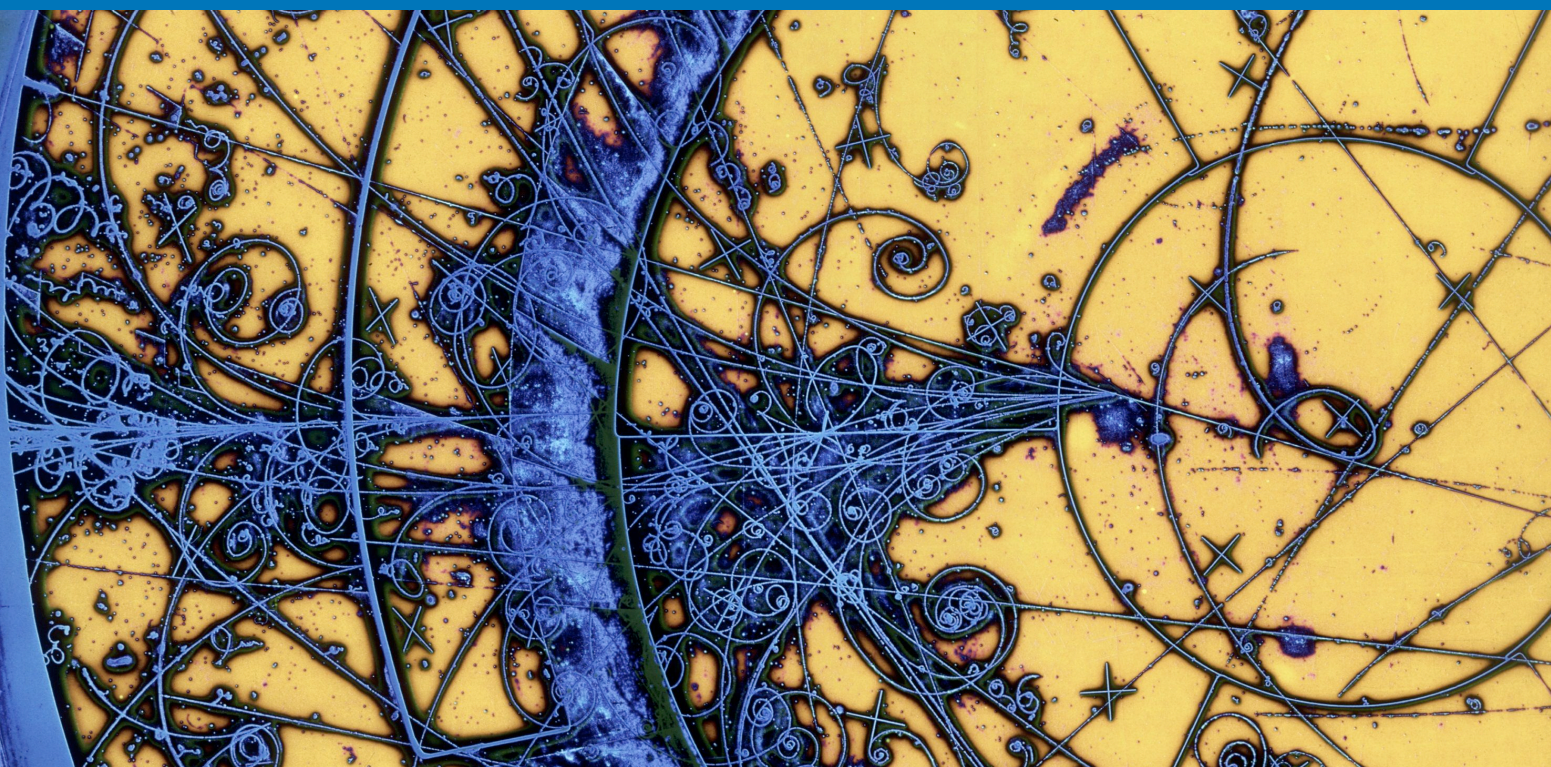




AMPLITUDES

Lecture 2

Pierpaolo Mastrolia (UNIPD, INFN)
Vatican Observatory School
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Scattering Amplitudes in Yang-Mills and Gravity

1. Photon decoupling and KK relations
2. Spinor helicity formalism and BCFW Recurrence Relation.

Kinematic Relations for Yang-Mills Amplitudes

3. Field-theory proof and Open-string monodromy derivation of BCJ relations

Kinematic Relations for Gravity Amplitudes

4. Color Kinematics Duality and Field-theory KLT relations

Amplitudes and Geometry

5. Intersection Theory for twisted DeRham co-homology.
6. KLT from twisted period/Riemann bilinear relations.

Spinor Helicity Formalism

We are going to develop a massless formalism, but the extension to the massive case is straightforward. Given a massive momentum P^μ , such that $P^2 \neq 0$, if q^μ , such that $q^2 = 0$ is an arbitrary massless momentum, we observe that p^μ , defined as

$$p^\mu = P^\mu - \frac{P^2}{2P \cdot q} q^\mu,$$

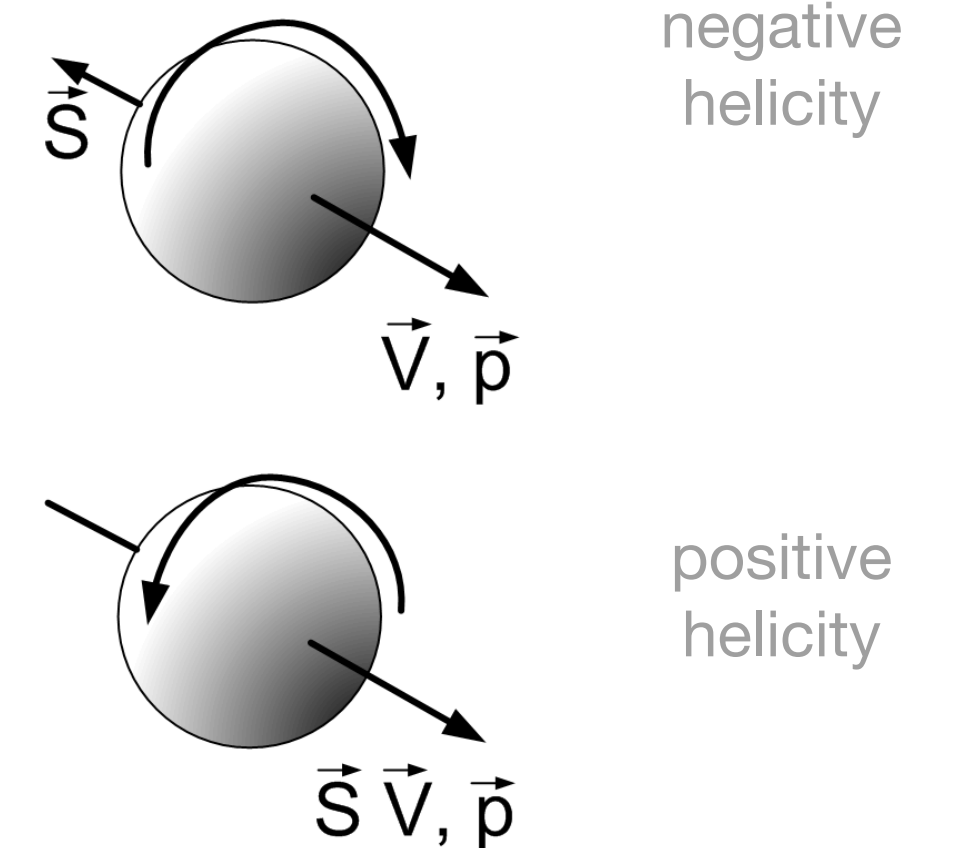
also satisfies $p^2 = 0$. Thus, our massive momentum P^μ can be decomposed into the two massless momenta q^μ and p^μ ,

$$P^\mu = p^\mu + \frac{P^2}{2P \cdot q} q^\mu.$$

Helicity The helicity of a particle is the projection of the spin onto the direction of motion, which is given by the momentum $\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$,

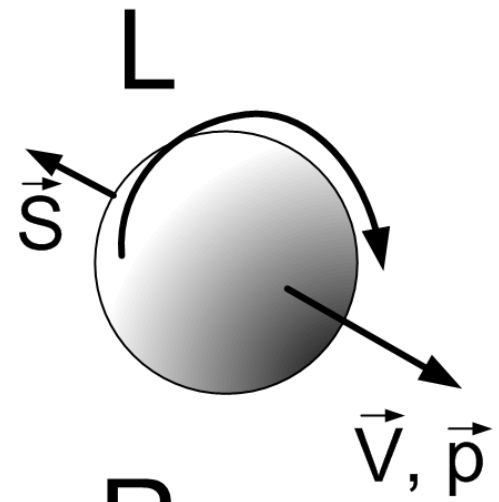
$$h(\hat{\mathbf{p}}) = \boldsymbol{\Sigma} \cdot \hat{\mathbf{p}} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix}.$$

It is well defined because, as an operator, it commutes with Dirac Hamiltonian, $[h(\hat{\mathbf{p}}), H_D] = 0$.

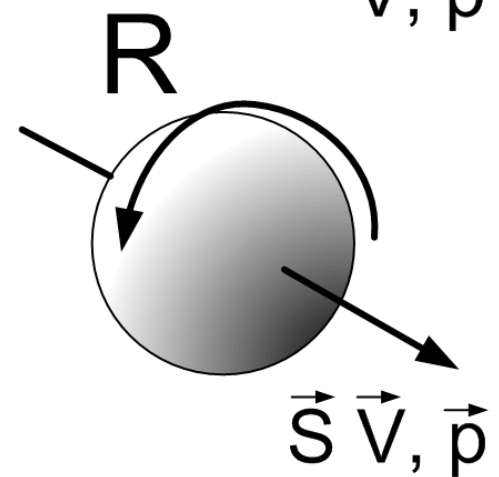


Spinor Helicity Formalism

negative
helicity



positive
helicity



Chirality Let us consider the Dirac equation in the momentum space,

$$\begin{array}{ll} \text{particle} & (\not{p} + m)u(p) = 0 \\ \text{antiparticle} & (\not{p} - m)v(p) = 0 \end{array} \quad \xrightarrow{m \rightarrow 0} \quad \begin{array}{l} \not{p}u(p) = 0 \\ \not{p}v(p) = 0 \end{array} .$$

It is clear that in the massless limit there is no difference between the two solutions

● **Chirality Projectors** $P_{\pm} = \frac{1 \pm \gamma_5}{2}$ equivalently $P_{L/R} = \left(\frac{1 \mp \gamma_5}{2} \right)$

● **Chiral Spinors** $u_{\pm}(p) = P_{\pm}u(p), \quad v_{\mp}(p) = P_{\pm}v(p),$
 $\bar{u}_{\pm}(p) = \bar{u}(p)P_{\mp}, \quad \bar{v}_{\mp}(p) = \bar{v}(p)P_{\mp}.$

● **(Bra-Ket) Notation**

● **spinors**

$$\begin{aligned} |i\rangle &= u_+(p_i) = v_-(p_i) \\ [i] &= u_-(p_i) = v_+(p_i), \end{aligned}$$

● **Massless Dirac Equation**

$$\begin{aligned} \not{p}_i |i\rangle &= 0 \\ \not{p}_i [i] &= 0 \end{aligned}$$

● **Adjoint spinors**

$$\begin{aligned} \langle i| &= \bar{u}_-(p_i) = \bar{v}_+(p_i), \\ [i| &= \bar{u}_+(p_i) = \bar{v}_-(p_i), \end{aligned}$$

● **Massless Adjoint Dirac Equation**

$$\begin{aligned} \langle i| \not{p}_i &= 0, \\ [i| \not{p}_i &= 0. \end{aligned}$$

Spinor Helicity Formalism

● Explicit representation of Chiral Spinors

- γ -matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix};$$

- chiral spinors

$$u_+ \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_+ \\ \chi_+ \end{pmatrix}, \quad u_- \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_- \\ -\chi_- \end{pmatrix},$$

where the two-component χ spinors are defined as

$$\chi_+ \equiv \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\phi_p} \end{pmatrix}, \quad \chi_- \equiv \begin{pmatrix} \sqrt{p^-} e^{-i\phi_p} \\ -\sqrt{p^+} \end{pmatrix},$$

where

$$p^\pm \equiv p^0 \pm p^3, \quad e^{\pm i\phi_p} = \frac{p^1 \pm ip^2}{\sqrt{p^+} \sqrt{p^-}}.$$

Spinor Helicity Formalism

● Mandelstam inv's

$$s_{ij} \equiv (p_i + p_j)^2 = 2p_i \cdot p_j$$

● Spinor Inner Products

$$\langle ij \rangle \equiv \bar{u}_-(p_i) u_+(p_j) = \sqrt{|s_{ij}|} e^{+i\phi_p} \qquad [ij] \equiv \bar{u}_+(p_i) u_-(p_j) = -\sqrt{|s_{ij}|} e^{-i\phi_p}$$

● Properties

Antisymmetry

$$\langle ij \rangle = -\langle ji \rangle, \qquad [ij] = -[ji],$$

Byproduct

$$\langle ii \rangle = [ii] = \langle ij \rangle = [ij] = 0$$

Spinor products vs Mandelstam inv's

$$s_{ij} = \langle ij \rangle [ji] = \langle ji \rangle [ij]$$

Negative momentum spinors

$$\langle i(-j) \rangle = i \langle ij \rangle, \qquad [i(-j)] = i [ij]$$

imaginary unit

Spinor Helicity Formalism

● **Spinor Outer product** $|i\rangle [i| = \frac{(1 + \gamma_5)\not{p}_i}{2}, \quad |i] \langle i| = \frac{(1 - \gamma_5)\not{p}_i}{2}, \quad \Rightarrow$ ● **Slashed momenta**

$$\not{p}_i = |i\rangle [i| + |i] \langle i|$$

● **Massless Real 4-momentum**

$$p_i^2 = 0,$$

$$p_i^\mu \equiv \frac{\langle i | \gamma^\mu | i \rangle}{2} = \frac{[i | \gamma^\mu | i \rangle}{2}$$

like-like spinor sandwich <--> **real** momenta

● **Special Identities**

1. charge conjugation

$$[i | \gamma^\mu | j \rangle = \langle j | \gamma^\mu | i]$$

$$p_{k\mu} [i | \gamma^\mu | j \rangle = [i | \not{k} | j \rangle = [ik] \langle kj \rangle = \langle jk \rangle [ki] = \langle j | \not{k} | i \rangle = p_{k\mu} \langle j | \gamma^\mu | i \rangle$$

2. Fierz identity

$$[i | \gamma^\mu | j \rangle [k | \gamma_\mu | l \rangle = 2 [ik] \langle lj \rangle$$

$$\begin{aligned} [i | \gamma^\mu \not{p}_j \not{p}_l \gamma_\mu | k \rangle &= [i | 4p_j \cdot p_l | k \rangle = 4p_j \cdot p_l [i k] = 2 \langle jl \rangle [lj] [i k] \\ &= [i | \gamma^\mu | j \rangle [j | l \rangle \langle l | \gamma_\mu | k \rangle \end{aligned}$$

