}] = |n].$$

arbitrary parameters y_1, y_2

$$\hat{p}_1(z) = p_1 - z y_1 q_1, \quad q_1^\mu = \epsilon_{1n}^\mu = \frac{\langle 1 | \gamma^\mu | n]}{2}$$

$$\hat{p}_2(z) = p_2 - z y_2 q_2,$$

$$\hat{p}_n(z) = p_n + z y_1 q_1 + z y_2 q_2, \quad q_2^\mu = \epsilon_{2n}^\mu = \frac{\langle 2 | \gamma^\mu | n]}{2}$$

BCJ relations / the proof

● Deformed BCJ sum

Define

$$I_n(z) = \sum_{i=2}^{n-1} \left(\sum_{j=i+1}^n 2\hat{p}_2(z) \cdot \hat{p}_j(z) \right) A_n(\hat{1}, 3, \dots, i, \hat{2}_g, i+1, \dots, n-1, \hat{n}).$$

At $z = 0$, this becomes exactly the BCJ combination:

$$I_n(0) = \sum_{i=2}^{n-1} \left(\sum_{j=i+1}^n 2p_2 \cdot p_j \right) A_n(1, 3, \dots, i, 2_g, i+1, \dots, n-1, n).$$

Thus the desired result is

$$I_n(0) = 0.$$

At tree level, $I_n(z)$ is a rational function of z . Consider the contour integral

$$\frac{1}{2\pi i} \oint \frac{dz}{z} I_n(z).$$

The pole at $z = 0$ gives $I_n(0)$. If there is no contribution from infinity, then

$$I_n(0) = - \sum_{\substack{z=z_\alpha \\ z_\alpha \neq 0}} \text{Res} \frac{I_n(z)}{z}. \quad \text{[#]}$$

Therefore the proof reduces to two statements:

there is no boundary term at $z = \infty$,

all finite residues on the right-hand side of [#] vanish.

BCJ relations / the proof

● INDUCTION PROOF

We now prove [#] by induction in the number of external particles.

● 3-point amplitude: *the base case*

At three points, the relation is

$$2p_2 \cdot p_3 A_3(1, 2_g, 3) = 0. \quad p_2^2 = 0$$

Using momentum conservation,

$$p_1 + p_2 + p_3 = 0,$$

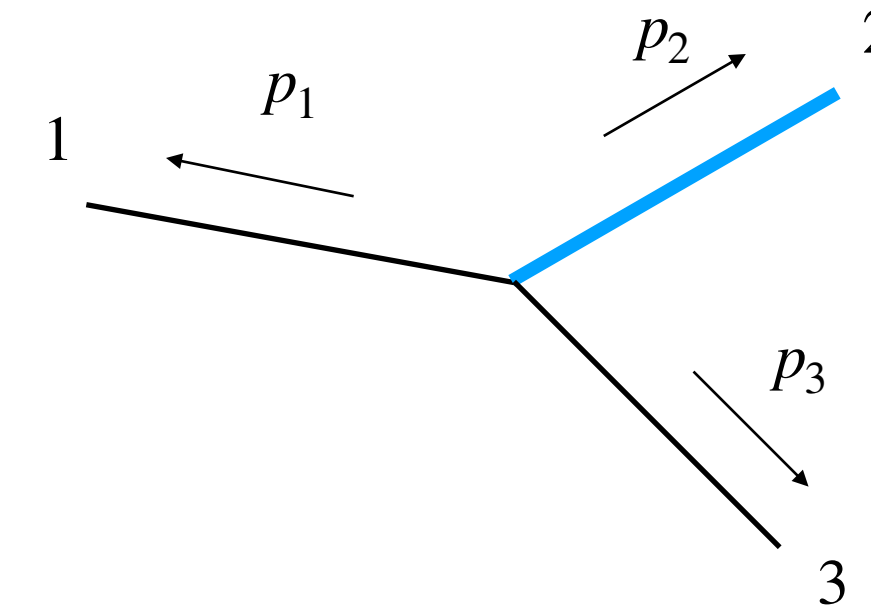
we have

$$2p_2 \cdot p_3 = (p_2 + p_3)^2 - p_2^2 - p_3^2 = p_1^2 - p_3^2.$$

If legs 1 and 3 are a quark–antiquark pair, they have equal mass, so $p_1^2 = p_3^2$. If the amplitude is purely gluonic, both are massless. Therefore

$$2p_2 \cdot p_3 = 0,$$

and the relation holds.



● Induction Hypothesis

Assume that the fundamental BCJ relation holds for all primitive QCD amplitudes with fewer than n external legs. We prove it for n legs.

Because there is no boundary term at infinity, Cauchy's theorem gives

$$I_n(0) = - \sum_{z_\alpha \neq 0} \text{Res}_{z=z_\alpha} \frac{I_n(z)}{z}.$$

The finite poles occur when a deformed internal momentum goes on shell:

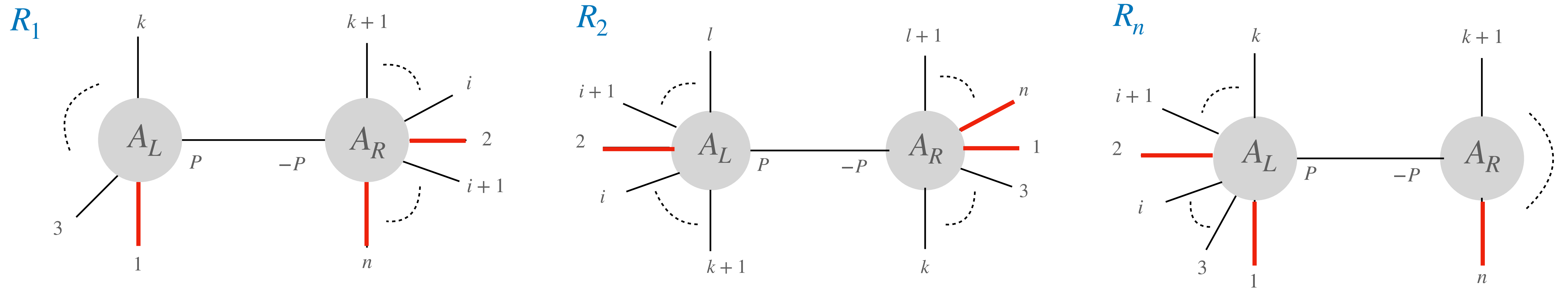
$$\hat{P}^2(z_\alpha) = m_P^2.$$

At such a pole, the amplitude factorises into lower-point primitive amplitudes:

$$A_n(z) \longrightarrow \sum_{\lambda} A_L(z_\alpha) \frac{i}{P^2 - m_P^2} A_R(z_\alpha),$$

where the sum is over the physical states of the internal particle.

It remains to show that the sum of all residues vanishes.



● Residue analysis

The three shifted legs are 1 , 2_g and n . A factorisation channel can separate one of these shifted legs from the other two. It is convenient to group the residues into three classes:

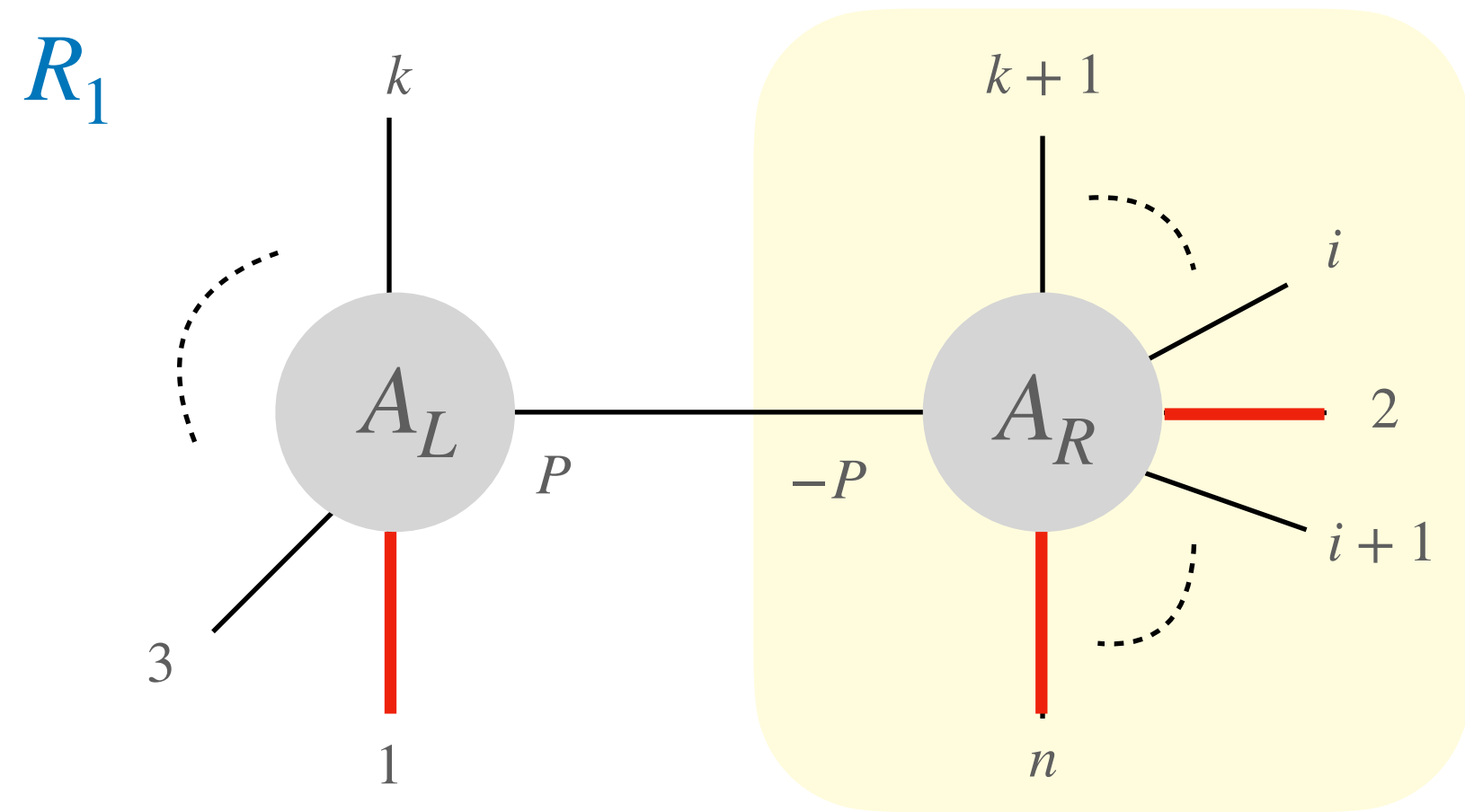
$$I_n(0) = R_1 + R_2 + R_n,$$

where:

- R_1 contains residues in which leg 1 is separated from 2_g and n ;
- R_n contains residues in which leg n is separated from 1 and 2_g ;
- R_2 contains residues in which the gluon 2_g is separated from 1 and n .

