Quantization of Black Holes

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CONTENTS OF THE SIX LECTURES

Contents

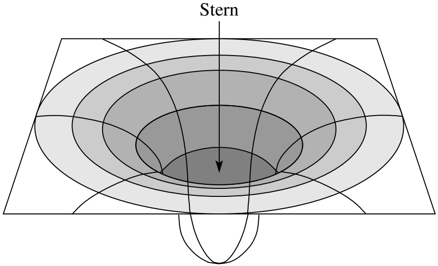
The dark stars of Michell and Laplace

John Michell 1784:

Hence . . . if the semi-diameter of a sphaere of the same density with the sun were to exceed that of the sun in the proportion of 500 to 1, a body falling from an infinite height towards it, would have acquired at its surface a greater velocity than that of light, and consequently, supposing light to be attracted by the same force in proportion to its vis inertiae, with other bodies, all light emitted from such a body would be made to return towards it, by its own proper gravity.

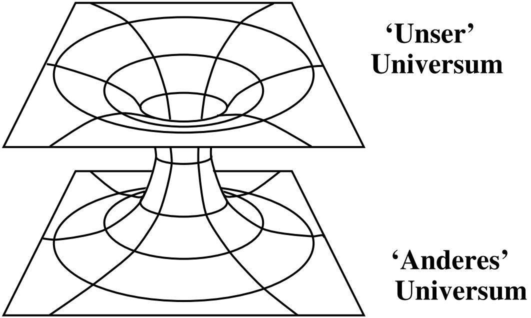
Independent prediction by Pierre-Simon Laplace 1796

Schwarzschild geometry



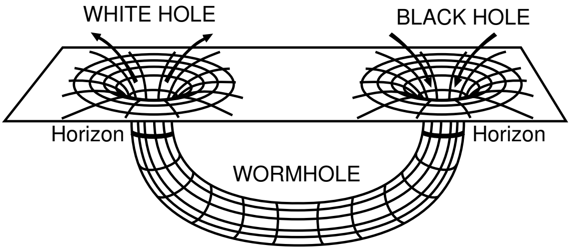
From: C. Kiefer, *Gravitation* (S. Fischer 2002)

Einstein–Rosen bridge



From: C. Kiefer, Thermodynamics of black holes and Hawking radiation (IOP 1999)

A wormhole



From: C. Kiefer, Thermodynamics of black holes and Hawking radiation (IOP 1999)

This wormhole is not suitable for travel!

A wormhole suitable for travel

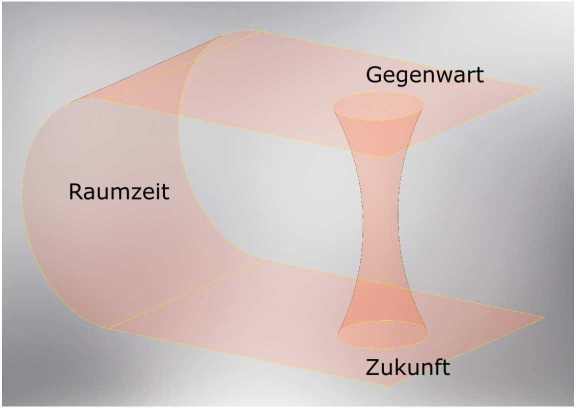
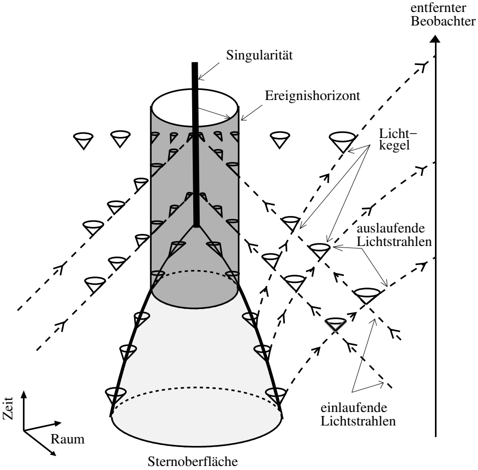


Figure credit: Sabine Graf

Collapse to a black hole



Event horizon at $R_{\mathrm{S}} = \frac{2GM}{c^2}$ (Schwarzschild radius);
inside event horizon: "'Zur Zeit wird hier der Raum"'