3. Schouten identity

$$\langle ab \rangle \langle cd \rangle = \langle ad \rangle \langle cb \rangle + \langle ac \rangle \langle bd \rangle$$

and analogously for the square brackets.

$$\langle c | \not{a} \not{b} | d \rangle + \langle c | \not{b} \not{a} | d \rangle = 2p_a \cdot p_b \langle cd \rangle$$

$$\langle ca \rangle [ab] \langle bd \rangle + \langle cb \rangle [ba] \langle ad \rangle = \langle ab \rangle [ba] \langle cd \rangle$$

● **How would you express Dirac Equation?**

Spinor Helicity Formalism

● Massless Gauge Bosons Polarisation

$$k^2 = q^2 = 0$$

boson momentum

$$\epsilon_{\mu}^{-}(k, q) \equiv -\frac{[q | \gamma^{\mu} | k \rangle}{\sqrt{2} [qk]}, \quad \epsilon_{\mu}^{+}(k, q) \equiv \frac{\langle q | \gamma^{\mu} | k]}{\sqrt{2} \langle qk \rangle},$$

reference momentum

different spinor sandwich <--> **complex** momenta

● Transversality from Dirac Eqn. $\epsilon_{\mu}^{\pm} k^{\mu} = 0, \quad \epsilon_{\mu}^{\pm} q^{\mu} = 0$

● slashed polarisation

$$\not{\epsilon}_{-}(k, q) = \frac{\sqrt{2}}{[kq]} (|k\rangle [q| + |q\rangle \langle k|)$$

$$\not{\epsilon}_{+}(k, q) = \frac{\sqrt{2}}{\langle qk \rangle} (|q\rangle [k| + |k\rangle \langle q|)$$

$$\langle i | \not{\epsilon}_{-}(k, q) | j] = \langle i | \gamma_{\mu} | j] \epsilon_{-}^{\mu}(k, q) = -\langle i | \gamma_{\mu} | j] \frac{[q | \gamma^{\mu} | k \rangle}{\sqrt{2} [qk]} = \left\langle i \left| \left(\frac{\sqrt{2}}{[kq]} |k\rangle [q| \right) \right| j \right]$$

$$[i | \not{\epsilon}_{-}(k, q) | j \rangle = [i | \gamma_{\mu} | j \rangle \epsilon_{-}^{\mu}(k, q) = -[i | \gamma_{\mu} | j \rangle \frac{[q | \gamma^{\mu} | k \rangle}{\sqrt{2} [qk]} = \left[i \left| \left(\frac{\sqrt{2}}{[kq]} |q\rangle \langle k| \right) \right| j \right\rangle$$

● Properties

conjugation $(\epsilon_{\mu}^{+})^{*} = \epsilon_{\mu}^{-}$

spacelikeness $\epsilon^{+} \cdot (\epsilon^{+})^{*} = \epsilon^{+} \cdot \epsilon^{-} = -\frac{1}{2} \frac{\langle q | \gamma_{\mu} | k]}{\langle qk \rangle} \frac{[q | \gamma^{\mu} | k \rangle}{[qk]} = -\frac{1}{2} \frac{2 \langle qk \rangle [qk]}{\langle qk \rangle [qk]} = -1,$

masslessness $\epsilon^{+} \cdot (\epsilon^{-})^{*} = \epsilon^{+} \cdot \epsilon^{+} = \frac{1}{2} \frac{\langle q | \gamma_{\mu} | k]}{\langle qk \rangle} \frac{\langle q | \gamma^{\mu} | k]}{\langle qk \rangle} = \frac{1}{2} \frac{2 \langle qq \rangle [kk]}{\langle qk \rangle \langle qk \rangle} = 0,$

$$\epsilon^{-} \cdot (\epsilon^{+})^{*} = \epsilon^{-} \cdot \epsilon^{-} = 0.$$

Spinor Helicity Formalism

● Properties (continued)

- polarizations differing for the reference momentum

$$\begin{aligned}
 \epsilon_{\mu}^{+}(k, \tilde{q}) - \epsilon_{\mu}^{+}(k, q) &= \frac{\langle \tilde{q} | \gamma^{\mu} | k \rangle}{\sqrt{2} \langle \tilde{q} k \rangle} - \frac{\langle q | \gamma^{\mu} | k \rangle}{\sqrt{2} \langle q k \rangle} = - \frac{\langle k q \rangle [q \tilde{q}] \langle \tilde{q} | \gamma_{\mu} | k \rangle + \langle k \tilde{q} \rangle [\tilde{q} q] \langle q | \gamma_{\mu} | k \rangle}{\sqrt{2} \langle \tilde{q} k \rangle \langle q k \rangle [q \tilde{q}]} \\
 &= - \frac{\langle k | \not{q} \not{\tilde{q}} \gamma_{\mu} | k \rangle + \langle k | \not{\tilde{q}} \not{q} \gamma_{\mu} | k \rangle}{\sqrt{2} \langle \tilde{q} k \rangle \langle q k \rangle [q \tilde{q}]} = \frac{2 (q \cdot \tilde{q}) \langle k | \gamma_{\mu} | k \rangle}{\sqrt{2} \langle \tilde{q} k \rangle \langle q k \rangle [q \tilde{q}]} \\
 &= - \frac{\langle \tilde{q} q \rangle [q \tilde{q}]}{\sqrt{2} \langle \tilde{q} k \rangle \langle q k \rangle [q \tilde{q}]} \langle k | \gamma_{\mu} | k \rangle = \sqrt{2} \frac{\langle q \tilde{q} \rangle}{\langle \tilde{q} k \rangle \langle q k \rangle} k^{\mu}.
 \end{aligned}$$

———— gauge transform

- polarization sum

$$\sum_{\lambda=\pm} \epsilon_{\mu}^{\lambda}(k, q) \left(\epsilon_{\nu}^{\lambda}(k, q) \right)^{*} = -g_{\mu\nu} + \frac{k_{\mu} q_{\nu} + k_{\nu} q_{\mu}}{k \cdot q},$$

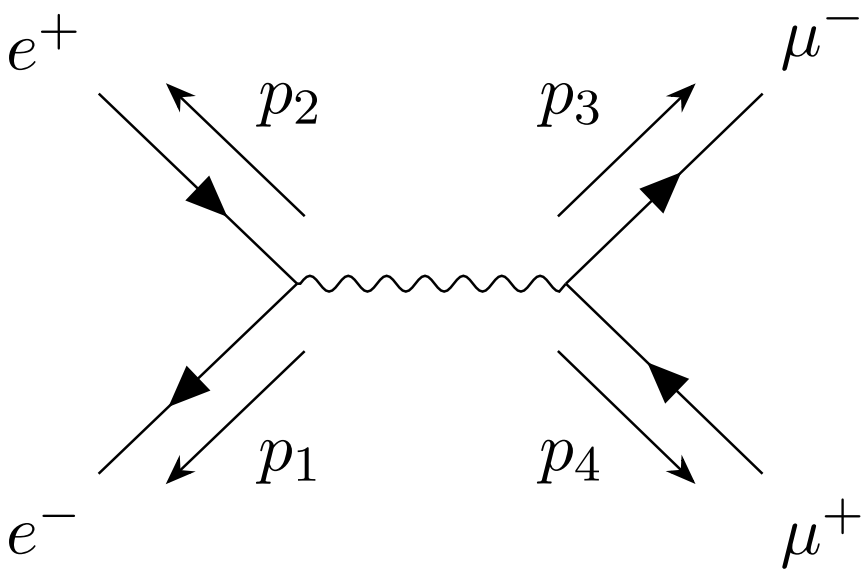
less trivial to prove

└──────────┬──────────┘
numerator of propagator in the axial/physical gauge

(unpolarised) Electron-Muon Scattering
 $e^+ e^- \rightarrow \mu^+ \mu^-$

all outgoing momenta

$p_1 + p_2 + p_3 + p_4 = 0.$



massless approximation, $p_i^2 = 0$

$s_{ij} \equiv (p_i + p_j)^2 = 2p_i \cdot p_j, \quad q^2 \equiv (p_1 + p_2)^2 = s_{12} = s_{34}.$

$$i\mathcal{M} = \bar{u}(1)_{h_1} (ie\gamma^\mu) v(2)_{h_2} \left(-\frac{ig_{\mu\nu}}{q^2}\right) \bar{u}(3)_{h_3} (ie\gamma^\nu) v(4)_{h_4} = i\mathcal{M} \left(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}\right) = e^2 a_4 \left(h_1 \ h_2 \ h_3 \ h_4\right)$$

$$a_4(+ - + -) = i \frac{[1 | \gamma_\mu | 2] [3 | \gamma^\mu | 4]}{s_{12}} \stackrel{(2.20)}{=} 2i \frac{[13] \langle 42 \rangle \langle 34 \rangle}{[12] \langle 21 \rangle \langle 34 \rangle} = 2i \frac{[1 | \cancel{p_3} | 4] \langle 42 \rangle}{[12] \langle 21 \rangle \langle 34 \rangle} = 2i \frac{[1 | \cancel{p_2} | 4] \langle 24 \rangle}{[12] \langle 21 \rangle \langle 34 \rangle} = -2i \frac{\langle 24 \rangle^2}{\langle 12 \rangle \langle 34 \rangle},$$

$$|a_4(+ - + -)|^2 = a_4(+ - + -) a_4^*(+ - + -) = \left(-2i \frac{\langle 24 \rangle^2}{\langle 12 \rangle \langle 34 \rangle}\right) \left(2i \frac{[42]^2}{[21] [43]}\right) = 4 \frac{s_{24}^2}{s_{12} s_{34}}$$

parity

charge conjugation

$\langle ij \rangle \leftrightarrow [ji],$
fermion $\leftrightarrow$ anti-fermion.

(unpolarised) Electron-Muon Scattering $e^+ e^- \rightarrow \mu^+ \mu^-$

$$a_4(+ - - +) = a_4(+ - + -) \Big|_{\substack{3 \leftrightarrow 4 \\ (\text{ch. conj.})}} = -2i \frac{\langle 23 \rangle^2}{\langle 12 \rangle \langle 43 \rangle},$$

$$a_4(- + - +) = a_4(+ - + -) \Big|_{\substack{\langle ij \rangle \leftrightarrow [ji] \\ (\text{parity})}} = -2i \frac{[42]^2}{[12] [34]},$$

$$a_4(- + + -) = a_4(+ - - +) \Big|_{\substack{\langle ij \rangle \leftrightarrow [ji] \\ (\text{parity})}} = -2i \frac{[23]^2}{[12] [43]}.$$

● Unpolarised Squared Amplitude

Incoherent summation (vanishing interference terms)

$$\begin{aligned} \sum_{\text{helicity}} |\mathcal{M}|^2 &= e^4 \left\{ |a_4(+ - + -)|^2 + |a_4(+ - - +)|^2 + |a_4(- + - +)|^2 + |a_4(- + + -)|^2 \right\} \\ &= 8e^4 \frac{s_{24}^2 + s_{23}^2}{s_{12}s_{34}} = 8(4\pi\alpha)^2 \frac{t^2 + u^2}{s^2}. \end{aligned}$$

(polarised) Compton Scattering

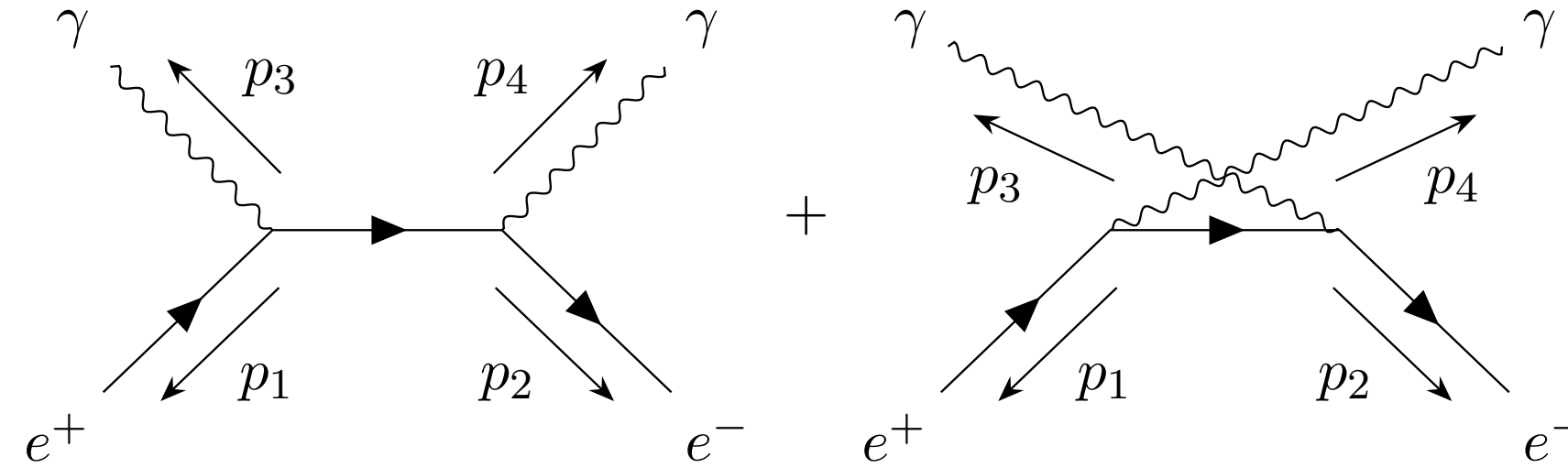
$$e^+_+ e^-_- \rightarrow \gamma\gamma$$

all outgoing momenta

$$p_1 + p_2 + p_3 + p_4 = 0.$$

Right-handed positron

Left-handed electron



$$i\mathcal{M} = (ie)^2 \left\langle 2 \left| \left\{ \not{\epsilon}_4 \frac{i \not{p}_{24}}{s_{24}} \not{\epsilon}_3 + \not{\epsilon}_3 \frac{i \not{p}_{23}}{s_{23}} \not{\epsilon}_4 \right\} \right| 1 \right]$$

There are four possible choices for the photon polarizations, but it is easy to see that both configurations in which the photons have the same helicity lead to a vanishing contribution. For

- the case $\gamma_+ \gamma_+$ we choose the reference momentum $r_i = p_2, i = 3, 4$ in both polarization vectors.