We show separately that

$$R_1 = 0, \quad R_2 = 0, \quad R_n = 0.$$



The R_1 residues

In the R_1 class, a typical factorisation term has the form

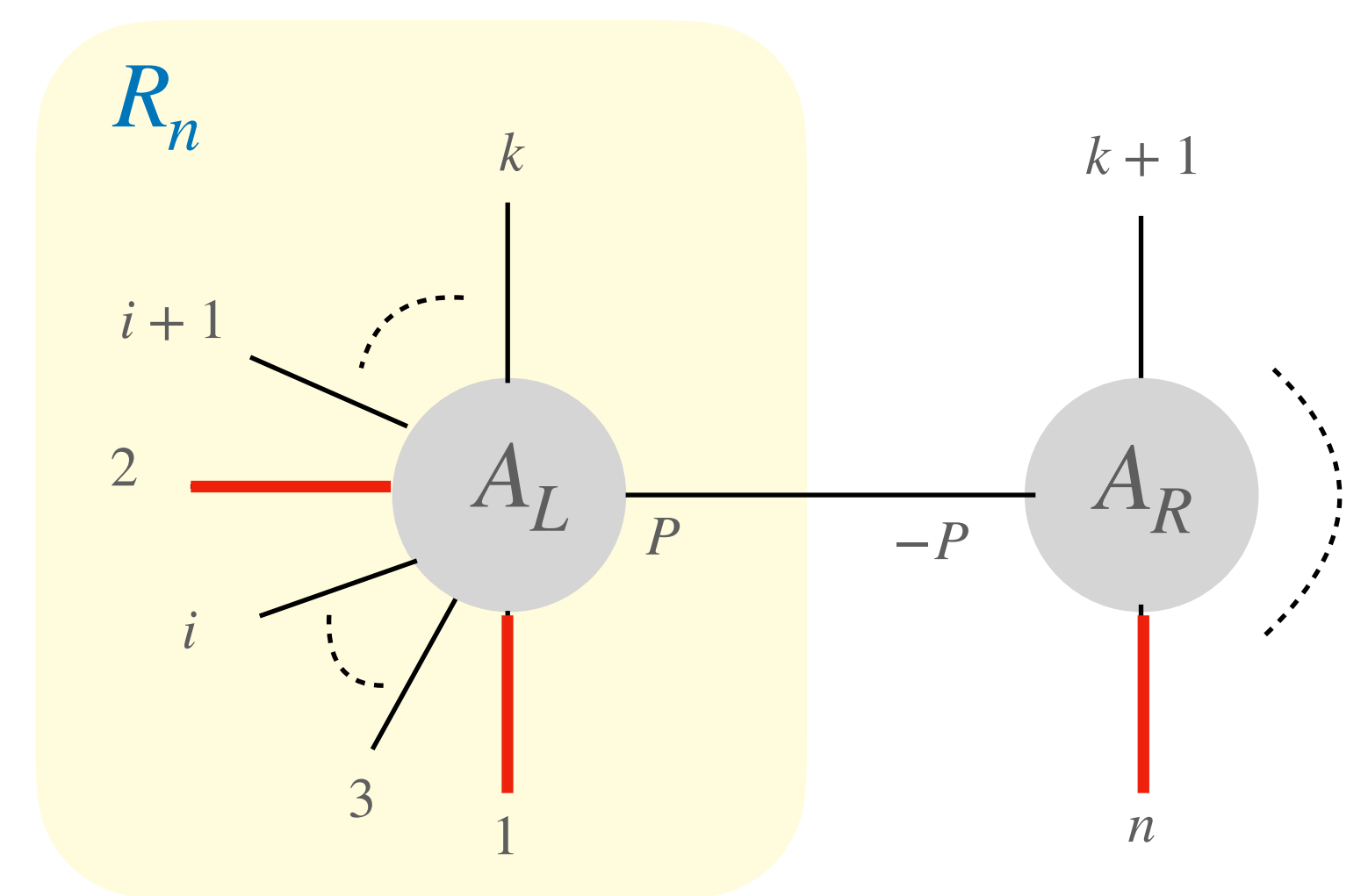
$$A_L(\hat{1}, \dots, \hat{P}) \frac{i}{P^2 - m_P^2} A_R(-\hat{P}, \dots, \hat{2}_g, \dots, \hat{n}).$$

The sum over insertions of $\hat{2}_g$ appears entirely inside the right subamplitude. After reorganising the BCJ coefficients, the relevant sum is precisely a lower-point BCJ relation acting on A_R :

$$\sum_i \left(\sum_j 2\hat{p}_2 \cdot \hat{p}_j \right) A_R(-\hat{P}, \dots, i, \hat{2}_g, i+1, \dots, \hat{n}).$$

The right subamplitude has fewer than n external legs. Therefore the induction hypothesis implies that this sum vanishes. Hence

$$\boxed{R_1 = 0.}$$



The R_n residues

The R_n residues are analogous. The factorisation channel separates leg n , and the sum over insertions of $\hat{2}_g$ appears inside the left subamplitude:

$$A_L(\hat{1}, \dots, i, \hat{2}_g, i+1, \dots, \hat{P}) \frac{i}{P^2 - m_P^2} A_R(-\hat{P}, \dots, \hat{n}).$$

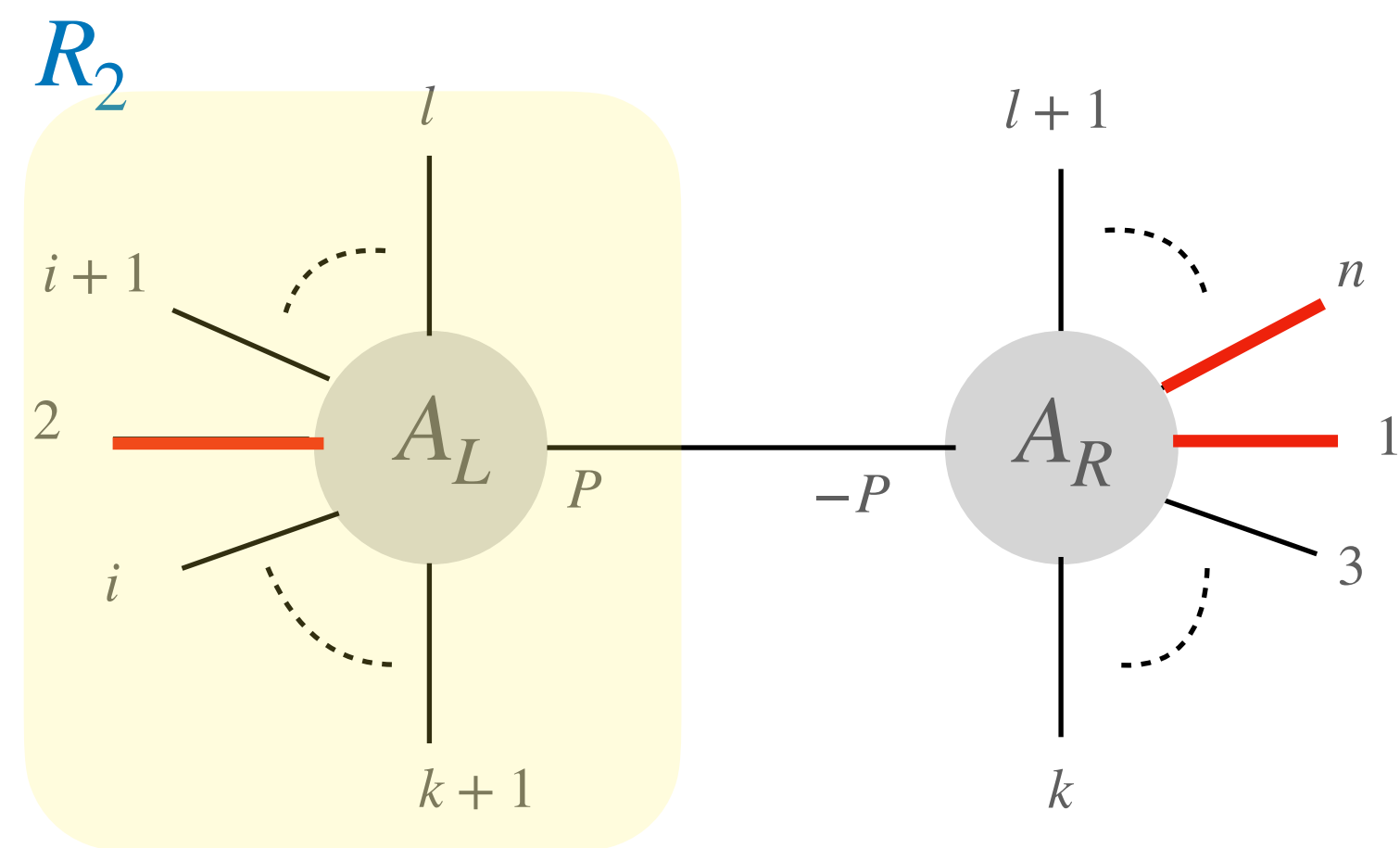
In this case it is convenient to use the equivalent BCJ form

[momentum conservation]

$$\sum_i \left(\sum_{j=1}^i 2\hat{p}_2 \cdot \hat{p}_j \right) A_L(\hat{1}, \dots, i, \hat{2}_g, i+1, \dots, \hat{P}) = 0.$$