Black holes and gravitational waves

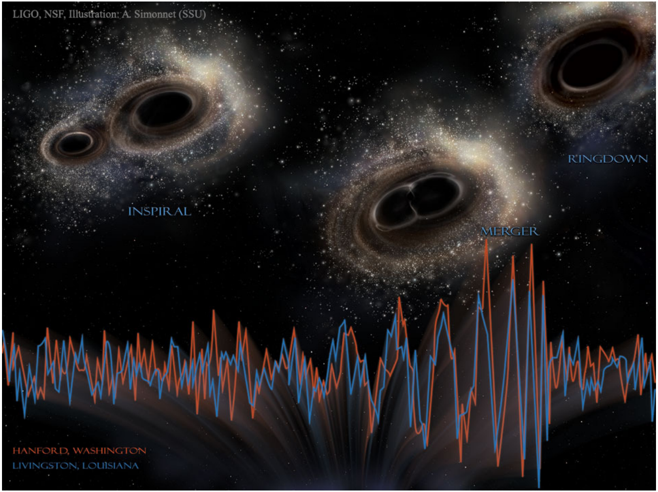


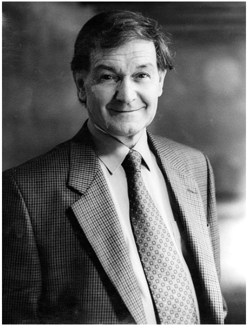
Figure credit: LIGO, NSF, Aurore Simonnet

First direct observation of gravitational waves in September 2015!

Cosmic censorship

Roger Penrose 1969:

In a realistic gravitational collapse, an event horizon forms which prevents the singularity to be seen from the outside. ("Nature abhors naked singularities.")



No-hair theorem

Stationary black holes within Einstein-Maxwell theory are uniquely characterized by three parameters:

- Mass M
- Angular momentum J
- Electric charge q

The most general solution is the Kerr–Newman solution. Of main astrophysical significance is the Kerr solution, which is completely characterized by M and J.

Black holes and thermodynamics

There are analogies between the laws of thermodynamics and the laws of black-hole mechanics. In these analogies, temperature is analogous to the surface gravity, and entropy is analogous to the area of the black hole.

If this is more than just an analogy, a fundamental length scale must enter. The only candidate is the Planck length, which results from a combination of gravitational constant G, speed of light c, and the quantum of action $\hbar$ (see Lecture I).

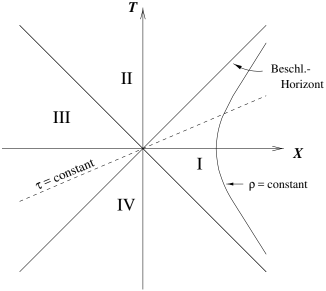
Hawking radiation

- Black holes radiate with a temperature proportional to $\hbar$, called *Hawking temperature* (Stephen Hawking 1974)

$$\boxed{T_{\mathrm{BH}} = \frac{\hbar c^3}{8\pi G k_{\mathrm{B}} M} \approx 6.2 \times 10^{-8} \frac{M_\odot}{M}\ \mathrm{K}}$$

- They have a finite lifetime: Cygnus X-1, for example, evaporates after about 10^{68} years, which corresponds to 10^{58} times the present age of the Universe!

Flat space–time



An accelerated observer in a flat space–time (no gravity) experiences the vacuum as filled with particles having the temperature (*Unruh temperature*)

$$T_{\mathrm{DU}} = \frac{\hbar a}{2\pi k_{\mathrm{B}} c} \approx 4.05 \times 10^{-25}\, a \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] \ \mathrm{K}$$

Is thermodynamics more fundamental than gravity?

- Black holes have a finite lifetime:

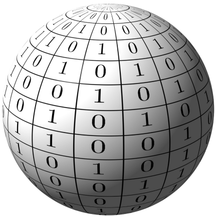
$$\tau_{\rm BH} \sim 10^{67} \left(\frac{M_0}{M_\odot} \right)^3 \text{ years};$$

  for observed (stellar and supermassive) black holes, this is much bigger than the age of the Universe
- Hawking's approximation breaks down when the mass of the black hole approaches the Planck mass
- If after black-hole evaporation only thermal radiation remains, there is the *problem of information loss*

Entropy of black holes

If black holes have a temperature, they also have an entropy:

$$S_{\mathrm{BH}} = k_{\mathrm{B}} \frac{Ac^3}{4\hbar G} \equiv k_{\mathrm{B}} \frac{A}{4 l_{\mathrm{P}}^2} \overset{\text{Schwarzschild}}{\approx} 1.07 \times 10^{77} k_{\mathrm{B}} \left(\frac{M}{M_{\odot}} \right)^2$$



Quantum gravity?

Is information lost?

There are three alternatives:

- Information is indeed lost – radiation is always strictly thermal;
- information is always preserved, but not seen in Hawking's approximation;
- after evaporation, the black hole leaves a relict.

Final answer comes from Quantum Gravity!