$$\epsilon_{3+}^\mu = \frac{\langle r_3 | \gamma^\mu | 3 \rangle}{\sqrt{2} \langle r_3 3 \rangle} \Big|_{r_3=p_2}, \quad \epsilon_{4+}^\mu = \frac{\langle r_4 | \gamma^\mu | 4 \rangle}{\sqrt{2} \langle r_4 4 \rangle} \Big|_{r_4=p_2}$$

$$\not{\epsilon}_{3+} = \frac{\sqrt{2}}{\langle 23 \rangle} (|2\rangle [3| + |3\rangle \langle 2|), \quad \not{\epsilon}_{4+} = \frac{\sqrt{2}}{\langle 24 \rangle} (|2\rangle [4| + |4\rangle \langle 2|)$$

$$\begin{cases} \langle 2 | \not{\epsilon}_{3+} \sim \langle 22 \rangle [3| = 0, \\ \langle 2 | \not{\epsilon}_{4+} \sim \langle 22 \rangle [4| = 0, \end{cases} \Rightarrow i\mathcal{M}(1^+, 2^-, 3^+, 4^+) = 0.$$

$$i\mathcal{M} = (ie)^2 \left\langle 2 \left| \left\{ \not{\epsilon}_4 \frac{i\not{p}_{24}}{s_{24}} \not{\epsilon}_3 + \not{\epsilon}_3 \frac{i\not{p}_{23}}{s_{23}} \not{\epsilon}_4 \right\} \right| 1 \right]$$

- the case $\gamma_- \gamma_-$ we choose $r_i = p_1$, $i = 3, 4$

$$\epsilon_{3-}^\mu = \frac{\langle 3 | \gamma^\mu | r_3 \rangle}{\sqrt{2} \langle 3 r_3 \rangle} \Big|_{r_3=p_1}, \quad \epsilon_{4-}^\mu = \frac{\langle 4 | \gamma^\mu | r_4 \rangle}{\sqrt{2} \langle 4 r_4 \rangle} \Big|_{r_4=p_1},$$

$$\not{\epsilon}_{3-} = \frac{\sqrt{2}}{[31]} (|3\rangle [1| + |1\rangle \langle 3|), \quad \not{\epsilon}_{4-} = \frac{\sqrt{2}}{[41]} (|4\rangle [1| + |1\rangle \langle 4|)$$

$$\begin{cases} \not{\epsilon}_{3-} |1\rangle \sim [11] |3\rangle = 0, \\ \not{\epsilon}_{4-} |1\rangle \sim [11] |4\rangle = 0, \end{cases} \Rightarrow i\mathcal{M}(1^+, 2^-, 3^-, 4^-) = 0$$

Important:

Before moving to the other helicity configurations, notice that all reference momenta have been already chosen for all photon polarisation states -- they must hold for the whole calculation

(polarised) Compton Scattering

$$e_+^+ e_-^- \rightarrow \gamma\gamma$$

$$i\mathcal{M} = (ie)^2 \left\langle 2 \left| \left\{ \not{\epsilon}_4 \frac{i\not{p}_{24}}{s_{24}} \not{\epsilon}_3 + \not{\epsilon}_3 \frac{i\not{p}_{23}}{s_{23}} \not{\epsilon}_4 \right\} \right| 1 \right\rangle$$

- the case $\gamma_+ \gamma_-$

$$\epsilon_{3+}^\mu = \frac{\langle r_3 | \gamma^\mu | 3 \rangle}{\sqrt{2} \langle r_3 3 \rangle} \Big|_{r_3=p_2} \quad \epsilon_{4-}^\mu = \frac{\langle 4 | \gamma^\mu | r_4 \rangle}{\sqrt{2} \langle 4 r_4 \rangle} \Big|_{r_4=p_1} \quad \not{\epsilon}_{3+} = \frac{\sqrt{2}}{\langle 23 \rangle} (|2\rangle [3| + |3\rangle \langle 2|), \quad \not{\epsilon}_{4-} = \frac{\sqrt{2}}{[41]} (|4\rangle [1| + |1\rangle \langle 4|)$$

$$\not{\epsilon}_{4-} |1\rangle \sim (|4\rangle [1| + |1\rangle \langle 4|) |1\rangle = 0 \quad \Rightarrow \quad \text{Therefore the second diagram does not contribute}$$

$$i\mathcal{M}(1^+, 2^-, 3^+, 4^-) = -e^2 \left\langle 2 \left| \not{\epsilon}_{4-} \frac{i\not{p}_{24}}{s_{24}} \not{\epsilon}_{3+} \right| 1 \right\rangle = -2ie^2 \frac{\langle 24 \rangle [1| (|2\rangle \langle 2| + |4\rangle \langle 4|) |2\rangle [31]}{[41] \langle 23 \rangle \langle 24 \rangle [42]} = -2ie^2 \frac{\langle 24 \rangle [13]}{\langle 23 \rangle [24]} = 2ie^2 \frac{\langle 24 \rangle^2}{\langle 13 \rangle \langle 23 \rangle}.$$

- the case $\gamma_- \gamma_+$

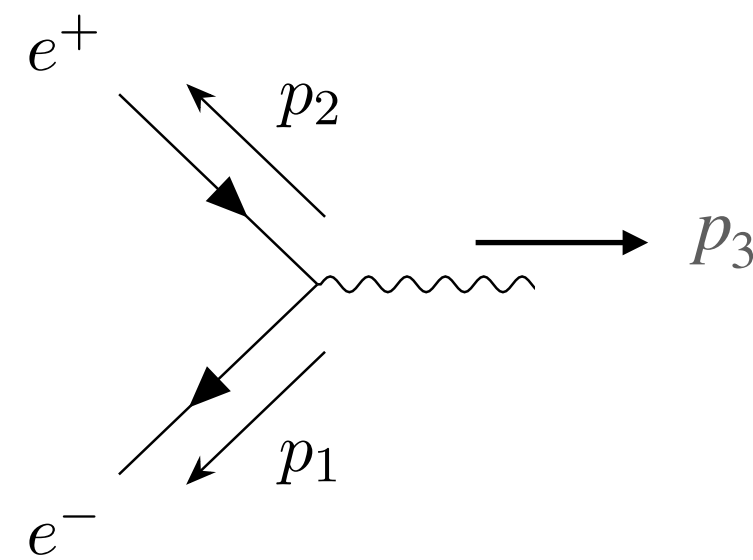
is left for you

3-point Amplitude

$$e^+_+ e^-_- \rightarrow \gamma$$

• in QED

$$p_1 + p_2 + p_3 = 0$$



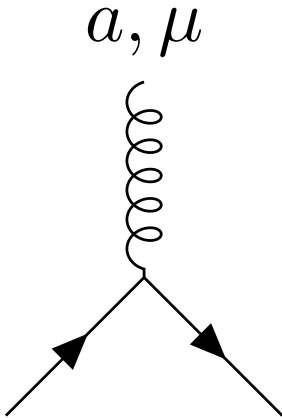
$$\begin{aligned}
 i\mathcal{M}(1_f^+, 2_{\bar{f}}^-, 3_\gamma^+) &= (-ie) \langle 2 \gamma^\mu 1 \rangle \frac{1}{\sqrt{2}} \frac{\langle r \gamma_\mu 3 \rangle}{\langle r 3 \rangle} \\
 &= -i\sqrt{2}e \frac{[13] \langle r 2 \rangle}{\langle r 3 \rangle} \\
 &= -i\sqrt{2}e \frac{[13] \langle r 2 \rangle [21]}{\langle r 3 \rangle [21]} \\
 &= -i\sqrt{2}e \frac{[13] \langle r \not{2} 1 \rangle}{\langle r 3 \rangle [21]} \\
 &= i\sqrt{2}e \frac{[13] \langle r (\not{1} + \not{3}) 1 \rangle}{\langle r 3 \rangle [21]} \\
 &= i\sqrt{2}e \frac{[13] \langle r 3 \rangle [31]}{\langle r 3 \rangle [21]} \\
 &= i\sqrt{2}e \frac{[31]^2}{[12]}.
 \end{aligned}$$

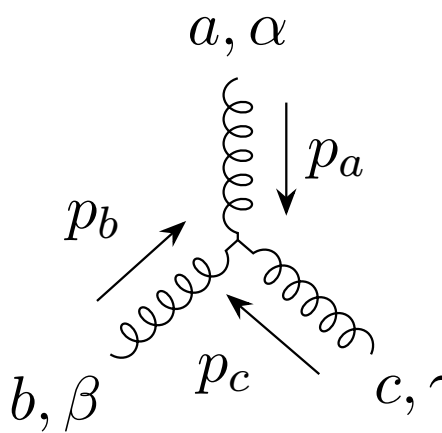
From QED to Color Ordered QCD

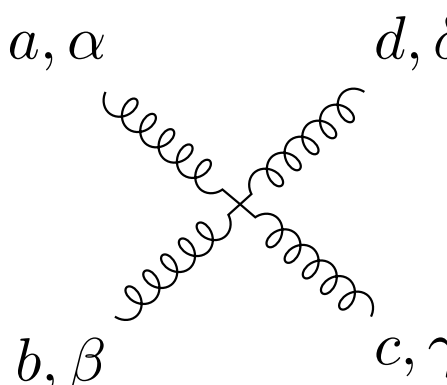
● Feynman Rules of QCD: color and kinematics

gluon $a, \mu \text{ } \text{-----} \text{ } b, \nu = \delta^{ab} G_{\gamma, \mu \nu}.$

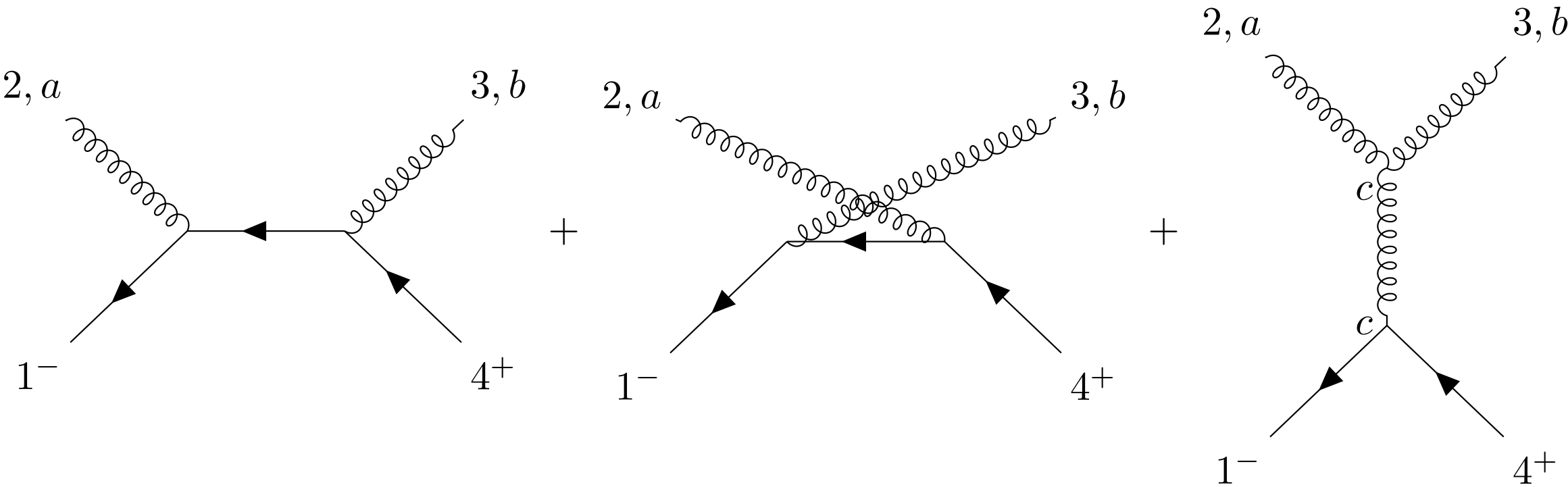
quark $a \text{ } \text{---} \text{ } b = \delta^{ab} G_F.$


 $= i g_s t^a \gamma^\mu.$


 $= g_s f_{abc} [g_{\alpha \beta} (p_a - p_b)_\gamma + g_{\beta \gamma} (p_b - p_c)_\alpha + g_{\gamma \alpha} (p_c - p_a)_\beta].$


 $= -i g_s^2 [f^{abe} f^{cde} (g^{\alpha \gamma} g^{\beta \delta} - g^{\alpha \delta} g^{\beta \gamma}) + f^{ace} f^{bde} (g^{\alpha \beta} g^{\gamma \delta} - g^{\alpha \delta} g^{\beta \gamma}) + f^{ade} f^{bce} (g^{\alpha \beta} g^{\gamma \delta} - g^{\alpha \gamma} g^{\beta \delta})]$

● Example-1 $q\bar{q} \rightarrow gg$



the color coefficients of each diagram: $t^a t^b$ $t^b t^a$ $-g f^{abc} t^c = (ig)(if^{abc} t^c) = ig(t^a t^b - t^b t^a)$

the amplitude structure: $i\mathcal{M} = i\mathbf{M}(1234)T^aT^b + i\mathbf{M}(1324)T^bT^a$

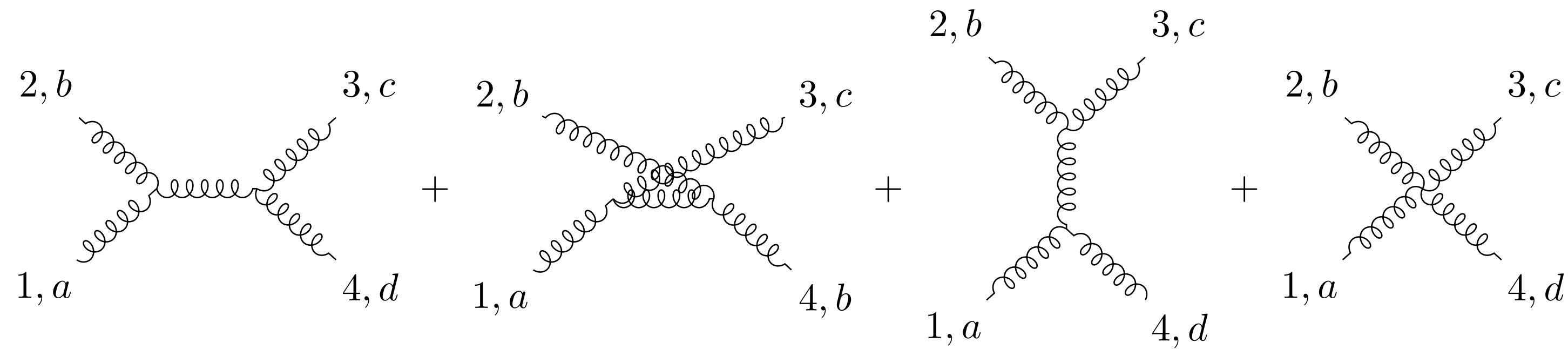


$T^a = \sqrt{2}t^a$ $\text{Tr}(T^a T^b) = \delta^{ab}$

From QED to Color Ordered QCD

● Example-2

$gg \rightarrow gg$



$$\begin{aligned} i\mathcal{M} = & i\mathbf{M}(1234) \cdot \text{tr}[T^a T^b T^c T^d] + i\mathbf{M}(1243) \cdot \text{tr}[T^a T^b T^d T^c] \\ & + i\mathbf{M}(1324) \cdot \text{tr}[T^a T^c T^b T^d] + i\mathbf{M}(1342) \cdot \text{tr}[T^a T^c T^d T^b] \\ & + i\mathbf{M}(1423) \cdot \text{tr}[T^a T^d T^b T^c] + i\mathbf{M}(1432) \cdot \text{tr}[T^a T^d T^c T^b] \end{aligned}$$