This is again a lower-point BCJ relation. By the induction hypothesis,

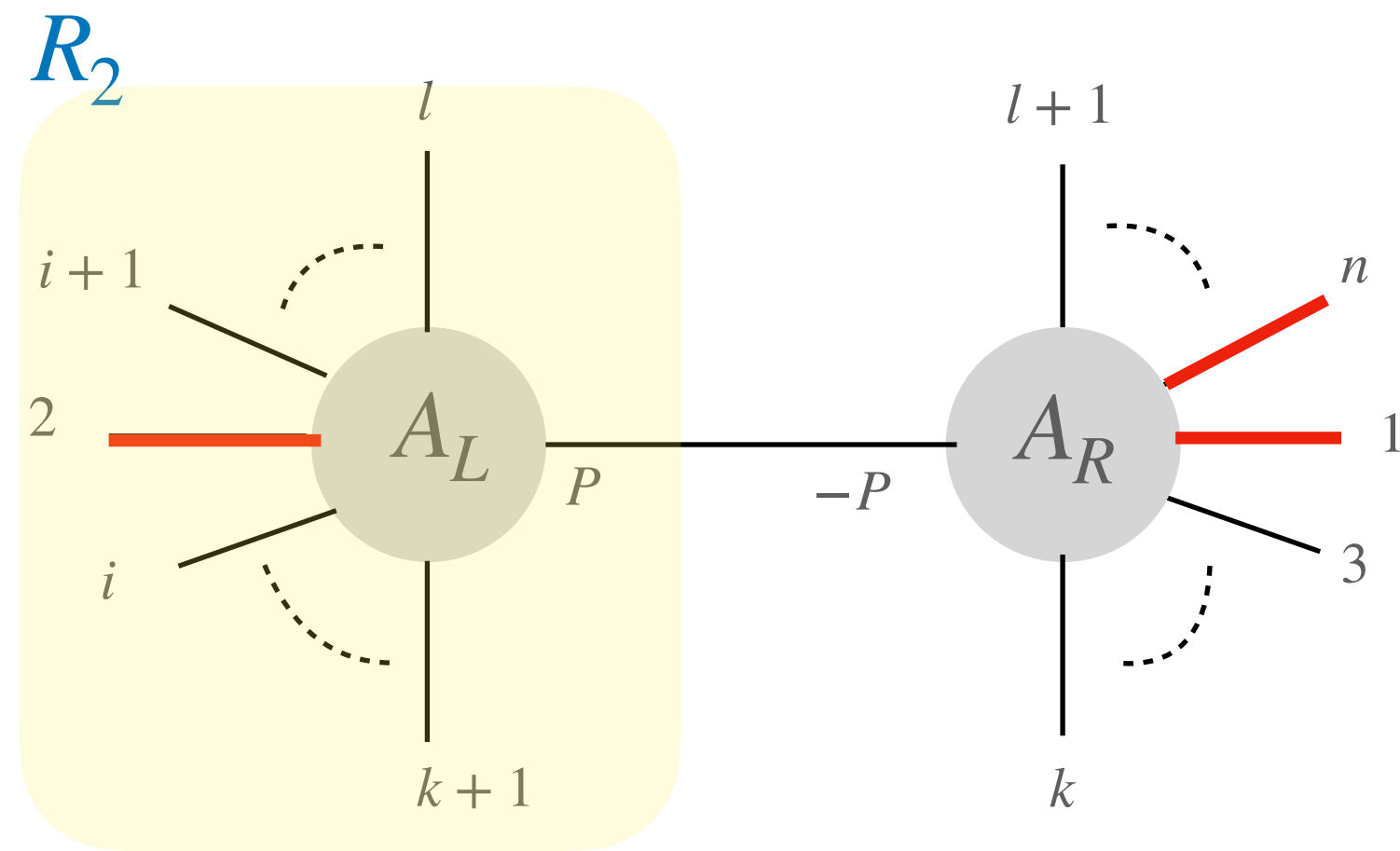
$$\boxed{R_n = 0.}$$



The R_2 residues

The BCJ coefficient into two pieces:

$$\sum_{j=i+1}^n 2\hat{p}_2 \cdot \hat{p}_j = \underbrace{\sum_{j=i+1}^l 2\hat{p}_2 \cdot \hat{p}_j}_{\text{inside } A_L} + \underbrace{\sum_{j=l+1}^n 2\hat{p}_2 \cdot \hat{p}_j}_{\text{outside } A_L}.$$



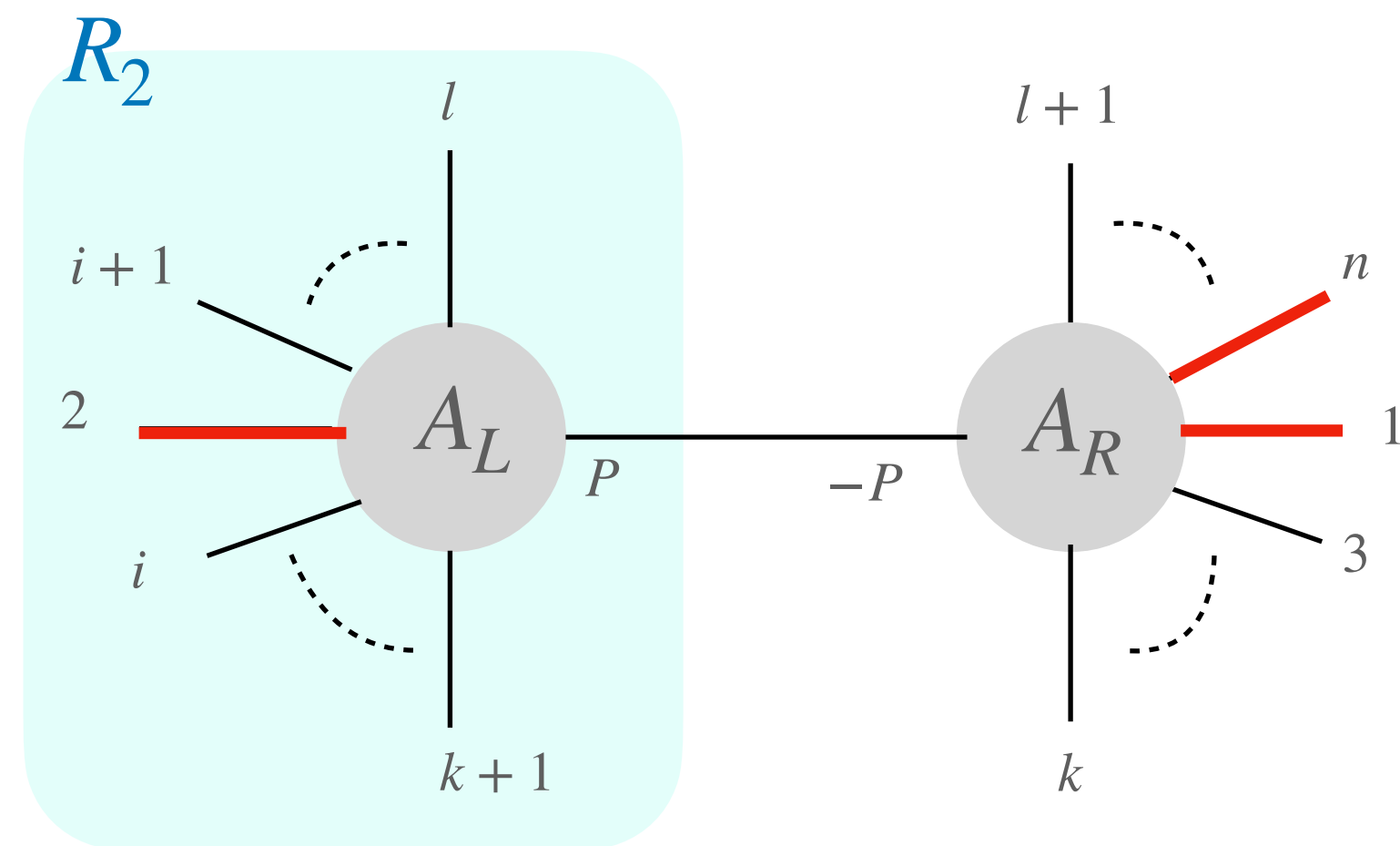
The R_2 residues

The BCJ coefficient into two pieces:

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The first part gives a lower-point BCJ relation on the left subamplitude and therefore vanishes by induction.

$$\sum_i \sum_{j=i+1}^l 2\hat{p}_2 \cdot \hat{p}_j \, A_L(\hat{P}, k+1, \dots, i, \hat{2}_g, i+1, \dots, l) = 0$$



The R_2 residues

The BCJ coefficient into two pieces:

$$\sum_{j=i+1}^n 2\hat{p}_2 \cdot \hat{p}_j = \underbrace{\sum_{j=i+1}^l 2\hat{p}_2 \cdot \hat{p}_j}_{\text{inside } A_L} + \underbrace{\sum_{j=l+1}^n 2\hat{p}_2 \cdot \hat{p}_j}_{\text{outside } A_L}.$$

The first part gives a lower-point BCJ relation on the left subamplitude and therefore vanishes by induction.

$$\sum_i \sum_{j=i+1}^l 2\hat{p}_2 \cdot \hat{p}_j A_L(\hat{P}, k+1, \dots, i, \hat{2}_g, i+1, \dots, l) = 0$$

The second part is independent of the insertion position i . It factors out of the sum over insertions of $\hat{2}_g$, leaving

$$\underbrace{\sum_i \sum_{j=l+1}^n 2\hat{p}_2 \cdot \hat{p}_j A_L}_{\text{[exchanging sums]}} = \sum_{j=l+1}^n 2\hat{p}_2 \cdot \hat{p}_j \underbrace{\sum_i A_L(\hat{P}, k+1, \dots, i, \hat{2}_g, i+1, \dots, l)}_{\text{the } U(1)\text{-decoupling identity}} = 0 \quad \Rightarrow \quad \boxed{R_2 = 0.}$$

4-gluon BCJ relation

The fundamental BCJ relation with leg 2 moved through the ordering is

$$(s_{23} + s_{24})A_4(1, 2, 3, 4) + s_{24}A_4(1, 3, 2, 4) = 0.$$

Using momentum conservation,

$$s_{21} + s_{23} + s_{24} = 0,$$

we have

$$s_{23} + s_{24} = -s_{12}.$$

Therefore the four-point BCJ relation may be written as

$$\boxed{s_{12}A_4(1, 2, 3, 4) - s_{24}A_4(1, 3, 2, 4) = 0.}$$

The amplitudes

$$A_4(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.$$

For the ordering (1, 3, 2, 4),

$$A_4(1^-, 3^+, 2^-, 4^+) = \frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \langle 41 \rangle}.$$

Since $\langle 32 \rangle = -\langle 23 \rangle$, this is

$$A_4(1, 3, 2, 4) = -\frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 23 \rangle \langle 24 \rangle \langle 41 \rangle}.$$

Therefore

$$\frac{A_4(1, 3, 2, 4)}{A_4(1, 2, 3, 4)} = -\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}.$$

Spinor identity from momentum conservation

Momentum conservation gives

$$p_1 + p_2 + p_3 + p_4 = 0.$$

Acting with $\langle 3|$ from the left and $|2\rangle$ from the right gives

$$\langle 3|p_1 + p_2 + p_3 + p_4|2\rangle = 0.$$

The p_2 and p_3 terms vanish, so

$$\langle 31 \rangle [12] + \langle 34 \rangle [42] = 0.$$

Equivalently,

$$\frac{[21]}{[42]} = -\frac{\langle 34 \rangle}{\langle 13 \rangle}.$$

Hence

$$\frac{s_{12}}{s_{24}} = \frac{\langle 12 \rangle [21]}{\langle 24 \rangle [42]} = -\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 24 \rangle}.$$

Thus

$$\frac{A_4(1, 3, 2, 4)}{A_4(1, 2, 3, 4)} = \frac{s_{12}}{s_{24}}.$$

Therefore

$$s_{12}A_4(1, 2, 3, 4) - s_{24}A_4(1, 3, 2, 4) = 0.$$

This proves the four-point BCJ relation.