Stockholm conference, August 2015

Simple model: Collapse of a thin dust shell

- Spherically-symmetric thin shell consisting of particles with zero rest mass ("null dust shell");
- Classical theory: collapse to a black hole, or expansion from a white hole (usually excluded for thermodynamical reasons)
- Our quantization will lead to a singularity-free quantum state ("superposition of black and white hole")

(Hájíček and C.K. 2001)

Dynamics of a null dust shell

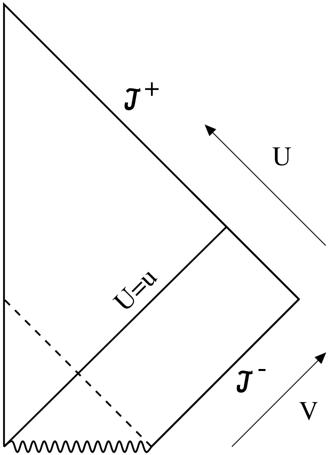


Figure: Penrose diagram for the outgoing shell in the classical theory. The shell is at $U = u$.

Approach here: reduced quantization

- Separation of variables into pure gauge degrees of freedom ('embedding variables') and physical degrees of freedom (plus the respective canonical momenta)
- General existence of this 'Kuchař decomposition' can be shown by making a transformation to the standard ADM phase space of general relativity (Hájíček and Kijowski 2000)
- In this construction, a formal 'background manifold' plays a crucial role.

Wave packets

Exact time evolution for a wave packet describing the shell:

$$\Psi_{\kappa\lambda}(t,r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{\mathrm{i}}{(\lambda + \mathrm{i}t + \mathrm{i}r)^{\kappa+1}} - \frac{\mathrm{i}}{(\lambda + \mathrm{i}t - \mathrm{i}r)^{\kappa+1}} \right]$$

Important consequence:

$$\lim_{r\to 0} \Psi_{\kappa\lambda}(t,r) = 0$$

This means that the probability of finding the shell at vanishing radius is zero! In this sense, the singularity is avoided in the quantum theory. The quantum shell bounces and re-expands, and no event horizon forms.

Expectation value and variance of the shell radius:

$$\langle R_0 \rangle_{\kappa\lambda} := 2G\langle E \rangle_{\kappa\lambda} = (2\kappa + 1)\frac{l_{\rm P}^2}{\lambda},$$

$$\Delta(R_0)_{\kappa\lambda} = 2G\Delta E_{\kappa\lambda} = \sqrt{2\kappa + 1}\frac{l_{\rm P}^2}{\lambda}$$

It turns out that the wave packet can be squeezed below its Schwarzschild radius if its energy is greater than the Planck energy—a genuine quantum effect!

“Superposition of black and white hole”

Astrophysical relevance?

Central question: what is the timescale $t_{\rm b}$ for shell collapse and re-expansion?

- Ambrus and Hájíček (2005): $t_{\rm b}$ is of order M, which would be too short for an observational significance of the model;
- later investigations (e.g. in loop quantum gravity) led to other timescales,[1] e.g. $t_{\rm b} \propto M^2$;
- question also relevant for the LTB model, see below.

[1] See e.g. D. Malafarina, *Universe* **3** (2017) 2,48 for a review.

The Lemaître–Tolman–Bondi (LTB) model

LTB model: spherically-symmetric solution of the Einstein equations with non-rotating dust of mass density ϵ as its source (for constant density we have the special case of the Oppenheimer–Snyder scenario).

$$\mathrm{d}s^2 \quad = \quad -c^2\mathrm{d}\tau^2 + \frac{R'^2(\rho)}{1+2f(\rho)}\,\mathrm{d}\rho^2 + R^2(\rho)\,\mathrm{d}\Omega^2\,,$$

$$\text{with} \quad \frac{8\pi G}{c^2}\epsilon = \frac{F'}{R^2R'} \quad \text{and} \quad \frac{\dot{R}^2}{c^2} = \frac{F}{R} + 2f,$$

where τ is the dust proper time and ρ the radial coordinate that labels the dust shells comprising the dust cloud; $F(\rho)$ is twice the active gravitational mass inside the shell with label ρ.

Topic here: Quantization – how to proceed?

As in the case of the collapsing shell, we seek for a unitary evolution (here with respect to the dust proper time τ).
Schrödinger quantization:

$$P_o \to \hat{P}_o = -\mathrm{i}\hbar \frac{d}{dR_o}.$$

The operator $\hat{R}_o$ acts by multiplication. (In the following we will suppress the subscript o.)
Hamilton operator:

$$\hat{H} = \frac{\hbar^2}{2}\, R^{-1+a+b} \frac{d}{dR} R^{-a} \frac{d}{dR} R^{-b},$$

where a and b encode factor ordering ambiguities. Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial \Psi(R,\tau)}{\partial \tau} = \hat{H}\Psi(R,\tau)$$

We impose square-integrability on wave functions and let them evolve unitarily according to a self-adjoint Hamiltonian. This corresponds to enforcing probability conservation in dust proper time.