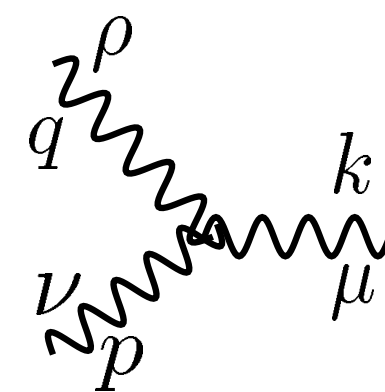
From QED to Color Ordered QCD

● Color Ordered Partial Amplitudes

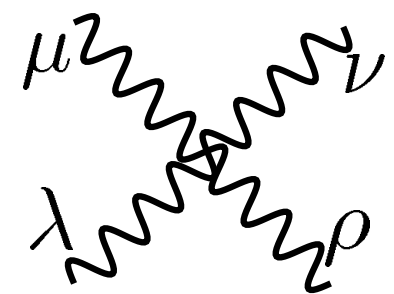
$$iM_{QCD} = \sum_j i\mathbf{M}_j^{(\text{color ordered})} \times (\text{color factor})_j$$

● Color Ordered Feynman Rules of QCD

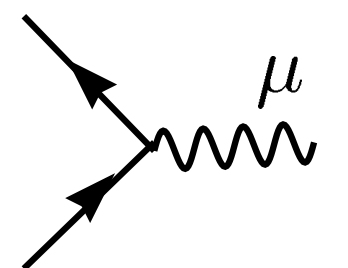
Modified (simpler) Feynman Rules



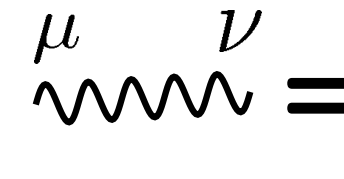
$$= \frac{i}{\sqrt{2}} \left(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho \right)$$



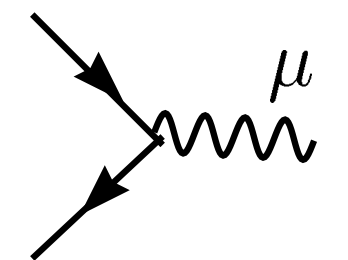
$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$



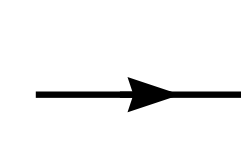
$$= \frac{i}{\sqrt{2}}\gamma_\mu$$



$$= -i\frac{\eta_{\mu\nu}}{p^2}$$

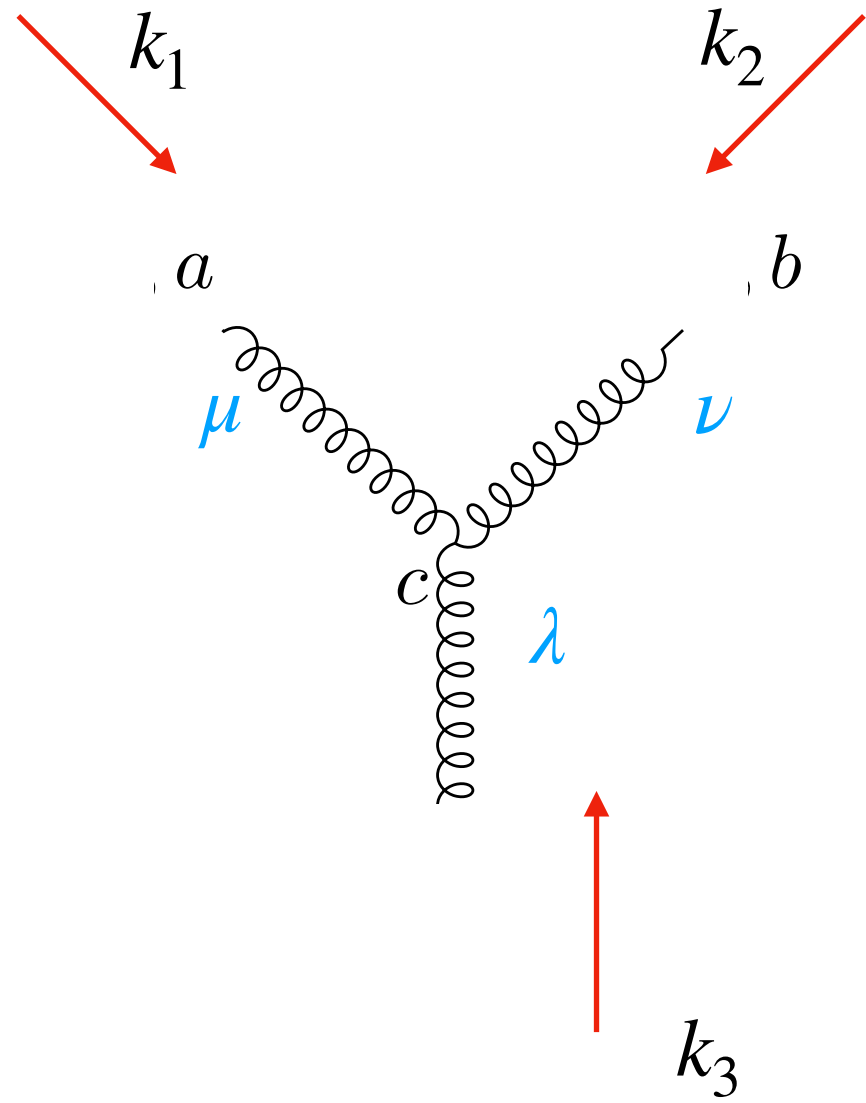


$$= -\frac{i}{\sqrt{2}}\gamma_\mu$$



$$= \frac{i}{\not{p}}$$

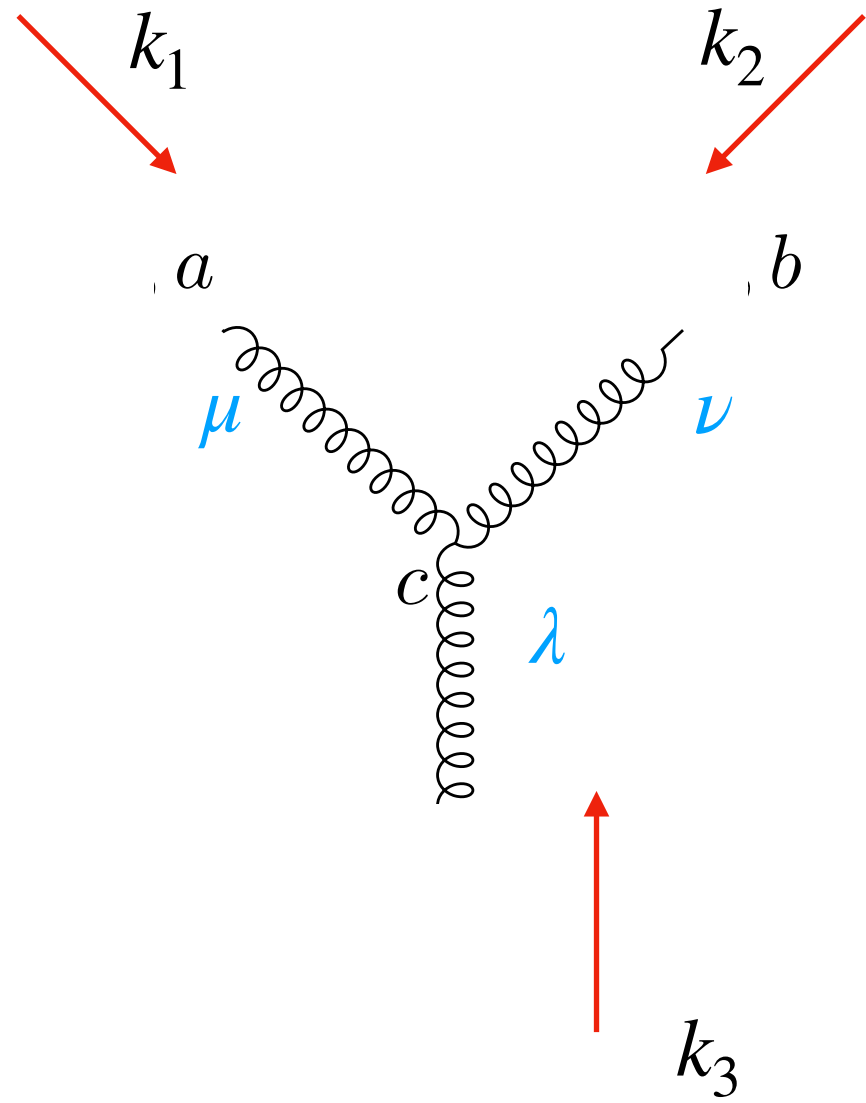
3-gluon Amplitude



$$\left(\frac{i}{\sqrt{2}}\right) \epsilon(1)^\mu \epsilon(2)^\nu \epsilon(3)^\lambda \left[g_{\mu\nu} (k_1 - k_2)_\lambda + g_{\nu\lambda} (k_2 - k_3)_\mu + g_{\lambda\mu} (k_3 - k_1)_\nu \right]$$

$$= \left(\frac{i}{\sqrt{2}}\right) \left[\underbrace{\epsilon(1) \cdot \epsilon(2) (k_1 - k_2) \cdot \epsilon(3)}_{\text{blue}} + \underbrace{\epsilon(2) \cdot \epsilon(3) (k_2 - k_3) \cdot \epsilon(1)}_{\text{green}} + \underbrace{\epsilon(3) \cdot \epsilon(1) (k_3 - k_1) \cdot \epsilon(2)}_{\text{red}} \right]$$

3-gluon Amplitude



$$\left(\frac{i}{\sqrt{2}}\right) \epsilon(1)^\mu \epsilon(2)^\nu \epsilon(3)^\lambda \left[g_{\mu\nu} (k_1 - k_2)_\lambda + g_{\nu\lambda} (k_2 - k_3)_\mu + g_{\lambda\mu} (k_3 - k_1)_\nu \right]$$

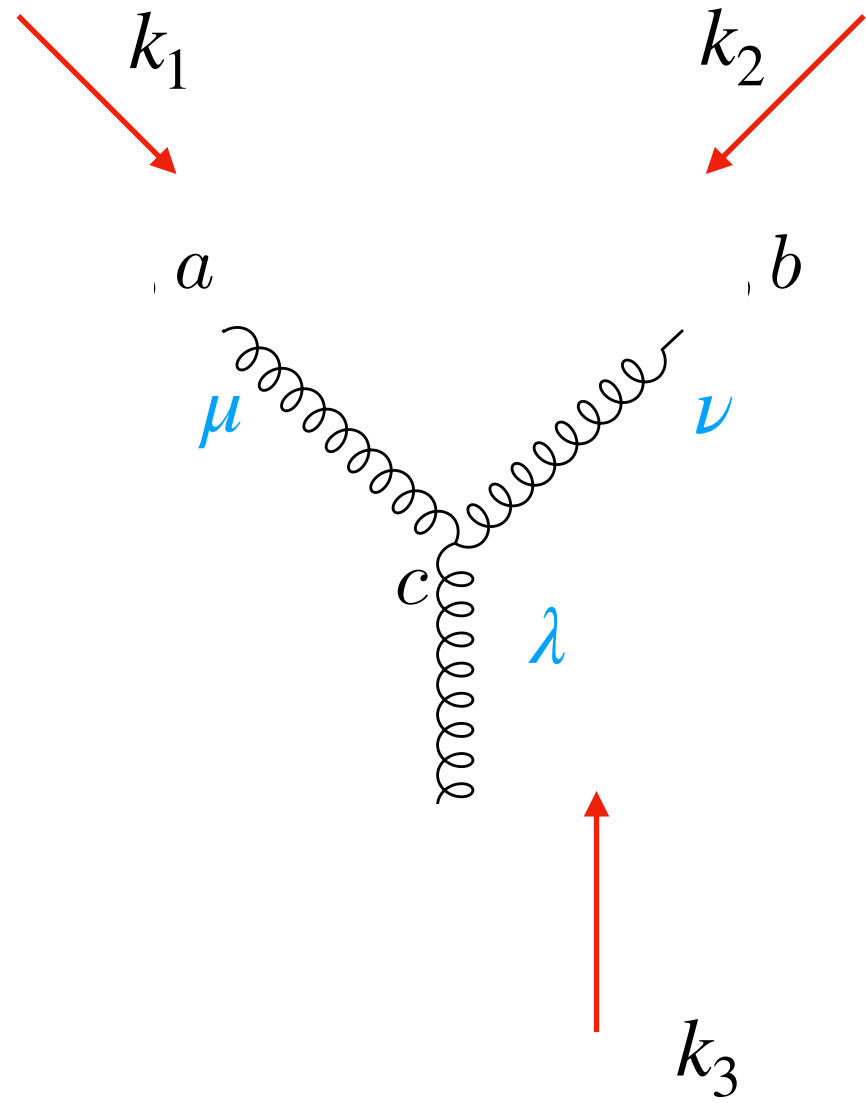
$$= \left(\frac{i}{\sqrt{2}}\right) \left[\underbrace{\epsilon(1) \cdot \epsilon(2) (k_1 - k_2) \cdot \epsilon(3)}_{\text{blue}} + \underbrace{\epsilon(2) \cdot \epsilon(3) (k_2 - k_3) \cdot \epsilon(1)}_{\text{green}} + \underbrace{\epsilon(3) \cdot \epsilon(1) (k_3 - k_1) \cdot \epsilon(2)}_{\text{red}} \right]$$

$$\epsilon_{-}(1) \cdot \epsilon_{-}(2) = \frac{(-1)}{\sqrt{2}} \frac{[r_1 | \gamma^\mu | 1\rangle}{[r_1 1]} \frac{(-1)}{\sqrt{2}} \frac{[r_2 | \gamma^\mu | 2\rangle}{[r_2 2]} = \frac{1}{2} \frac{2 [r_1 r_2] \langle 21 \rangle}{[r_1 1][r_2 2]} = \frac{[r_1 r_2] \langle 21 \rangle}{[r_1 1][r_2 2]}$$

$$(k_1 - k_2) \cdot \epsilon_{+}(3) = (k_1 - k_2)^\lambda \frac{1}{\sqrt{2}} \frac{\langle r_3 | \gamma^\lambda | 3 \rangle}{\langle r_3 3 \rangle} = \frac{1}{\sqrt{2}} \frac{\langle r_3 | (k_1 - k_2) | 3 \rangle}{\langle r_3 3 \rangle}$$

$$\epsilon_{-}(1) \cdot \epsilon_{-}(2) (k_1 - k_2) \cdot \epsilon_{+}(3) = \frac{1}{\sqrt{2}} \frac{[r_1 r_2] \langle 21 \rangle}{[r_1 1][r_2 2]} \frac{\langle r_3 | (k_1 - k_2) | 3 \rangle}{\langle r_3 3 \rangle}$$