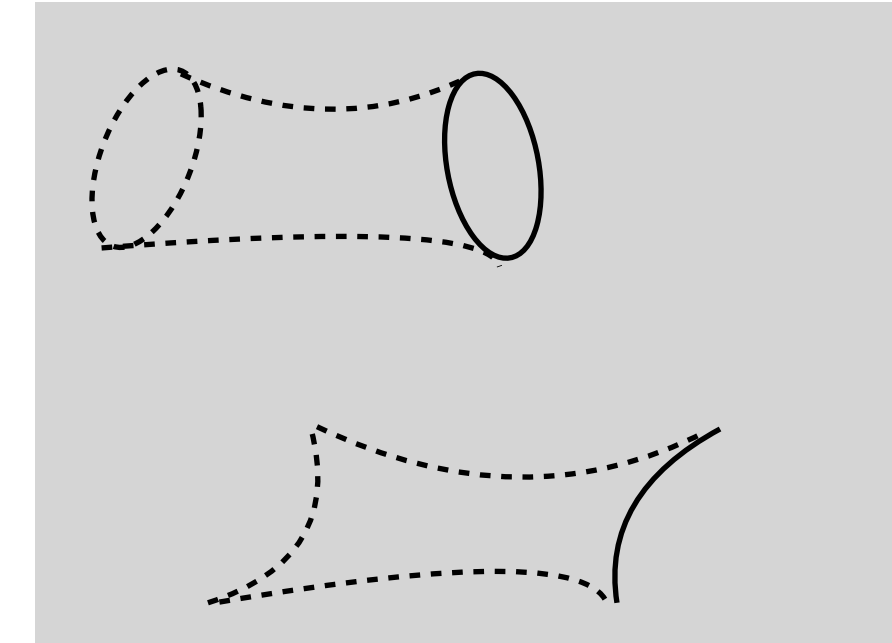
String-theory proof of BCJ relation

String Amplitudes in a nutshell

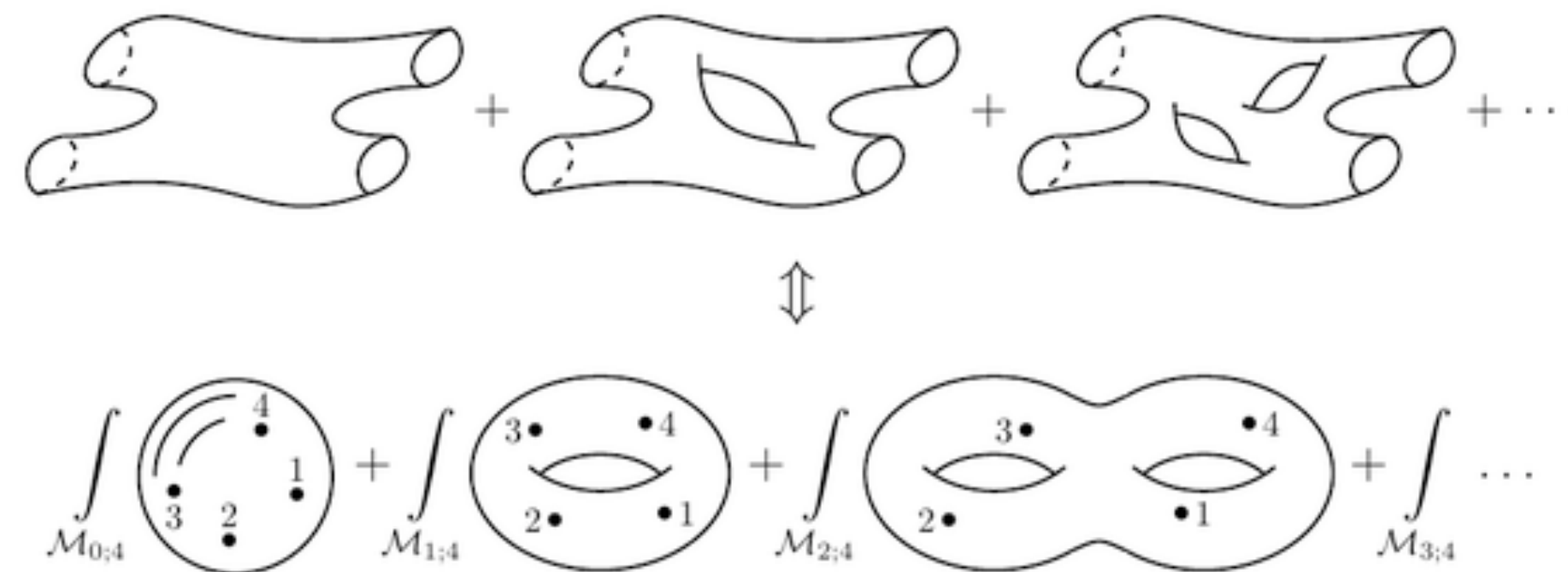
5 Tree-level topology summary

The basic topological distinction is therefore

- closed string : circle evolving in time $\Rightarrow$ closed worldsheet,
- open string : interval evolving in time $\Rightarrow$ worldsheet with boundary.



● **Perturbative expansion**
= **genus expansion.**

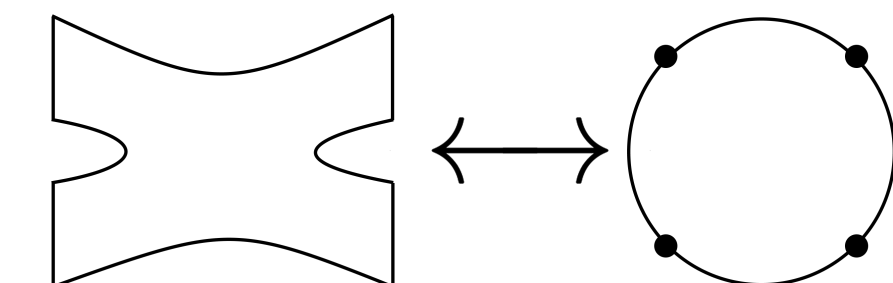
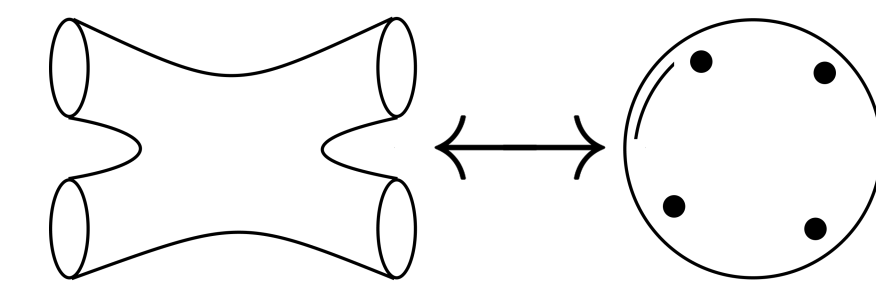


At tree level this gives

closed strings at tree level: S^2 with punctures

and

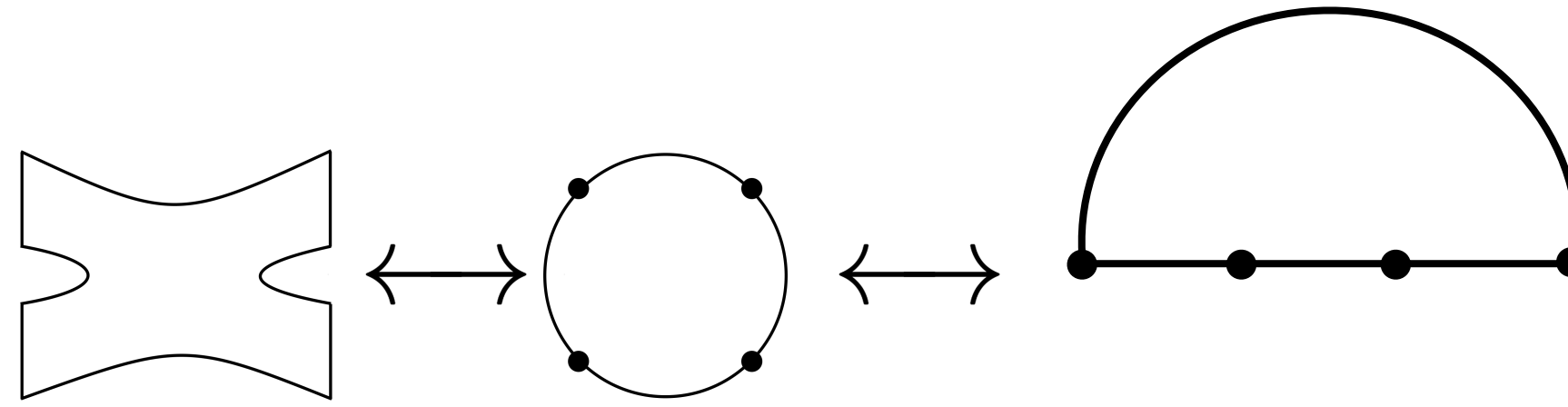
open strings at tree level: D^2 with boundary punctures.



Open String monodromy derivation of BCJ Relations

Goal: from disk integrals of open-string partial amplitudes to field theory BCJ relations in the limit: $\alpha' \rightarrow 0$.

● Open-string partial Amplitudes on the disk



After mapping the disk to the upper half-plane, the vertex operators sit on the real axis.

Color-ordered open-string Amplitudes from integrals of vertex positions in a fixed ordering:

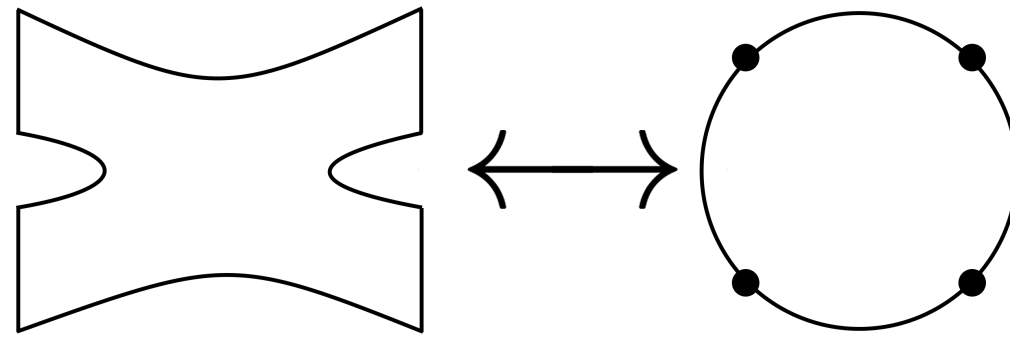
$$A_n(1, 2, \dots, n) = \int_{0 < x_2 < \dots < x_{n-2} < 1} \prod_{i=2}^{n-2} dx_i \prod_{1 \leq i < j \leq n-1} |x_i - x_j|^{\alpha' k_i \cdot k_j} F_n$$

└ External string state type and polarisations
└ Koba-Nielsen kernel

using the $SL(2, \mathbb{R})$ invariance to fix three insertion points, $x_1 = 0$, $x_{n-1} = 1$, $x_n = +\infty$.

Monodromy relations for 4-point Amplitudes

● 4-point amplitude: the integrand



$$n = 4 \quad x_1 = 0 \quad x_3 = 1 \quad x_4 = \infty$$

$$A(1,2,3,4) = \int_{0 < x_2 < 1} dx_2 \prod_{1 \leq i < j \leq 3} |x_i - x_j|^{\alpha' k_i \cdot k_j} = \int_{0 < x_2 < 1} dx_2 |x_1 - x_2|^{\alpha' k_1 \cdot k_2} |x_2 - x_3|^{\alpha' k_2 \cdot k_3}$$

$$= \int_{0 < x_2 < 1} dx_2 |-x_2|^{\alpha' k_1 \cdot k_2} |x_2 - 1|^{\alpha' k_2 \cdot k_3}$$

$$= \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}$$

$$|x_i - x_j| = \begin{cases} x_i - x_j, & \text{if } x_i > x_j \\ x_j - x_i, & \text{if } x_i < x_j \end{cases}$$

Basic monodromy idea

● KN-factor multivaluedness

The key observation is that the factors $|x_i - x_j|^{\alpha' k_i \cdot k_j}$ are multi-valued functions in the complex plane.