Singularity avoidance for wave packets

For a wide class of wave packets, the probability for the outermost dust shell to be in the classically singular configuration $R = 0$ is zero.

One explicit example:

$$\Psi(R,\tau) = \sqrt{3}\left(\frac{\sqrt{2}}{3}\right)^{\frac{1}{3}|1+a|+1} \frac{\Gamma\left(\frac{1}{6}|1+a| + \frac{\kappa}{2} + 1\right)}{\sqrt{\Gamma(\kappa+1)}\Gamma\left(\frac{1}{3}|1+a|+1\right)} R^{\frac{1}{2}(1+a+|1+a|+2b)}$$
$$\times \frac{\lambda^{\frac{1}{2}(\kappa+1)}}{(\frac{\lambda}{2} - i\tau)^{\frac{1}{6}|1+a|+\frac{\kappa}{2}+1}} \, {}_1F_1\left(\frac{1}{6}|1+a| + \frac{\kappa}{2} + 1; \frac{1}{3}|1+a|+1; -\frac{2R^3}{9(\frac{\lambda}{2} - i\tau)}\right)$$

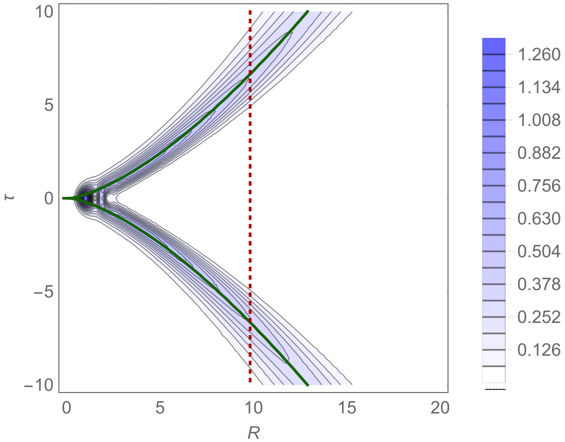


Figure: Probability amplitude for R as given by $R^{1-a-2b}\,|\Psi(R,\tau)|^2$, compared to the classical trajectories (full green line) and the exterior apparent horizon (dotted red line), with $a = 2$ and $b = 1$

C. Kiefer and T. Schmitz, *Phys. Rev. D* **99**, 126010 (2019)

- Discussion of Oppenheimer–Snyder models with flat[2] and non-flat[3] Friedmann sections describing the interior of the dust cloud: Again, for certain parameter values, there is a bounce of wave packets as seen by a stationary observer.
- Lifetime of bouncing solution (for an exterior observer) turns out to be proportional to M^3 (same order as black-hole evaporation time); but more recent investigations suggest $\tau_{\mathrm{b}} \propto M$ (Schmitz 2020) as in some models of loop quantum gravity

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[2] T. Schmitz, *Phys. Rev. D* **101**, 026016 (2020)

[3] C. Kiefer and H. Mohaddes, *Phys. Rev. D* **107**, 126006 (2023)

Quantum Oppenheimer–Snyder scenario

Closed slices: Bounce and oscillations inside the horizons: eternal lifetime for the comoving observer

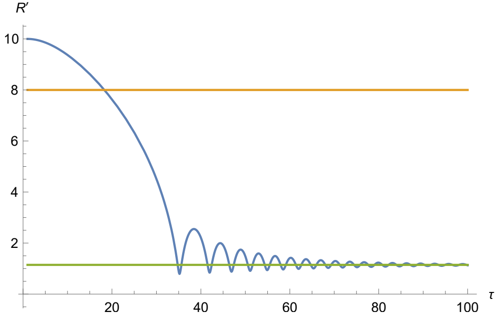

Figure: Graph in the $R-\tau$ space. The orange line represents the Schwarzschild radius $R = 2GM/c^2$, the green line the location where equilibrium is reached after oscillations, where $\dot{R} = \ddot{R} = 0$.
From Kiefer and Mohaddes, *op. cit.*

Reflections

- One can construct quantum models for gravitational collapse which are singularity-free. There is a unitary evolution from a collapsing to an expanding wave packet (bouncing solution). If generally true, this would solve the cosmic-censorship problem.
- Lifetime of black-and-white hole? Compatible with observations? Relevance for primordial black holes?
- Fate of naked singularities?
- Implementation of Hawking radiation? Fate of horizon and relevance for information-loss problem? Most likely, the horizon disappears.