3-gluon Amplitude



$$\left(\frac{i}{\sqrt{2}}\right) \epsilon(1)^\mu \epsilon(2)^\nu \epsilon(3)^\lambda \left[g_{\mu\nu} (k_1 - k_2)_\lambda + g_{\nu\lambda} (k_2 - k_3)_\mu + g_{\lambda\mu} (k_3 - k_1)_\nu \right]$$

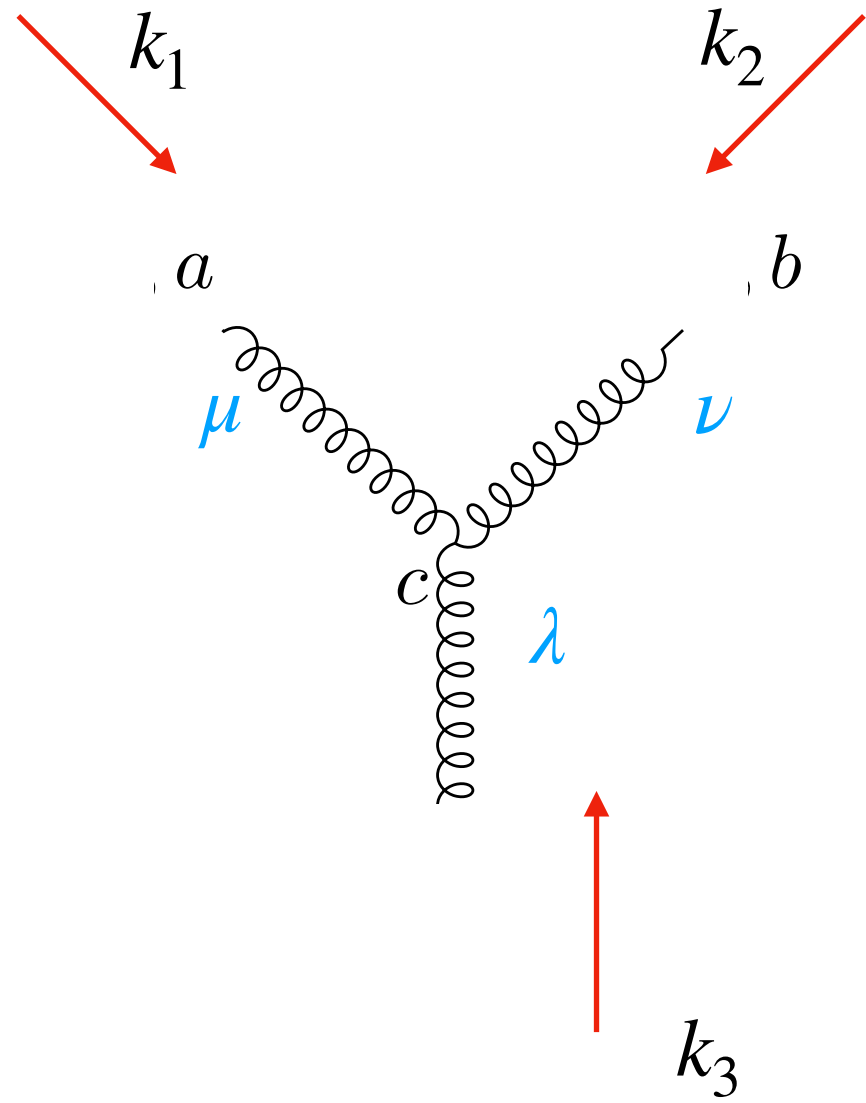
$$= \left(\frac{i}{\sqrt{2}}\right) \left[\underbrace{\epsilon(1) \cdot \epsilon(2) (k_1 - k_2) \cdot \epsilon(3)}_{\text{blue}} + \underbrace{\epsilon(2) \cdot \epsilon(3) (k_2 - k_3) \cdot \epsilon(1)}_{\text{green}} + \underbrace{\epsilon(3) \cdot \epsilon(1) (k_3 - k_1) \cdot \epsilon(2)}_{\text{red}} \right]$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) = \frac{(-1)}{\sqrt{2}} \frac{[r_2 | \gamma^\mu | 2\rangle}{[r_2 2]} \frac{(1)}{\sqrt{2}} \frac{\langle r_3 | \gamma^\mu | 3\rangle}{\langle r_3 3\rangle} = (-1) \frac{[r_2 3] \langle r_3 2\rangle}{[r_2 2] \langle r_3 3\rangle}$$

$$(k_2 - k_3) \cdot \epsilon_{-}(1) = (k_2 - k_3)^\lambda \frac{(-1)}{\sqrt{2}} \frac{[r_1 | \gamma^\lambda | 1\rangle}{[r_1 1]} = \frac{-1}{\sqrt{2}} \frac{[r_1 | (k_2 - k_3) | 1\rangle}{[r_1 1]}$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) (k_2 - k_3) \cdot \epsilon_{-}(1) = \frac{1}{\sqrt{2}} \frac{[r_2 3] \langle r_3 2\rangle}{[r_2 2] \langle r_3 3\rangle} \frac{[r_1 | (k_2 - k_3) | 1\rangle}{[r_1 1]}$$

3-gluon Amplitude



$$\left(\frac{i}{\sqrt{2}}\right) \epsilon(1)^\mu \epsilon(2)^\nu \epsilon(3)^\lambda \left[g_{\mu\nu} (k_1 - k_2)_\lambda + g_{\nu\lambda} (k_2 - k_3)_\mu + g_{\lambda\mu} (k_3 - k_1)_\nu \right]$$

$$= \left(\frac{i}{\sqrt{2}}\right) \left[\underbrace{\epsilon(1) \cdot \epsilon(2) (k_1 - k_2) \cdot \epsilon(3)}_{\text{blue}} + \underbrace{\epsilon(2) \cdot \epsilon(3) (k_2 - k_3) \cdot \epsilon(1)}_{\text{green}} + \underbrace{\epsilon(3) \cdot \epsilon(1) (k_3 - k_1) \cdot \epsilon(2)}_{\text{red}} \right]$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) = \frac{(-1)}{\sqrt{2}} \frac{[r_2 | \gamma^\mu | 2\rangle}{[r_2 2]} \frac{(1)}{\sqrt{2}} \frac{\langle r_3 | \gamma^\mu | 3\rangle}{\langle r_3 3\rangle} = (-1) \frac{[r_2 3] \langle r_3 2\rangle}{[r_2 2] \langle r_3 3\rangle}$$

$$(k_2 - k_3) \cdot \epsilon_{-}(1) = (k_2 - k_3)^\lambda \frac{(-1)}{\sqrt{2}} \frac{[r_1 | \gamma^\lambda | 1\rangle}{[r_1 1]} = \frac{-1}{\sqrt{2}} \frac{[r_1 | (k_2 - k_3) | 1\rangle}{[r_1 1]}$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) (k_2 - k_3) \cdot \epsilon_{-}(1) = \frac{1}{\sqrt{2}} \frac{[r_2 3] \langle r_3 2\rangle}{[r_2 2] \langle r_3 3\rangle} \frac{[r_1 | (k_2 - k_3) | 1\rangle}{[r_1 1]}$$

$$\epsilon_{+}(3) \cdot \epsilon_{-}(1) (k_3 - k_1) \cdot \epsilon_{-}(2) = \frac{1}{\sqrt{2}} \frac{[r_1 3] \langle r_3 1\rangle}{[r_1 1] \langle r_3 3\rangle} \frac{[r_2 | (k_3 - k_1) | 2\rangle}{[r_2 2]}$$

3-gluon Amplitude

$$\epsilon_{-}(1) \cdot \epsilon_{-}(2) (k_1 - k_2) \cdot \epsilon_{+}(3) = \frac{1}{\sqrt{2}} \frac{[r_1 r_2] \langle 21 \rangle}{[r_1 1][r_2 2]} \frac{\langle r_3 | (k_1 - k_2) | 3 \rangle}{\langle r_3 3 \rangle}$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) (k_2 - k_3) \cdot \epsilon_{-}(1) = \frac{1}{\sqrt{2}} \frac{[r_2 3] \langle r_3 2 \rangle}{[r_2 2] \langle r_3 3 \rangle} \frac{[r_1 | (k_2 - k_3) | 1 \rangle}{[r_1 1]}$$

$$\epsilon_{+}(3) \cdot \epsilon_{-}(1) (k_3 - k_1) \cdot \epsilon_{-}(2) = \frac{1}{\sqrt{2}} \frac{[r_1 3] \langle r_3 1 \rangle}{[r_1 1] \langle r_3 3 \rangle} \frac{[r_2 | (k_3 - k_1) | 2 \rangle}{[r_2 2]}$$

$$r_1 = k_2, r_2 = k_1, r_3 = ? ,$$

$$\epsilon_{-}(2) \cdot \epsilon_{+}(3) (k_2 - k_3) \cdot \epsilon_{-}(1) = \frac{1}{\sqrt{2}} \frac{[13] \langle r_3 2 \rangle}{[12] \langle r_3 3 \rangle} \frac{[2 | (-k_3) | 1 \rangle}{[21]} \rightarrow 0$$

$$\epsilon_{+}(3) \cdot \epsilon_{-}(1) (k_3 - k_1) \cdot \epsilon_{-}(2) = \frac{1}{\sqrt{2}} \frac{[23] \langle r_3 1 \rangle}{[21] \langle r_3 3 \rangle} \frac{[1 | (k_3) | 2 \rangle}{[12]} \rightarrow 0$$

$$\epsilon_{-}(1) \cdot \epsilon_{-}(2) (k_1 - k_2) \cdot \epsilon_{+}(3) = \frac{1}{\sqrt{2}} \frac{[21] \langle 21 \rangle}{[21][12]} \frac{\langle r_3 | (k_1 - k_2) | 3 \rangle}{\langle r_3 3 \rangle} = \frac{1}{\sqrt{2}} \frac{\langle 21 \rangle}{[12]} \frac{\langle r_3 | (k_1 - (-k_1 - k_3)) | 3 \rangle}{\langle r_3 3 \rangle} = \frac{1}{\sqrt{2}} \frac{\langle 21 \rangle}{[12]} \frac{2 \langle r_3 | k_1 | 3 \rangle}{\langle r_3 3 \rangle}$$

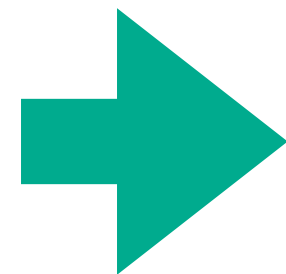
$$= \frac{2}{\sqrt{2}} \frac{\langle 21 \rangle}{[12]} \frac{\langle r_3 | k_1 | 3 \rangle}{\langle r_3 3 \rangle} = \frac{2}{\sqrt{2}} \frac{\langle 21 \rangle}{[12]} \frac{\langle r_3 1 \rangle [13]}{\langle r_3 3 \rangle}$$

$$r_3 = k_2$$

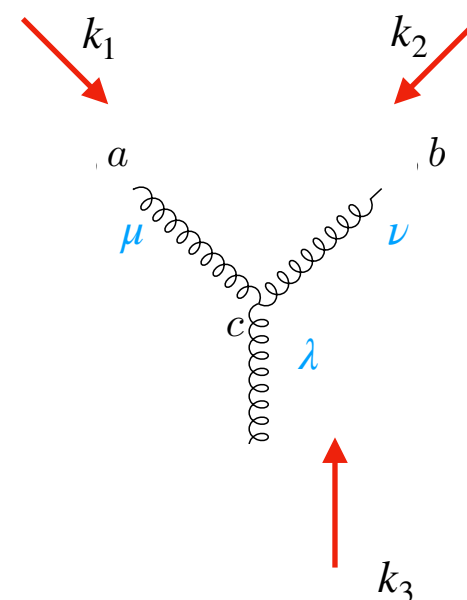
$$\rightarrow \frac{2}{\sqrt{2}} \frac{\langle 21 \rangle \langle 21 \rangle [13]}{[12] \langle 23 \rangle} = \frac{2}{\sqrt{2}} \frac{\langle 21 \rangle \langle 21 \rangle [13] \langle 31 \rangle}{[12] \langle 23 \rangle \langle 31 \rangle} = \frac{2}{\sqrt{2}} \frac{\langle 21 \rangle \langle 21 \rangle (-1) [12] \langle 21 \rangle}{[12] \langle 23 \rangle \langle 31 \rangle} = \frac{2}{\sqrt{2}} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

Multiplying by

$$\left(\frac{i}{\sqrt{2}} \right)$$



$$i\mathcal{M}(1_g^-, 2_g^-, 3_g^+) =$$

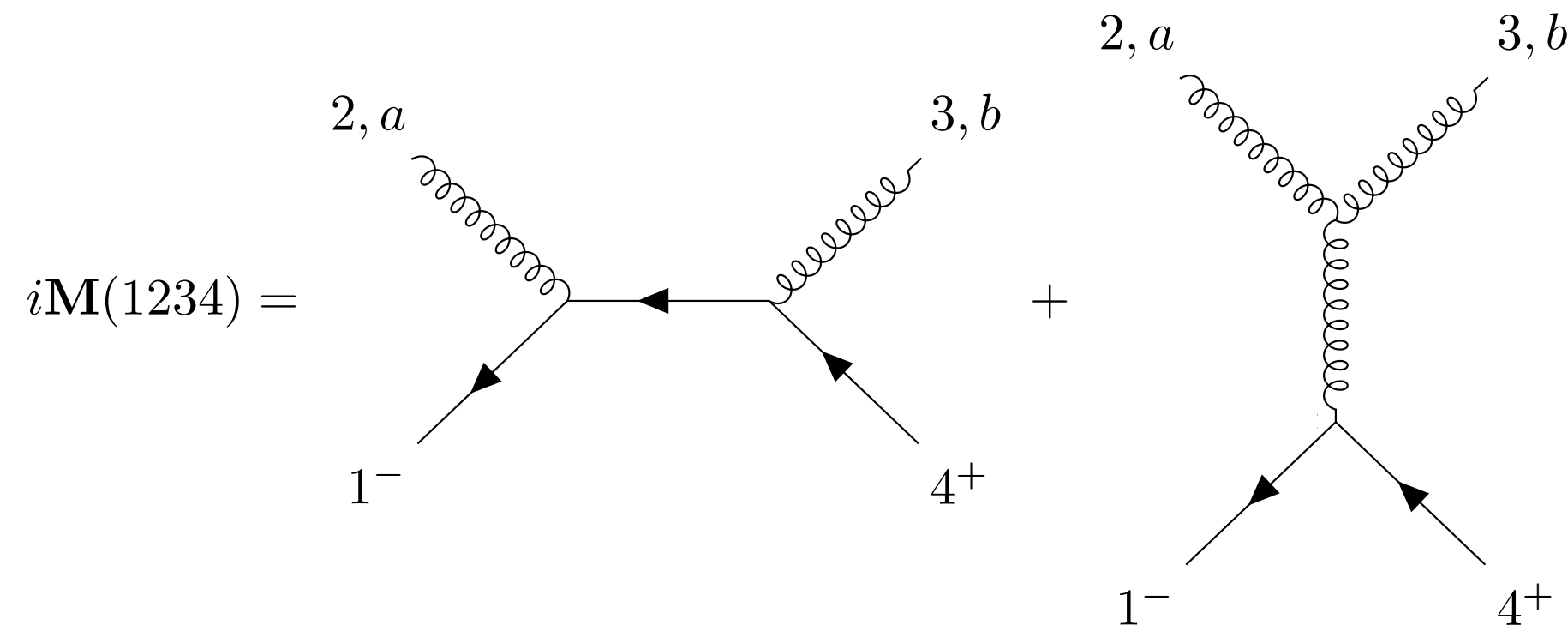


$$= i \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} = i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$$