● Branch points and phases

If an integration contour is moved across a branch point, the integrand picks up a phase $e^{i\pi\alpha' k_i \cdot k_j}$

● Monodromy relations from contour deformations

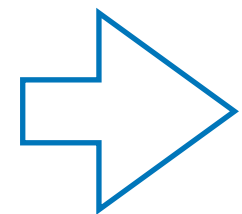
By analytically continuing one integration variable and deforming its contour, one obtains linear relations among different color orderings of the same open-string amplitude.

● Algorithm

The simplest derivation is obtained by starting from the amplitude $A_n(2, 1, 3, 4, \dots, n)$

and moving the integration contour of x_2 from the region $x_2 < 0$

through the ordered intervals $0 < x_2 < x_3, \quad x_3 < x_2 < x_4, \quad \dots, \quad x_{n-1} < x_2 < \infty.$



Each time x_2 passes one of the points x_j , the contour picks up a phase. The resulting relation expresses an amplitude with leg 2 before leg 1 in terms of amplitudes where leg 2 is inserted in all possible positions between legs 1 and n .

Monodromy relations for 4-point Amplitudes

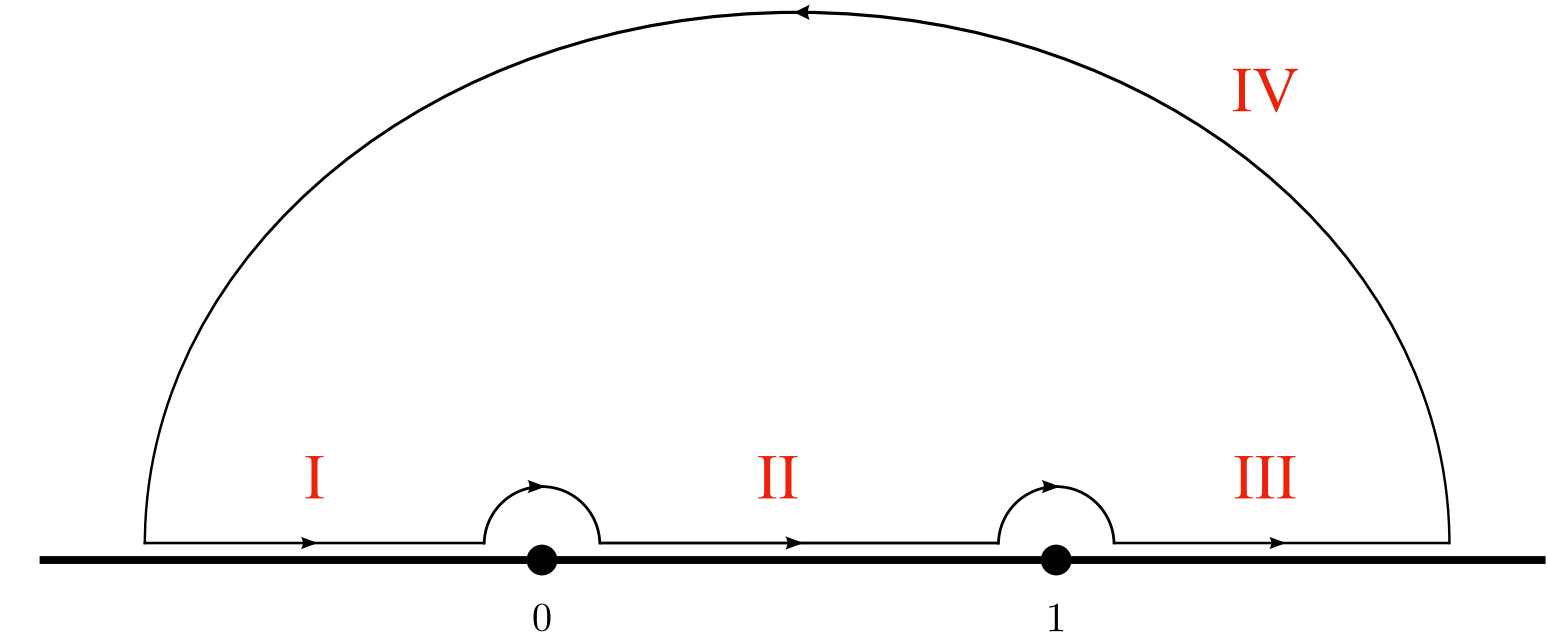
The four-point open-string amplitudes are very simple, they involve only one integral,

$$\begin{aligned}
 \text{I} \quad \mathcal{A}_4(2, 1, 3, 4) &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} , \\
 \text{II} \quad \mathcal{A}_4(1, 2, 3, 4) &= \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} , \\
 \text{III} \quad \mathcal{A}_4(1, 3, 2, 4) &= \int_1^{\infty} dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (x_2 - 1)^{\alpha' k_2 \cdot k_3} .
 \end{aligned}
 \tag{\#}$$

If we make an analytic continuation of the x_2 variable to the complex plane we can consider integrating the integrand of $\mathcal{A}_4(2, 1, 3, 4)$ over the closed contour shown in figure

Since no poles lie inside the contour the residue theorem tells us that this integral is zero. Taking the contour to infinity the upper semicircle will not contribute and we have

$$\begin{aligned}
 0 &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} + \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &\quad + \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} .
 \end{aligned}$$



Contour of x_2 -integration over the integrand of $\mathcal{A}_4(2, 1, 3, 4)$.

$x_1 = 0$	$x_3 = 1$	$x_4 = \infty$
$x_2 < x_1$	$x_1 < x_2 < x_3$	$x_3 < x_2 < \infty$
$x_2 < 0$	$0 < x_2 < 1$	$1 < x_2 < \infty$

$$|x_i - x_j| = \begin{cases} x_i - x_j , & \text{if } x_i > x_j \\ x_j - x_i , & \text{if } x_i < x_j \end{cases}$$

Monodromy relations for 4-point Amplitudes

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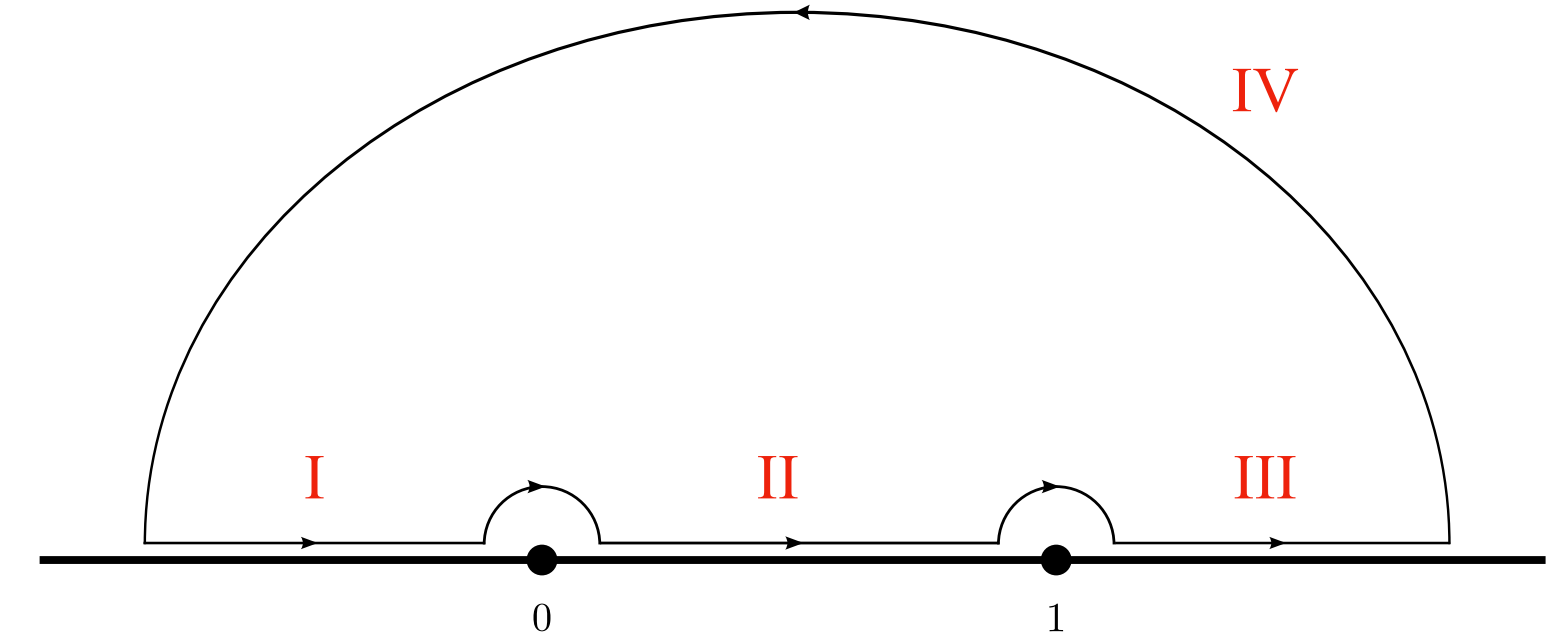
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$$\begin{aligned}
 0 &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} + \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &\quad + \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}.
 \end{aligned}$$

Comparing with the expressions in [#] this can be rewritten in terms of the open-string amplitudes as

$$\mathcal{A}_4(2, 1, 3, 4) = - \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} - \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}$$



Contour of x_2 -integration over the integrand of $\mathcal{A}_4(2, 1, 3, 4)$.

$$\begin{array}{lll}
 x_1 = 0 & x_3 = 1 & x_4 = \infty \\
 x_2 < x_1 & x_1 < x_2 < x_3 & x_3 < x_2 < \infty \\
 x_2 < 0 & 0 < x_2 < 1 & 1 < x_2 < \infty
 \end{array}$$

$$|x_i - x_j| = \begin{cases} x_i - x_j, & \text{if } x_i > x_j \\ x_j - x_i, & \text{if } x_i < x_j \end{cases}$$

Monodromy relations for 4-point Amplitudes

The four-point open-string amplitudes are very simple, they involve only one integral,

$$\begin{aligned}
 \text{I} \quad \mathcal{A}_4(2, 1, 3, 4) &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}, \\
 \text{II} \quad \mathcal{A}_4(1, 2, 3, 4) &= \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}, \\
 \text{III} \quad \mathcal{A}_4(1, 3, 2, 4) &= \int_1^{\infty} dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (x_2 - 1)^{\alpha' k_2 \cdot k_3}.
 \end{aligned}
 \tag{#}$$