Color Ordered Amplitudes in QCD

● Example-1

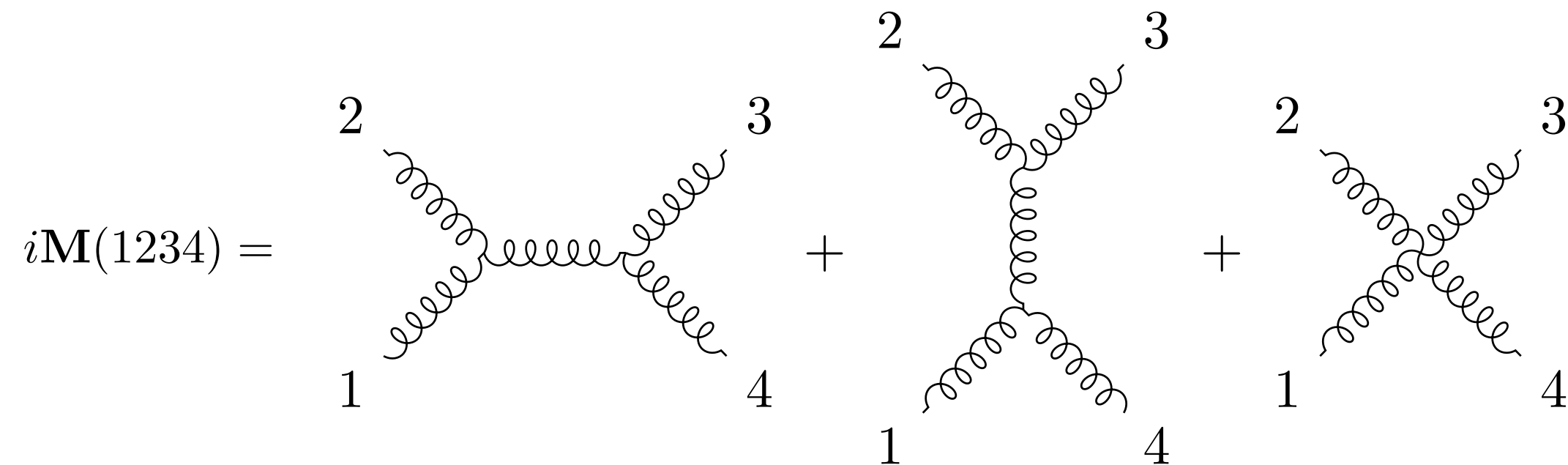
$$q\bar{q} \rightarrow gg$$



$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \left[\langle 1\gamma \cdot \epsilon(2) \frac{i(1+2)}{s_{12}} \gamma \cdot \epsilon(3) 4 \rangle \right. \\ \left. + \frac{-i}{s_{23}} \langle 1\gamma^\lambda 4 \rangle [\epsilon(2) \cdot \epsilon(3)(2-3)_\lambda \right. \\ \left. + \epsilon_\lambda(3)(2 \cdot 3 + 2) \cdot \epsilon(2) + \epsilon_\lambda(2)(-2 \cdot 2 - 3) \cdot \epsilon(3)] \right]$$

● Example-2

$$gg \rightarrow gg$$



$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \left[\frac{-i}{s_{14}} [\epsilon(4) \cdot \epsilon(1)(4-1)^\lambda + \epsilon^\lambda(1)(21+4) \cdot \epsilon(4) + \epsilon^\lambda(4)(-24-1) \cdot \epsilon(1)] \right. \\ \cdot [\epsilon(2) \cdot \epsilon(3)(2-3)_\lambda + \epsilon_\lambda(3)(23+2) \cdot \epsilon(2) + \epsilon_\lambda(2)(-22-3) \cdot \epsilon(3)] \\ \left. + \frac{-i}{s_{12}} [\epsilon(1) \cdot \epsilon(2)(1-2)^\lambda + \epsilon^\lambda(2)(22+1) \cdot \epsilon(1) + \epsilon^\lambda(1)(-21-2) \cdot \epsilon(2)] \right. \\ \cdot [\epsilon(3) \cdot \epsilon(4)(3-4)_\lambda + \epsilon_\lambda(4)(24+3) \cdot \epsilon(3) + \epsilon_\lambda(3)(-23-4) \cdot \epsilon(2)] \\ \left. + (-i)[2\epsilon(1) \cdot \epsilon(3) \epsilon(2) \cdot \epsilon(4) - \epsilon(1) \cdot \epsilon(2) \epsilon(3) \cdot \epsilon(4) - \epsilon(1) \cdot \epsilon(4) \epsilon(2) \cdot \epsilon(3)] \right]$$

Maximally Helicity Violating (MHV) Amplitudes

● n-particle scattering processes

● n-gluon amplitudes

$$gg \rightarrow g_1 \dots g_{n-2}$$

$$i\mathbf{M}(\underbrace{1^+, \dots, n^+}_{\text{same helicity}}) = 0 = i\mathbf{M}(\underbrace{1^-, 2^+, 3^+, \dots, (n-1)^+, n^+}_{\text{one minus}})$$

$$\text{MHV} \quad i\mathbf{M}(g_+(1) \dots g_-(i) \dots g_-(j) \dots g_+(n)) = ig^{n-2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}$$

$$\overline{\text{MHV}} \quad i\mathbf{M}(g_-(1) \dots g_+(i) \dots g_+(j) \dots g_-(n)) = (-1)^n ig^{n-2} \frac{[ij]^4}{[12][23] \dots [(n-1)n][n1]}$$

Parke-Taylor formulas

● quark-gluon amplitudes

$$q\bar{q} \rightarrow g_1 \dots g_{n-2}$$

(depending just on one type of spinor-product)

$$i\mathbf{M}(1_q^-, \underbrace{2^+, 3^+, \dots, (n-1)^+, n_{\bar{q}}^+}_{\text{same helicity}}) = 0.$$

$$\text{MHV} \quad i\mathbf{M}(q_-(1)g_+(2) \dots g_-(i) \dots g_+(n-1)\bar{q}_+(n)) = ig^{n-2} \frac{\langle 1i \rangle^3 \langle ni \rangle}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}$$

$$\overline{\text{MHV}} \quad i\mathbf{M}(q_-(1)g_-(2) \dots g_+(i) \dots g_-(n-1)\bar{q}_+(n)) = (-1)^{n-1} ig^{n-2} \frac{[1i][ni]^3}{[12][23] \dots [(n-1)n][n1]}$$

Maximally Helicity Violating (MHV) Amplitudes

prefactors and coupling dropped

● 3-gluon MHV amplitude

[illegible]

● 4-gluon MHV amplitude

[illegible]

● 5-gluon MHV amplitude

Little Group Transform

● Little Group Transform on Spinors

The Little Group is a subgroup of Lorentz group leaving the momentum invariant:

$$\boxed{|k\rangle \rightarrow z |k\rangle \quad \& \quad |k] \rightarrow z^{-1} |k]} \quad \text{leave} \quad k^\mu \equiv \frac{\langle k | \gamma^\mu | k]}{2} \quad \text{invariant}$$

● Little Group Transform on Polarisations

$$\epsilon_\mu^-(k, q) \equiv -\frac{[q | \gamma^\mu | k \rangle}{\sqrt{2} [qk]} \rightarrow z^2 \epsilon_\mu^-(k, q) \quad \text{negative helicity polarisation} \quad \epsilon_\mu^+(k, q) \equiv \frac{\langle q | \gamma^\mu | k]}{\sqrt{2} \langle qk \rangle} \rightarrow z^{-2} \epsilon_\mu^+(k, q) \quad \text{negative helicity polarisation}$$

● Little Group Transform on Amplitudes

the rescaling goes as $z_j^{-2h_j}$ with the helicity h_j for the external particle j .

$$\boxed{i \mathcal{M}(1^{h_1}, \dots, n^{h_n}) \rightarrow \prod_{j=1}^n z_j^{-2h_j} i \mathcal{M}(1^{h_1}, \dots, n^{h_n})} \quad \text{Amplitudes' scaling}$$

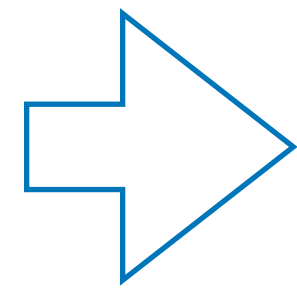
Little Group and 3-particle Amplitudes

● 3-particle Amplitude from Little Group

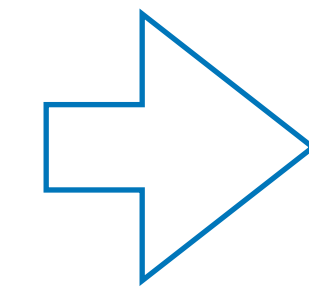
● **ansatz** $i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim \langle 12 \rangle^{n_3} \langle 23 \rangle^{n_1} \langle 31 \rangle^{n_2}$ or $i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim [12]^{n'_3} [23]^{n'_1} [31]^{n'_2}$ $n'_j = -n_j$ by CP

$n_1 + n_2 + n_3 \geq 0$ $n'_1 + n'_2 + n'_3 \geq 0,$

● **LG scaling** $i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim \langle 12 \rangle^{n_3} \langle 23 \rangle^{n_1} \langle 31 \rangle^{n_2} \rightarrow z_1^{n_3+n_2} z_2^{n_3+n_1} z_3^{n_1+n_2} i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \equiv z_1^{-2h_1} z_2^{-2h_2} z_3^{-2h_3} i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3})$



$$\begin{aligned} h_1 &= -\frac{n_3 + n_2}{2} \\ h_2 &= -\frac{n_3 + n_1}{2} \\ h_3 &= -\frac{n_1 + n_2}{2} \end{aligned}$$



$$\begin{aligned} n_1 &= h_1 - h_2 - h_3, \\ n_2 &= h_2 - h_3 - h_1, \\ n_3 &= h_3 - h_1 - h_2. \end{aligned}$$

● final behaviour

$$i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim \begin{cases} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} & h \leq 0, \\ [12]^{-h_3+h_1+h_2} [23]^{-h_1+h_2+h_3} [31]^{-h_2+h_3+h_1} & h \geq 0. \end{cases} \quad h = h_1 + h_2 + h_3$$

Little Group and 3-particle Amplitudes

$$i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim \begin{cases} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} & h \leq 0, \\ [12]^{-h_3+h_1+h_2} [23]^{-h_1+h_2+h_3} [31]^{-h_2+h_3+h_1} & h \geq 0. \end{cases} \quad h = h_1 + h_2 + h_3$$

● fermion-antifermion-boson Amplitudes

$$i\mathcal{M}(1_f^+, 2_{\bar{f}}^+, 3^+) \sim [23][31] \quad i\mathcal{M}(1_f^+, 2_{\bar{f}}^-, 3^+) \sim \frac{[31]^2}{[12]}$$

● 3-gluon Amplitudes

possible helicity configurations: $+++$, $++-$, $--+$ and $---$

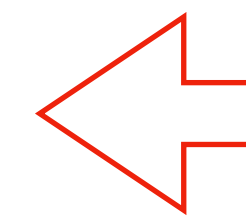
$$i\mathcal{M}(1^{a+}2^{b+}3^{c+}) \sim [12][23][31]$$

$$i\mathcal{M}(1^{a+}2^{b+}3^{c-}) \sim \frac{[12]^3}{[23][31]}$$

$$i\mathcal{M}(1^-2^-,3^-) \sim \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle$$

$$i\mathcal{M}(1^-2^-3^+) \sim \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

3-gluon MHV Amplitude



$$\begin{aligned} h_1 &= h_2 = -1 & h_3 &= 1 \\ h_3 - h_1 - h_2 &= 3 \\ h_1 - h_2 - h_3 &= -1 \\ h_2 - h_3 - h_1 &= -1 \end{aligned}$$

Little Group and 3-particle Amplitudes

$$i\mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) \sim \begin{cases} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} & h \leq 0, \\ [12]^{-h_3+h_1+h_2} [23]^{-h_1+h_2+h_3} [31]^{-h_2+h_3+h_1} & h \geq 0. \end{cases} \quad h = h_1 + h_2 + h_3$$

● fermion-antifermion-boson Amplitudes

$$i\mathcal{M}(1_f^+, 2_{\bar{f}}^+, 3^+) \sim [23][31] \quad i\mathcal{M}(1_f^+, 2_{\bar{f}}^-, 3^+) \sim \frac{[31]^2}{[12]}$$