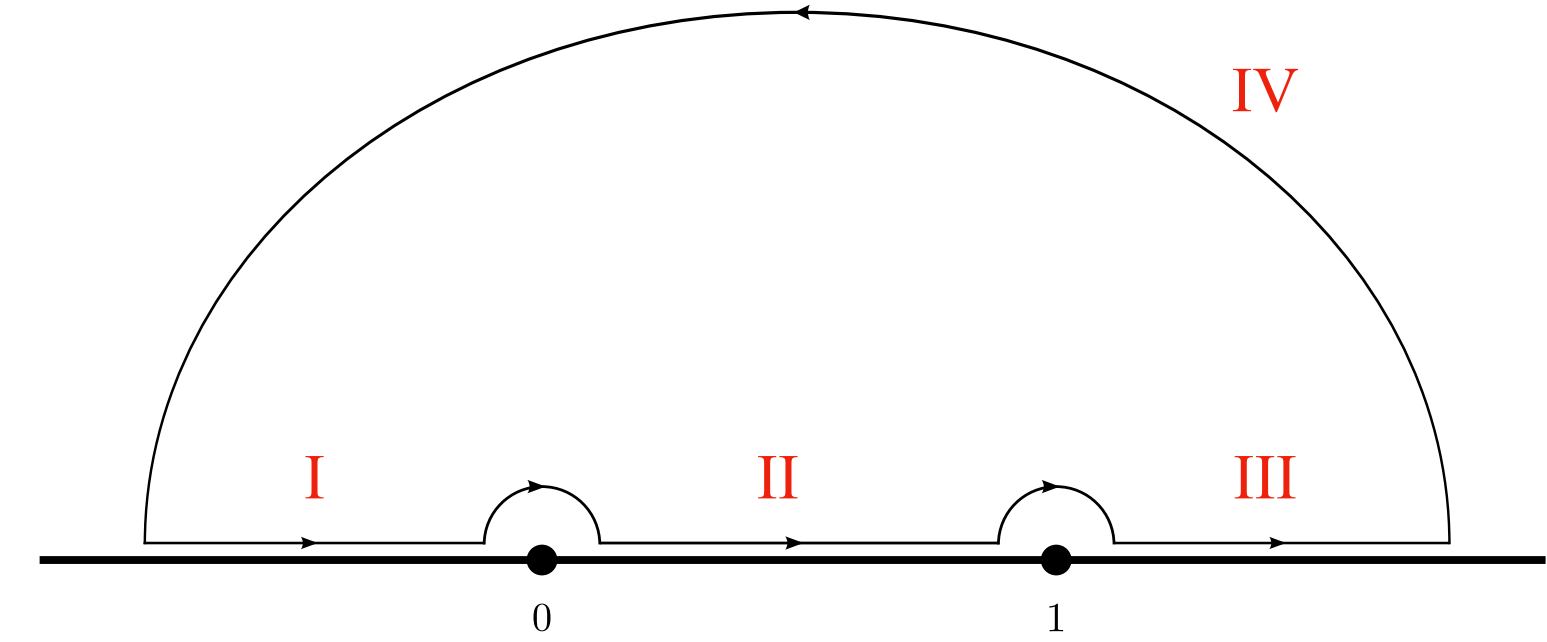
If we make an analytic continuation of the x_2 variable to the complex plane we can consider integrating the integrand of $\mathcal{A}_4(2, 1, 3, 4)$ over the closed contour shown in figure

Since no poles lie inside the contour the residue theorem tells us that this integral is zero. Taking the contour to infinity the upper semicircle will not contribute and we have

$$\begin{aligned}
 0 &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} + \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &\quad + \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}.
 \end{aligned}$$

Comparing with the expressions in [#] this can be rewritten in terms of the open-string amplitudes as

$$\begin{aligned}
 \mathcal{A}_4(2, 1, 3, 4) &= - \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} - \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &= - e^{i\pi \alpha' k_1 \cdot k_2} \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} - e^{i\pi \alpha' (k_1 \cdot k_2 + k_2 \cdot k_3)} \int_1^{\infty} dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (x_2 - 1)^{\alpha' k_2 \cdot k_3}
 \end{aligned}$$



Contour of x_2 -integration over the integrand of $\mathcal{A}_4(2, 1, 3, 4)$.

$x_1 = 0$	$x_3 = 1$	$x_4 = \infty$
$x_2 < x_1$	$x_1 < x_2 < x_3$	$x_3 < x_2 < \infty$
$x_2 < 0$	$0 < x_2 < 1$	$1 < x_2 < \infty$

$$|x_i - x_j| = \begin{cases} x_i - x_j, & \text{if } x_i > x_j \\ x_j - x_i, & \text{if } x_i < x_j \end{cases}$$

Monodromy relations for 4-point Amplitudes

The four-point open-string amplitudes are very simple, they involve only one integral,

$$\begin{aligned}
 \text{I} \quad \mathcal{A}_4(2, 1, 3, 4) &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}, \\
 \text{II} \quad \mathcal{A}_4(1, 2, 3, 4) &= \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}, \\
 \text{III} \quad \mathcal{A}_4(1, 3, 2, 4) &= \int_1^{\infty} dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (x_2 - 1)^{\alpha' k_2 \cdot k_3}.
 \end{aligned}
 \tag{#}$$

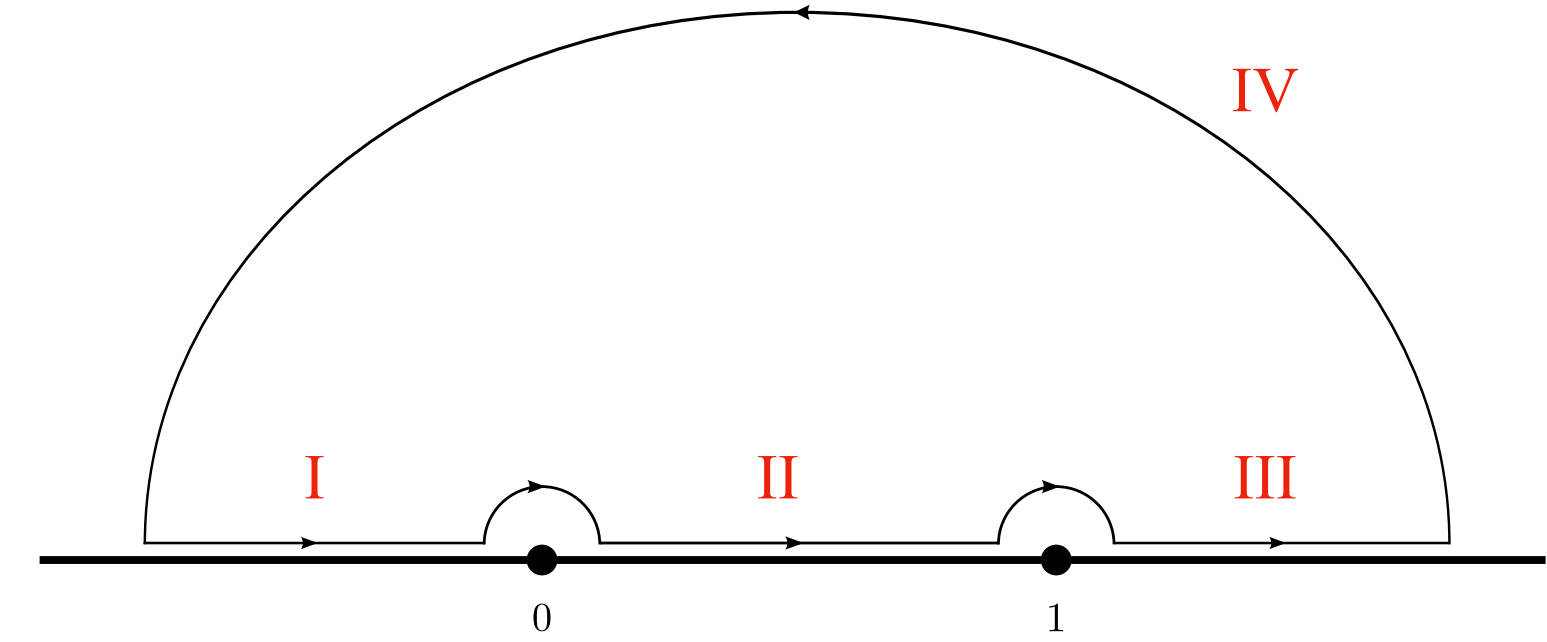
If we make an analytic continuation of the x_2 variable to the complex plane we can consider integrating the integrand of $\mathcal{A}_4(2, 1, 3, 4)$ over the closed contour shown in figure

Since no poles lie inside the contour the residue theorem tells us that this integral is zero. Taking the contour to infinity the upper semicircle will not contribute and we have

$$\begin{aligned}
 0 &= \int_{-\infty}^0 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} + \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &\quad + \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3}.
 \end{aligned}$$

Comparing with the expressions in [#] this can be rewritten in terms of the open-string amplitudes as

$$\begin{aligned}
 \mathcal{A}_4(2, 1, 3, 4) &= - \int_0^1 dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} - \int_1^{\infty} dx_2 (-x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} \\
 &= - e^{i\pi\alpha' k_1 \cdot k_2} \int_0^1 dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (1 - x_2)^{\alpha' k_2 \cdot k_3} - e^{i\pi\alpha' (k_1 \cdot k_2 + k_2 \cdot k_3)} \int_1^{\infty} dx_2 (x_2)^{\alpha' k_1 \cdot k_2} (x_2 - 1)^{\alpha' k_2 \cdot k_3} \\
 &= - e^{i\pi\alpha' k_1 \cdot k_2} \mathcal{A}_4(1, 2, 3, 4) - e^{i\pi\alpha' (k_1 \cdot k_2 + k_2 \cdot k_3)} \mathcal{A}_4(1, 3, 2, 4)
 \end{aligned}$$



Contour of x_2 -integration over the integrand of $\mathcal{A}_4(2, 1, 3, 4)$.

$x_1 = 0$	$x_3 = 1$	$x_4 = \infty$
$x_2 < x_1$	$x_1 < x_2 < x_3$	$x_3 < x_2 < \infty$
$x_2 < 0$	$0 < x_2 < 1$	$1 < x_2 < \infty$

$$|x_i - x_j| = \begin{cases} x_i - x_j, & \text{if } x_i > x_j \\ x_j - x_i, & \text{if } x_i < x_j \end{cases}$$

Monodromy relations for 4-point Amplitudes

$$A_4(2, 1, 3, 4) + e^{i\pi\alpha' k_1 \cdot k_2} A_4(1, 2, 3, 4) + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_4(1, 3, 2, 4) = 0.$$

● REAL part...

Taking the real part gives

$$A_4(2, 1, 3, 4) + \cos(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) \\ + \cos(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

In the field-theory limit $\alpha' \rightarrow 0$,

$$\cos(\pi\alpha' K) \rightarrow 1,$$

so the real part becomes

$$A_4(2, 1, 3, 4) + A_4(1, 2, 3, 4) + A_4(1, 3, 2, 4) = 0.$$

Monodromy relations for 4-point Amplitudes

$$A_4(2, 1, 3, 4) + e^{i\pi\alpha' k_1 \cdot k_2} A_4(1, 2, 3, 4) + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_4(1, 3, 2, 4) = 0.$$

● REAL part...

Taking the real part gives

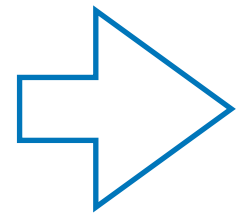
$$A_4(2, 1, 3, 4) + \cos(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) \\ + \cos(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

In the field-theory limit $\alpha' \rightarrow 0$,

$$\cos(\pi\alpha' K) \rightarrow 1,$$

so the real part becomes

$$A_4(2, 1, 3, 4) + A_4(1, 2, 3, 4) + A_4(1, 3, 2, 4) = 0.$$



● ...Photon decoupling relation

Monodromy relations for 4-point Amplitudes

$$A_4(2, 1, 3, 4) + e^{i\pi\alpha' k_1 \cdot k_2} A_4(1, 2, 3, 4) + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_4(1, 3, 2, 4) = 0.$$

● REAL part...

Taking the real part gives

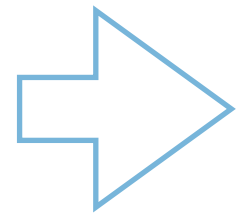
$$A_4(2, 1, 3, 4) + \cos(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) \\ + \cos(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

In the field-theory limit $\alpha' \rightarrow 0$,

$$\cos(\pi\alpha' K) \rightarrow 1,$$

so the real part becomes

$$A_4(2, 1, 3, 4) + A_4(1, 2, 3, 4) + A_4(1, 3, 2, 4) = 0.$$



● ...Photon decoupling relation

● IMAGINARY part

Taking the imaginary part

$$\sin(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

Using momentum conservation,

$$k_1 + k_2 + k_3 + k_4 = 0,$$

one has

$$k_2 \cdot (k_1 + k_3) = -k_2 \cdot k_4.$$

In the field-theory limit,

$$\sin(\pi\alpha' K) = \pi\alpha' K + \mathcal{O}((\alpha')^3),$$

so the imaginary part becomes

$$(k_1 \cdot k_2) A_4(1, 2, 3, 4) - (k_2 \cdot k_4) A_4(1, 3, 2, 4) = 0.$$

With

$$s_{ij} = 2k_i \cdot k_j,$$

this is

$$s_{12} A_4(1, 2, 3, 4) - s_{24} A_4(1, 3, 2, 4) = 0.$$

Equivalently,

$$s_{12} A_4(1, 2, 3, 4) + (s_{12} + s_{23}) A_4(1, 3, 2, 4) = 0.$$

Monodromy relations for 4-point Amplitudes

$$A_4(2, 1, 3, 4) + e^{i\pi\alpha' k_1 \cdot k_2} A_4(1, 2, 3, 4) + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_4(1, 3, 2, 4) = 0.$$

● REAL part...

Taking the real part gives

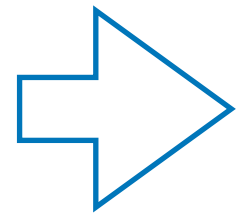
$$A_4(2, 1, 3, 4) + \cos(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) + \cos(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

In the field-theory limit $\alpha' \rightarrow 0$,

$$\cos(\pi\alpha' K) \rightarrow 1,$$

so the real part becomes

$$A_4(2, 1, 3, 4) + A_4(1, 2, 3, 4) + A_4(1, 3, 2, 4) = 0.$$



● ...Photon decoupling relation

● IMAGINARY part...

Taking the imaginary part

$$\sin(\pi\alpha' k_1 \cdot k_2) A_4(1, 2, 3, 4) + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_4(1, 3, 2, 4) = 0.$$

Using momentum conservation,

$$k_1 + k_2 + k_3 + k_4 = 0,$$

one has

$$k_2 \cdot (k_1 + k_3) = -k_2 \cdot k_4.$$

In the field-theory limit,

$$\sin(\pi\alpha' K) = \pi\alpha' K + \mathcal{O}((\alpha')^3),$$

so the imaginary part becomes

$$(k_1 \cdot k_2) A_4(1, 2, 3, 4) - (k_2 \cdot k_4) A_4(1, 3, 2, 4) = 0.$$

With

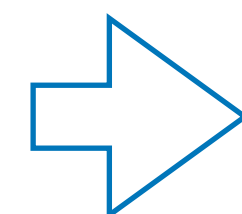
$$s_{ij} = 2k_i \cdot k_j,$$

this is

$$s_{12} A_4(1, 2, 3, 4) - s_{24} A_4(1, 3, 2, 4) = 0.$$

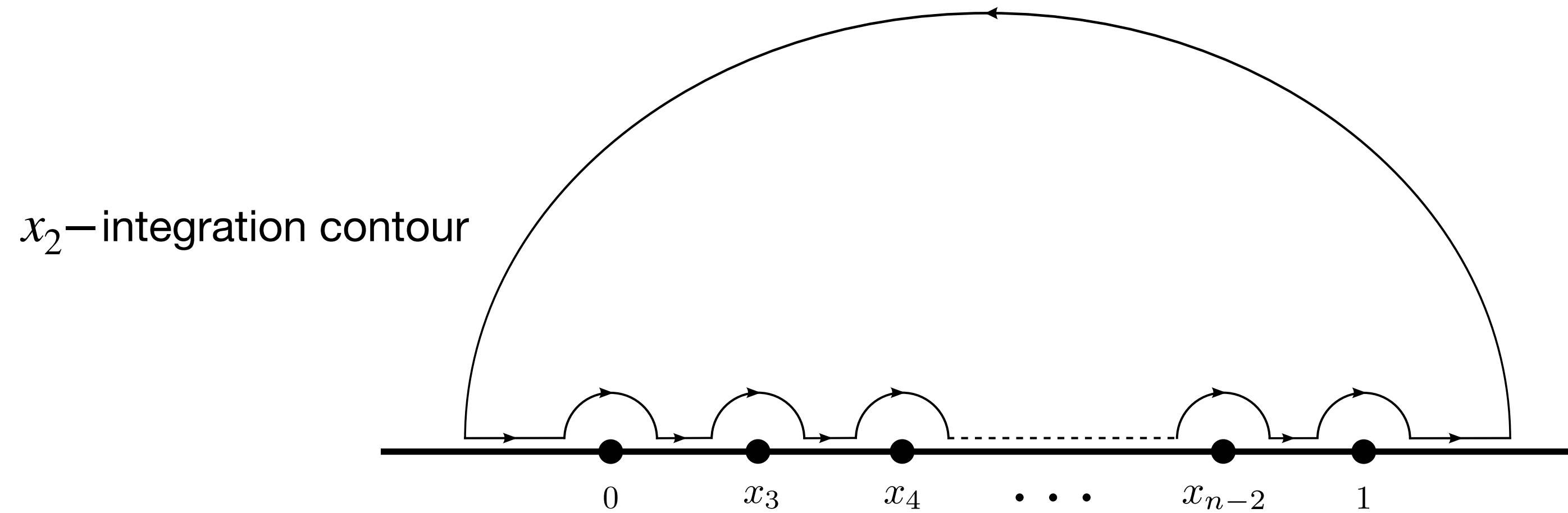
Equivalently,

$$s_{12} A_4(1, 2, 3, 4) + (s_{12} + s_{23}) A_4(1, 3, 2, 4) = 0.$$



● ...BCJ relation

Monodromy relations for n-point Amplitudes



The same contour argument works for arbitrary n . Starting from

$$A_n(2, 1, 3, 4, \dots, n),$$

move the contour of x_2 through the ordered points

$$0, x_3, x_4, \dots, x_{n-1} = 1.$$

Each crossing produces a phase. The result is

$$\begin{aligned} 0 = & A_n(2, 1, 3, \dots, n) + e^{i\pi\alpha' k_2 \cdot k_1} A_n(1, 2, 3, \dots, n) \\ & + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_n(1, 3, 2, 4, \dots, n) \\ & + \dots + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3 + \dots + k_{n-1})} A_n(1, 3, 4, \dots, n-1, 2, n). \end{aligned}$$

Monodromy relations for n-point Amplitudes

$$\begin{aligned} 0 = & A_n(2, 1, 3, \dots, n) + e^{i\pi\alpha' k_2 \cdot k_1} A_n(1, 2, 3, \dots, n) \\ & + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_n(1, 3, 2, 4, \dots, n) \\ & + \dots + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3 + \dots + k_{n-1})} A_n(1, 3, 4, \dots, n-1, 2, n). \end{aligned}$$

● REAL part: Photon Decoupling Relation

Taking the field-theory limit of the real part gives

$$\begin{aligned} 0 = & A_n(2, 1, 3, \dots, n) + A_n(1, 2, 3, \dots, n) + A_n(1, 3, 2, 4, \dots, n) \\ & + \dots + A_n(1, 3, 4, \dots, n-1, 2, n). \end{aligned}$$

This is a photon-decoupling identity: leg 2 is inserted in all possible positions relative to the ordered set $(1, 3, 4, \dots, n)$.

● IMAGINARY part: BCJ Relation

The field-theory limit of the imaginary part gives the fundamental BCJ relation

$$0 = \sum_{i=2}^{n-1} \left(\sum_{j=1}^i s_{2j} \right) A_n(1, 3, 4, \dots, i, 2, i+1, \dots, n).$$

##

Monodromy relations for n-point Amplitudes: *the general case*

The relation **##** shows one family of BCJ relations, but the full monodromy system is richer. More general contour deformations give relations involving two ordered sets

$$\beta = (\beta_1, \dots, \beta_r), \quad \alpha = (\alpha_1, \dots, \alpha_s).$$

A representative monodromy relation is

$$A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) = (-1)^r \prod_{1 \leq i < j \leq r} e^{i\pi \alpha' k_{\beta_i} \cdot k_{\beta_j}} \sum_{\sigma \in \text{OP}(\alpha \cup \beta^T)} \left(\prod e^{i\pi \alpha' k_{\alpha} \cdot k_{\beta}} \right) A_n(1, \sigma, n), \quad \mathbf{###}$$

where $\text{OP}(\alpha \cup \beta^T)$ denotes ordered permutations, or shuffles, of α with the reversed list β^T .