● 3-gluon Amplitudes

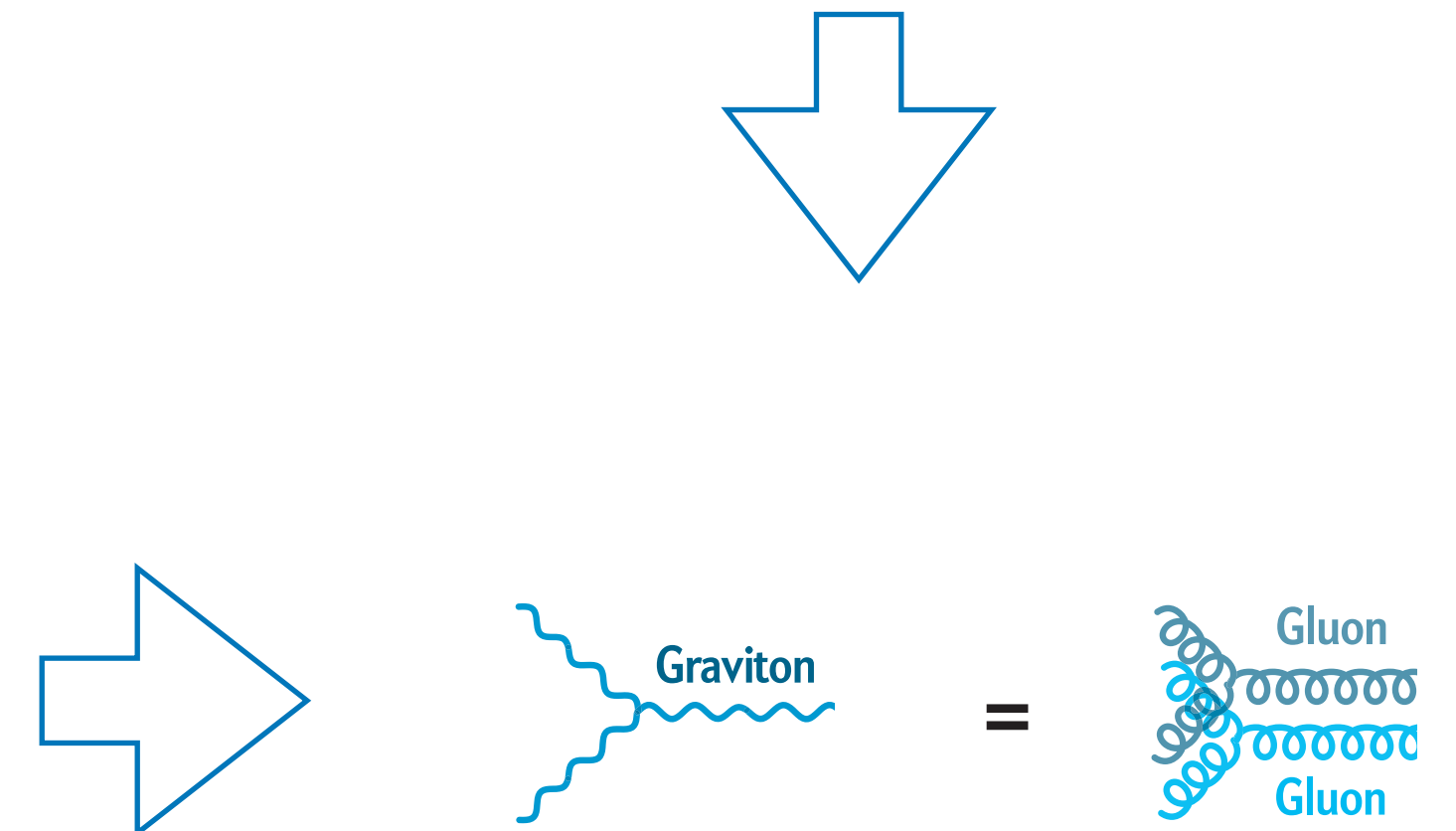
possible helicity configurations: $+++$, $++-$, $--+$ and $---$

$$\begin{aligned} i\mathcal{M}(1^{a+}2^{b+}3^{c+}) &\sim [12][23][31] & iM(1^-2^-,3^-) &\sim \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \\ i\mathcal{M}(1^{a+}2^{b+}3^{c-}) &\sim \frac{[12]^3}{[23][31]} & iM(1^-2^-3^+) &\sim \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \end{aligned}$$

● 3-Graviton Amplitudes (spin-2)

$$\begin{aligned} i\mathcal{M}(1_G^+, 2_G^+, 3_G^+) &\sim [12]^2[23]^2[31]^2 & iM(1_G^- 2_G^-, 3_G^-) &\sim \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2 \\ i\mathcal{M}(1_G^+, 2_G^+, 3_G^-) &\sim \frac{[12]^6}{[23]^2[31]^2} & iM(1_G^- 2_G^- 3_G^+) &\sim \frac{\langle 12 \rangle^6}{\langle 23 \rangle^2 \langle 31 \rangle^2} \end{aligned}$$

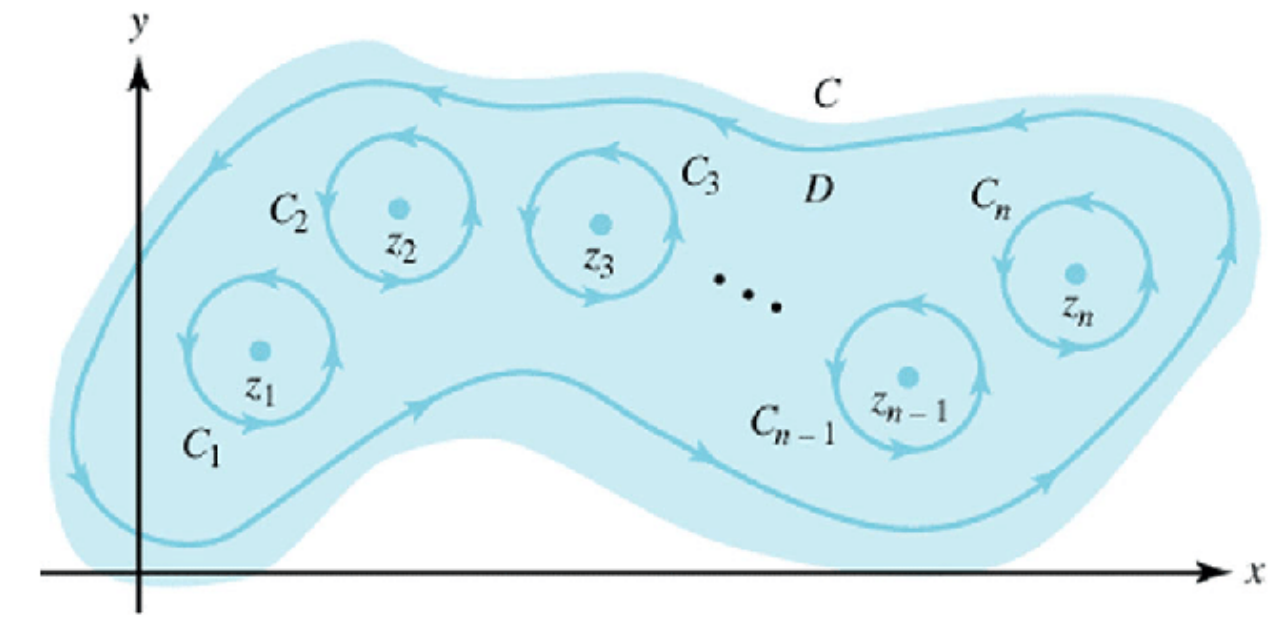
3-graviton MHV Amplitude



Residue Theorem and Partial Fractions Formula

Suppose that the function f is analytic within and on a positively oriented simple closed contour C except for a finite number of isolated singular points $\{z_j, j = 1, 2, \dots, N\}$ interior to C , then

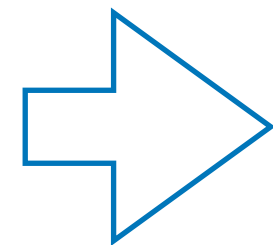
$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^N \operatorname{Res}_{z=z_j} f(z).$$



● Pole at Infinity

$$\frac{1}{2\pi i} \oint_C dz f(z) = \sum_i \operatorname{Res}_{z=z_i} f(z) = -\operatorname{Res}_{z=\infty} f(z),$$

● if $\operatorname{Res}_{z=\infty} f(z) = 0$



$$\operatorname{Res}_{z=0} f(z) = - \sum_i \operatorname{Res}_{z=z_i \neq 0} f(z)$$

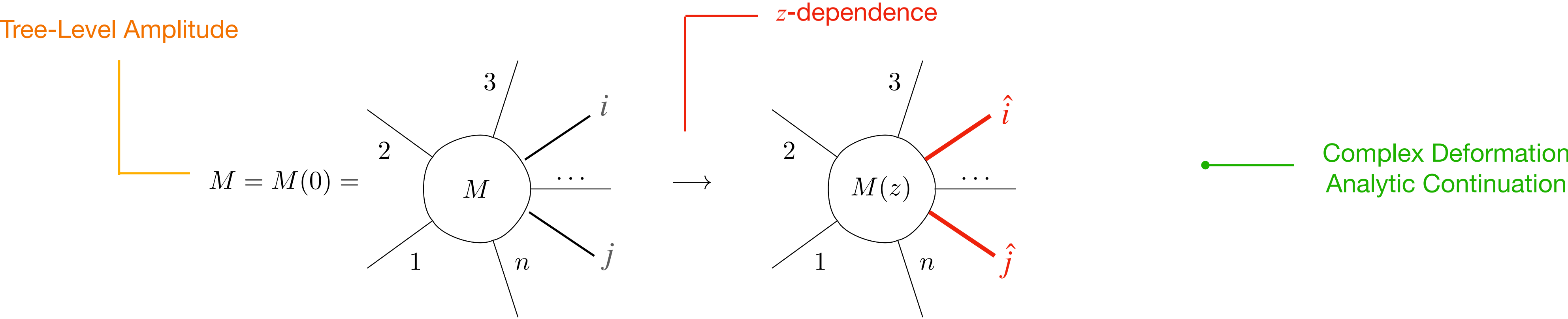
● Example

$$\oint \frac{dz}{z(z - z_1) \cdots (z - z_n)} = 0$$

$$\frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n} = \frac{1}{z_1(z_2 - z_1) \cdots (z_n - z_1)} + \frac{1}{(z_1 - z_2)z_2 \cdots (z_n - z_2)} + \cdots + \frac{1}{(z_1 - z_n) \cdots (z_{n-1} - z_n)z_n}.$$

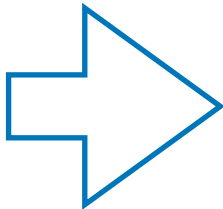
Residue Theorem and Tree-Level Amplitudes

(redefinition) $i\mathcal{M} \equiv M$



Residue Theorem

Let us define $f(z) = \frac{M(z)}{z}$ with $\text{Res}_{z=\infty} f(z) = 0$,

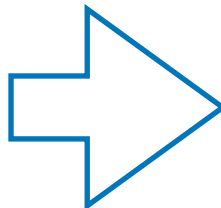


$$M = M(0) = \text{Res}_{z=0} f(z) = - \sum_i \text{Res}_{z=z_i \neq 0} \frac{M(z)}{z_i}.$$

Complex Momenta Deformation by Spinor Shift

Choose two legs i and j

$$\begin{aligned} |\hat{i}\rangle &\equiv |i\rangle \\ |\hat{i}] &\equiv |i] + z |j] \\ |\hat{j}\rangle &\equiv |j\rangle - z |i\rangle \\ |\hat{j}] &\equiv |j] \end{aligned}$$



$$\begin{aligned} p_i^\mu &\rightarrow \hat{p}_i^\mu = \frac{\langle \hat{i} | \gamma^\mu | \hat{i} \rangle}{2} = \frac{\langle i | \gamma^\mu | i \rangle}{2} + z \frac{\langle i | \gamma^\mu | j \rangle}{2} = p_i^\mu + z \epsilon_{ij}^\mu, \\ p_j^\mu &\rightarrow \hat{p}_j^\mu = \frac{\langle \hat{j} | \gamma^\mu | \hat{j} \rangle}{2} = \frac{\langle j | \gamma^\mu | j \rangle}{2} - z \frac{\langle i | \gamma^\mu | j \rangle}{2} = p_j^\mu - z \epsilon_{ij}^\mu \end{aligned}$$

$$\epsilon_{ij}^\mu \equiv \frac{\langle i | \gamma^\mu | j \rangle}{2}$$

[complex momentum]

momentum conservation

$$\hat{p}_i + \hat{p}_j = p_i + p_j$$

masslessness

$$\hat{p}_i^2 = p_i^2 + \epsilon_{ij}^2 + 2z \epsilon_{ij} \cdot p_i = 0$$

Residue Theorem and Tree-Level Amplitudes

Poles We have seen that $f(z)$ has an obvious pole at $z = 0$. The other poles will come from denominators of propagators involved in a factorization of the amplitude of the form,

$$M(z) = \text{diagram} = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 \rightarrow \sum_{a,b} M_L(z) \frac{1}{\hat{Q}_{ab}^2} M_R(z)$$

NO pole in z
pole in z
NO pole in z

relevant for the residue thm

Propagator momentum:

$$Q_{ab}^\mu = \sum_{m=a}^b p_m^\mu \quad \rightarrow \quad \hat{Q}_{ab}^\mu = \sum_{m=a}^b p_m^\mu = p_a^\mu + \cdots + p_{j-1}^\mu + \hat{p}_j^\mu + p_{j+1}^\mu + \cdots + p_b^\mu = Q_{ab}^\mu - z\epsilon_{ij}^\mu,$$

$$\hat{Q}_{ab}^2 = \left(Q_{ab}^\mu - z\epsilon_{ij}^\mu \right)^2 = Q_{ab}^2 - 2z\epsilon_{ij} Q_{ab} = 0 \quad \Rightarrow \quad \frac{1}{\hat{Q}_{ab}^2} = - \frac{1}{\left(z - \frac{Q_{ab}^2}{2Q_{ab} \cdot \epsilon_{ij}} \right) \langle i | \not{Q}_{ab} | j \rangle}$$

simple pole @

$$z_{ab} = \frac{Q_{ab}^2}{2Q_{ab} \cdot \epsilon_{ij}} = \frac{Q_{ab}^2}{\langle i | \not{Q}_{ab} | j \rangle}$$

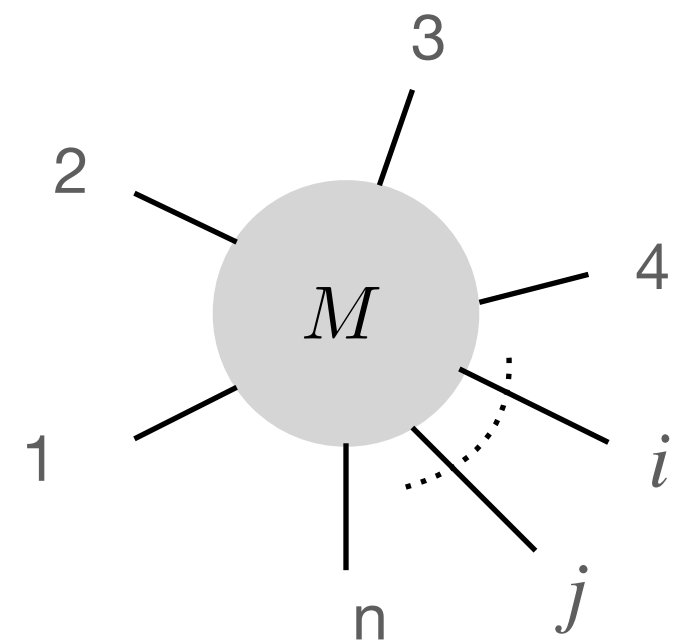
Residue Theorem and Tree-Level Amplitudes / Quadratic recurrence relation (BCFW)