● KK Relations

In the field-theory limit, the real part of **###** becomes the Kleiss-Kuijf relation

$$A_n(\beta_1, \dots, \beta_r, 1, \alpha_1, \dots, \alpha_s, n) = (-1)^r \sum_{\sigma \in \text{OP}(\alpha \cup \beta^T)} A_n(1, \sigma, n).$$

● BCJ Relations

The imaginary part gives generalized BCJ relations. Schematically,

$$0 = \sum_{\sigma \in \text{OP}(\alpha \cup \beta^T)} \left[\sum_{i=1}^r \sum_{\sigma_J < \sigma_{\beta_i}} s_{\beta_i J} \right] A_n(1, \sigma, n).$$

These generalized BCJ relations reduce the number of independent color-ordered amplitudes from the KK basis size

$$(n-2)!$$

to the BCJ basis size

$$(n-3)!$$

● Minimal bases

A convenient minimal basis is

$$A_n(1, \sigma(2, 3, \dots, n-2), n-1, n), \quad \sigma \in S_{n-3}.$$

EXTRA

5-gluon BCJ relation

At five points, the fundamental BCJ relation is

$$(s_{23} + s_{24} + s_{25})A_5(1, 2, 3, 4, 5) + (s_{24} + s_{25})A_5(1, 3, 2, 4, 5) + s_{25}A_5(1, 3, 4, 2, 5) = 0. \quad (21)$$

Using momentum conservation,

$$s_{21} + s_{23} + s_{24} + s_{25} = 0, \quad (22)$$

we may rewrite this as

$$\boxed{s_{12}A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23})A_5(1, 3, 2, 4, 5) - s_{25}A_5(1, 3, 4, 2, 5) = 0.} \quad (23)$$

The amplitudes

For the helicity configuration $1^-, 2^-, 3^+, 4^+, 5^+$, BCFW recursion from the three-point amplitudes gives

$$A_5(1, 2, 3, 4, 5) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}. \quad (24)$$

Similarly,

$$A_5(1, 3, 2, 4, 5) = \frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \langle 45 \rangle \langle 51 \rangle} = -\frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 23 \rangle \langle 24 \rangle \langle 45 \rangle \langle 51 \rangle}, \quad (25)$$

and

$$A_5(1, 3, 4, 2, 5) = \frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 25 \rangle \langle 51 \rangle} = -\frac{\langle 12 \rangle^4}{\langle 13 \rangle \langle 34 \rangle \langle 24 \rangle \langle 25 \rangle \langle 51 \rangle}. \quad (26)$$

Substitution into the BCJ combination

Define

$$\mathcal{B}_5 = s_{12}A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23})A_5(1, 3, 2, 4, 5) - s_{25}A_5(1, 3, 4, 2, 5). \quad (27)$$

Substituting the Parke–Taylor expressions, it is enough to show

$$\begin{aligned} 0 &= \frac{s_{12}}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} - \frac{s_{12} + s_{23}}{\langle 13 \rangle \langle 23 \rangle \langle 24 \rangle \langle 45 \rangle \langle 51 \rangle} \\ &\quad + \frac{s_{25}}{\langle 13 \rangle \langle 34 \rangle \langle 24 \rangle \langle 25 \rangle \langle 51 \rangle}. \end{aligned} \quad (28)$$

Multiplying by the common denominator

$$\langle 13 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 25 \rangle, \quad (29)$$

we need to prove that

$$N = 0, \quad (30)$$

where

$$N = s_{12} \frac{\langle 13 \rangle \langle 24 \rangle \langle 25 \rangle}{\langle 12 \rangle} - (s_{12} + s_{23}) \langle 34 \rangle \langle 25 \rangle + s_{25} \langle 23 \rangle \langle 45 \rangle. \quad (31)$$

Using $s_{ij} = \langle ij \rangle [ji]$, this becomes

$$\begin{aligned} N &= [21] \langle 13 \rangle \langle 24 \rangle \langle 25 \rangle - (\langle 12 \rangle [21] + \langle 23 \rangle [32]) \langle 34 \rangle \langle 25 \rangle \\ &\quad + \langle 25 \rangle [52] \langle 23 \rangle \langle 45 \rangle. \end{aligned} \quad (32)$$

Factoring out $\langle 25 \rangle$ gives

$$N = \langle 25 \rangle \left[[21] (\langle 13 \rangle \langle 24 \rangle - \langle 12 \rangle \langle 34 \rangle) - [32] \langle 23 \rangle \langle 34 \rangle + [52] \langle 23 \rangle \langle 45 \rangle \right]. \quad (33)$$

Now use Schouten,

$$\langle 13 \rangle \langle 24 \rangle - \langle 12 \rangle \langle 34 \rangle = \langle 23 \rangle \langle 14 \rangle. \quad (34)$$

Therefore

$$N = \langle 25 \rangle \langle 23 \rangle ([21] \langle 14 \rangle - [32] \langle 34 \rangle + [52] \langle 45 \rangle). \quad (35)$$

Final momentum-conservation identity

Momentum conservation gives

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0. \quad (36)$$

Acting with $\langle 4|$ from the left and $|2\rangle$ from the right gives

$$\langle 4|p_1 + p_2 + p_3 + p_4 + p_5|2\rangle = 0. \quad (37)$$

The p_2 and p_4 terms vanish, so

$$\langle 41 \rangle [12] + \langle 43 \rangle [32] + \langle 45 \rangle [52] = 0. \quad (38)$$

Using

$$\langle 41 \rangle = -\langle 14 \rangle, \quad \langle 43 \rangle = -\langle 34 \rangle, \quad [21] = -[12], \quad (39)$$

this becomes

$$[21] \langle 14 \rangle - [32] \langle 34 \rangle + [52] \langle 45 \rangle = 0. \quad (40)$$

Hence

$$N = 0, \quad (41)$$

and therefore

$$\boxed{s_{12}A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23})A_5(1, 3, 2, 4, 5) - s_{25}A_5(1, 3, 4, 2, 5) = 0.} \quad (42)$$

This proves the five-point BCJ relation.

4 Five-point monodromy relation

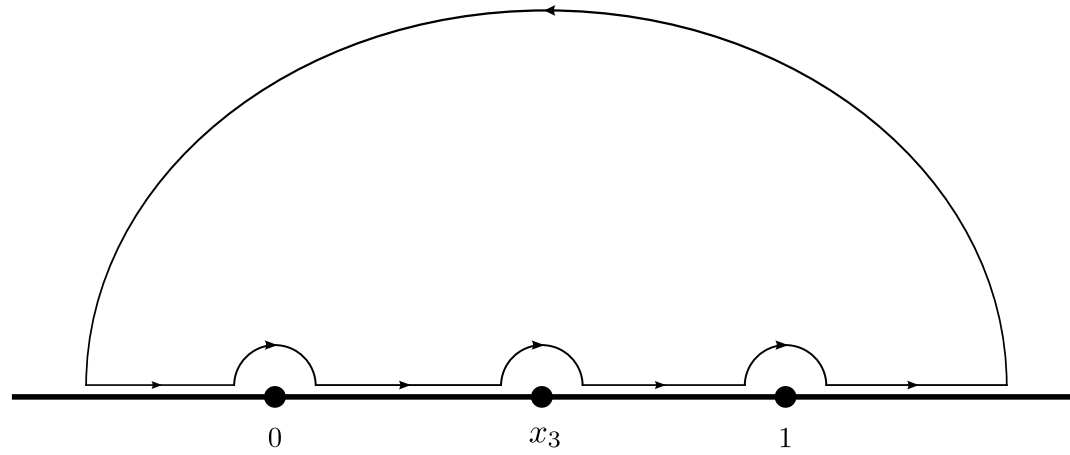


Figure 3.2: Contour of x_2 -integration over $I(x_2)$.

At five points, start from

$$A_5(2, 1, 3, 4, 5),$$

where $x_2 < 0 < x_3 < 1$. Deforming the x_2 contour across the branch points at 0, x_3 , and 1 gives

$$\begin{aligned} \mathcal{A}_5(2, 1, 3, 4, 5) &= \int_{-\infty}^0 dx_2 \int_0^1 dx_3 (-x_2)^{\alpha' k_1 \cdot k_2} (x_3)^{\alpha' k_1 \cdot k_3} \\ &\quad \times (1 - x_2)^{\alpha' k_4 \cdot k_2} (1 - x_3)^{\alpha' k_4 \cdot k_3} (x_3 - x_2)^{\alpha' k_2 \cdot k_3} \\ &\equiv \int_{-\infty}^0 dx_2 I(x_2). \end{aligned} \quad (3.14)$$

Again we analytically continue the x_2 -integration of $I(x_2)$ to the complex plane, and look at an integration over the contour shown in figure 3.2. Like for the four-point case this gives us a relation

$$\mathcal{A}_5(2, 1, 3, 4, 5) = - \int_0^{x_3} dx_2 I(x_2) - \int_{x_3}^1 dx_2 I(x_2) - \int_1^{\infty} dx_2 I(x_2), \quad (3.15)$$

with

$$\int_0^{x_3} dx_2 I(x_2) = e^{i\pi\alpha' k_1 \cdot k_2} \mathcal{A}_5(1, 2, 3, 4, 5), \quad (3.16)$$

$$\int_{x_3}^1 dx_2 I(x_2) = e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} \mathcal{A}_5(1, 3, 2, 4, 5), \quad (3.17)$$

$$\int_1^{\infty} dx_2 I(x_2) = e^{i\pi\alpha' k_2 \cdot (k_1 + k_3 + k_4)} \mathcal{A}_5(1, 3, 4, 2, 5), \quad (3.18)$$

and hence

$$\begin{aligned} \mathcal{A}_5(2, 1, 3, 4, 5) &= - e^{i\pi\alpha' k_1 \cdot k_2} \mathcal{A}_5(1, 2, 3, 4, 5) - e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} \mathcal{A}_5(1, 3, 2, 4, 5) \\ &\quad - e^{i\pi\alpha' k_2 \cdot (k_1 + k_3 + k_4)} \mathcal{A}_5(1, 3, 4, 2, 5). \end{aligned} \quad (3.19)$$

$$\begin{aligned} 0 &= A_5(2, 1, 3, 4, 5) + e^{i\pi\alpha' k_2 \cdot k_1} A_5(1, 2, 3, 4, 5) \\ &\quad + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3)} A_5(1, 3, 2, 4, 5) \\ &\quad + e^{i\pi\alpha' k_2 \cdot (k_1 + k_3 + k_4)} A_5(1, 3, 4, 2, 5). \end{aligned} \quad (12)$$

Real part: five-point photon decoupling

The real part gives a string-corrected photon-decoupling relation. In the field-theory limit all cosine factors go to one, giving

$$\begin{aligned} 0 &= A_5(2, 1, 3, 4, 5) + A_5(1, 2, 3, 4, 5) + A_5(1, 3, 2, 4, 5) \\ &\quad + A_5(1, 3, 4, 2, 5). \end{aligned} \quad (13)$$

Imaginary part: five-point BCJ

The imaginary part of (??) is

$$\begin{aligned} 0 &= \sin(\pi\alpha' k_2 \cdot k_1) A_5(1, 2, 3, 4, 5) \\ &\quad + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3)) A_5(1, 3, 2, 4, 5) \\ &\quad + \sin(\pi\alpha' k_2 \cdot (k_1 + k_3 + k_4)) A_5(1, 3, 4, 2, 5). \end{aligned} \quad (14)$$

Taking $\alpha' \rightarrow 0$ gives

$$\begin{aligned} 0 &= (k_2 \cdot k_1) A_5(1, 2, 3, 4, 5) + k_2 \cdot (k_1 + k_3) A_5(1, 3, 2, 4, 5) \\ &\quad + k_2 \cdot (k_1 + k_3 + k_4) A_5(1, 3, 4, 2, 5). \end{aligned} \quad (15)$$

Multiplying by 2 yields

$$\boxed{0 = s_{12} A_5(1, 2, 3, 4, 5) + (s_{12} + s_{23}) A_5(1, 3, 2, 4, 5) + (s_{12} + s_{23} + s_{24}) A_5(1, 3, 4, 2, 5).} \quad (16)$$

Using momentum conservation this is equivalent to the more standard form

$$\boxed{0 = (s_{23} + s_{24} + s_{25}) A_5(1, 2, 3, 4, 5) + (s_{24} + s_{25}) A_5(1, 3, 2, 4, 5) + s_{25} A_5(1, 3, 4, 2, 5).} \quad (17)$$