● Residue Theorem

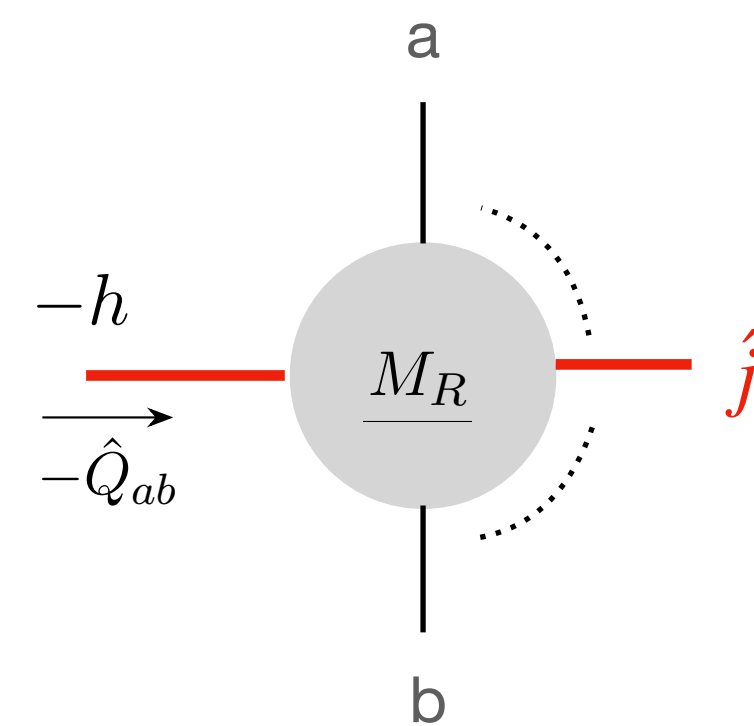
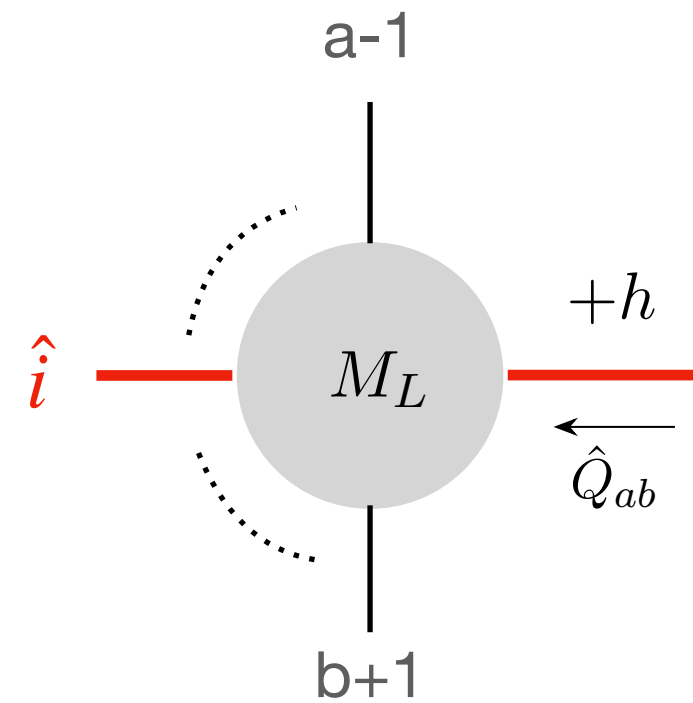
simple pole @

$$z_{ab} = \frac{Q_{ab}^2}{2Q_{ab} \cdot \epsilon_{ij}} = \frac{Q_{ab}^2}{\langle i | \not{Q}_{ab} | j \rangle}$$

$$M = \text{Res}_{z=0} \frac{M(z)}{z} = - \sum_{a,b} \text{Res}_{z=z_{a,b}} \frac{M(z)}{z} = \sum_{a,b} \frac{M_L(z=z_{ab}) M_R(z=z_{ab})}{z_{ab} \langle i | \not{Q}_{ab} | j \rangle} = \sum_{a,b} \frac{M_L(z=z_{ab}) M_R(z=z_{ab})}{Q_{ab}^2}$$



$$= \sum_{\substack{a,b \\ \text{cuts, helicity}}} \frac{1}{s_{a\dots b}} \times$$



Britto
Cachazo
Feng
Witten
Relation

Quadratic Relation: Building Tree-level Amplitudes from **lower-point** (simpler) **on-shell** Tree-Level Amplitudes

$$M_L(z) = M \left((b+1), \dots, \hat{i}, (a-1), \hat{Q}_{ab}^h \right)$$

$$M_R(z) = M \left(-\hat{Q}_{ab}^{-h}, a, \hat{j}, \dots, b \right).$$

(all incoming momenta)

$$s_{a\dots b} = Q_{ab}^2 = \left(\sum_{m=a}^b p_m \right)^2$$

BCFW Relation @ Work

● 5-gluon MHV Amplitude

$$\mathbf{M}_5 = \begin{array}{c} \hat{2}^- \\ \text{wavy} \\ 1^- \\ \text{wavy} \\ 5^+ \\ \text{wavy} \\ \hat{3}^+ \\ \text{wavy} \\ 4^+ \end{array} = \underbrace{1^- \text{---} \text{wavy} \text{---} \text{MHV} \text{---} \text{wavy} \text{---} \hat{Q}_{34}^+ \text{---} \frac{1}{s_{34}} \text{---} \text{wavy} \text{---} \overline{\text{MHV}} \text{---} \text{wavy} \text{---} \hat{3}^- \text{---} 4^+}_{\text{MHV}} + \begin{array}{c} \hat{2}^- \quad \hat{3}^+ \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 5^+ \quad 4^+ \end{array} + \begin{array}{c} \hat{2}^- \quad \hat{3}^+ \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 1^- \quad 5^+ \end{array} + \begin{array}{c} \hat{2}^- \quad \hat{3}^+ \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ 1^- \quad 5^+ \end{array}$$

$$= \frac{\langle 1\hat{2} \rangle^3}{\langle \hat{2}\hat{Q} \rangle \langle \hat{Q}5 \rangle \langle 51 \rangle} \frac{1}{s_{34}} \frac{[\hat{3}4]^3}{[4(-\hat{Q})][(-\hat{Q})\hat{3}]}, \quad \hat{Q} \equiv \hat{Q}_{34}$$

$$\begin{aligned} |\hat{2}\rangle &= |2\rangle, & |\hat{2}] &= |2] + z|3], \\ |\hat{3}\rangle &= |3\rangle - z|2\rangle, & |\hat{3}] &= |3], \\ \hat{p}_2 &= p_2 + z\epsilon_{23}, & \hat{p}_3 &= p_3 - z\epsilon_{23}, \end{aligned}$$

$$= - \frac{\langle 12 \rangle^3}{\langle \hat{2}\hat{Q} \rangle \langle \hat{Q}5 \rangle \langle 51 \rangle} \frac{1}{s_{23}} \frac{[34]^3}{[4\hat{Q}][\hat{Q}3]} = - \frac{1}{s_{34}} \frac{\langle 12 \rangle^3 [34]^3}{\langle 2|\hat{Q}|4 \rangle \langle 5|\hat{Q}|3 \rangle \langle 51 \rangle},$$

$$\begin{aligned} \langle 2|\hat{Q}|4 \rangle &= \langle 2|p_3 + p_4 - z\epsilon_{23}|4 \rangle = \langle 2|p_3|4 \rangle = \langle 23 \rangle [34], \\ \langle 5|\hat{Q}|3 \rangle &= \langle 5|p_3 + p_4 - z\epsilon_{23}|3 \rangle = \langle 5|p_4|3 \rangle = \langle 54 \rangle [43], \end{aligned}$$

$$\mathbf{M}_5 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}$$

1 term instead of 10 Feynman diagrams

BCFW Relation @ Work

● 4-graviton MHV Amplitude

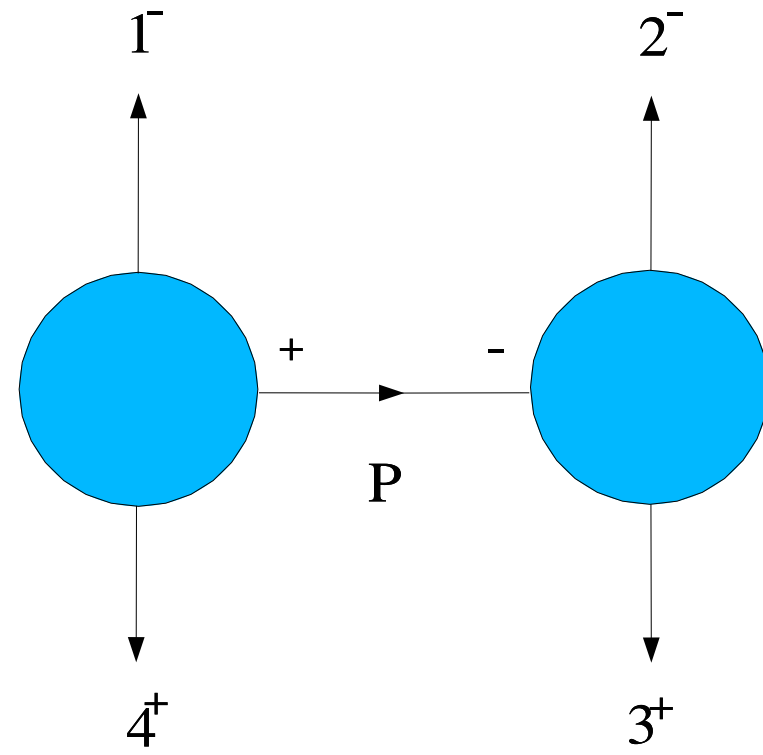


Figure 3: One of the two diagrams contributing to the recursion relation for the MHV amplitude $\mathcal{M}(1^-, 2^-, 3^+, 4^+)$. The other is obtained from this by cyclically permuting the labels $(3, 4)$ – i.e. swapping 3 with 4.

Shifting p_1 and p_2

$$\mathcal{M}^{(4)} = \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R \quad \mathcal{M}_L = \left(\frac{[\hat{P} 4]^3}{[4 1][1 \hat{P}]} \right)^2, \quad \mathcal{M}_R = \left(\frac{\langle \hat{P} 2 \rangle^3}{\langle 2 3 \rangle \langle 3 \hat{P} \rangle} \right)^2, \quad P^2 = (p_1 + p_4)^2, \quad \langle i \hat{P} \rangle = \frac{\langle i | P | 1 \rangle}{[\hat{P} 1]}$$

$$\Rightarrow \mathcal{M}^{(4)} = \frac{\langle 12 \rangle^6 [14]}{\langle 14 \rangle \langle 23 \rangle^2 \langle 34 \rangle^2}$$

$$\text{The full amplitude is } \mathcal{M}(1^-, 2^-, 3^+, 4^+) = \mathcal{M}^{(3)} + \mathcal{M}^{(4)} = \frac{\langle 12 \rangle^6 [14]}{\langle 14 \rangle \langle 23 \rangle^2 \langle 34 \rangle^2} + 3 \leftrightarrow 4$$

1 Soft limit and notation

Let an external massless momentum p_s become soft,

$$p_s \longrightarrow \varepsilon p_s, \quad \varepsilon \rightarrow 0.$$

In spinor variables we use the symmetric scaling

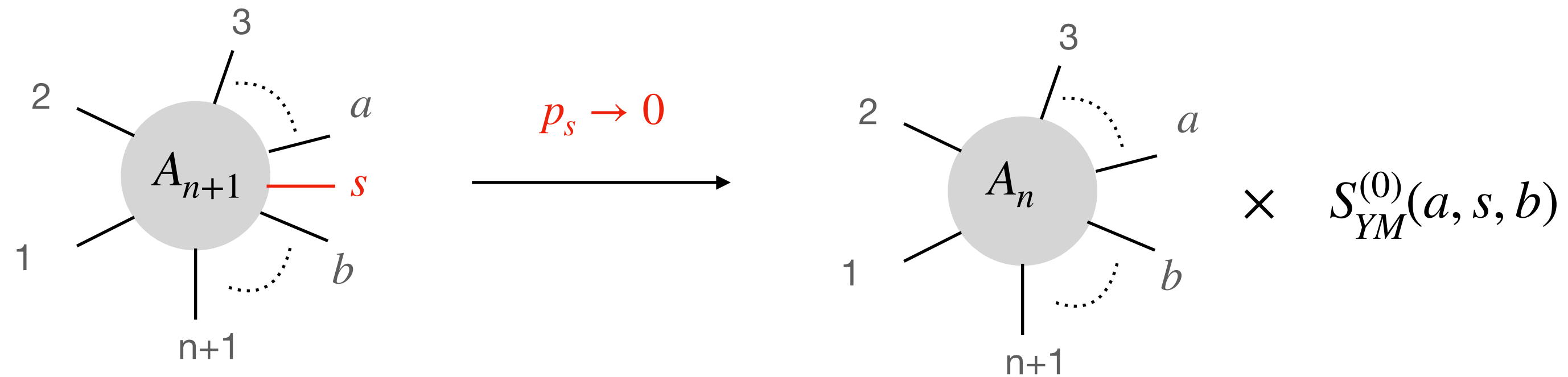
$$p_s^\mu \equiv \frac{\langle s | \gamma^\mu | s \rangle}{2} \quad |s\rangle \longrightarrow \sqrt{\varepsilon} |s\rangle, \quad |s] \longrightarrow \sqrt{\varepsilon} |s].$$

Then

$$\langle is \rangle = \mathcal{O}(\sqrt{\varepsilon}), \quad [is] = \mathcal{O}(\sqrt{\varepsilon}), \quad s_{is} = 2p_i \cdot p_s = \langle is \rangle [si] = \mathcal{O}(\varepsilon).$$

The leading soft behaviour of scattering amplitudes is universal: it depends only on the momentum and helicity of the soft particle and on the external hard particles, not on the detailed structure of the hard process. Historically, this universality is associated with Low's soft theorem for photons and Weinberg's soft theorem for photons and gravitons.

Soft Gluon Theorem



Consider a color-ordered Yang–Mills tree amplitude

$$A_{n+1}(\dots, a, s, b, \dots),$$

where the soft gluon s is inserted between adjacent hard legs a and b in that cyclic ordering. At leading order,

$$A_{n+1}(\dots, a, s^{h_s}, b, \dots) \longrightarrow S_{YM}^{(0)}(a, s^{h_s}, b) A_n(\dots, a, b, \dots).$$

The leading color-ordered soft factor depends only on the two legs adjacent to the soft gluon.

● **Soft gluon factors**

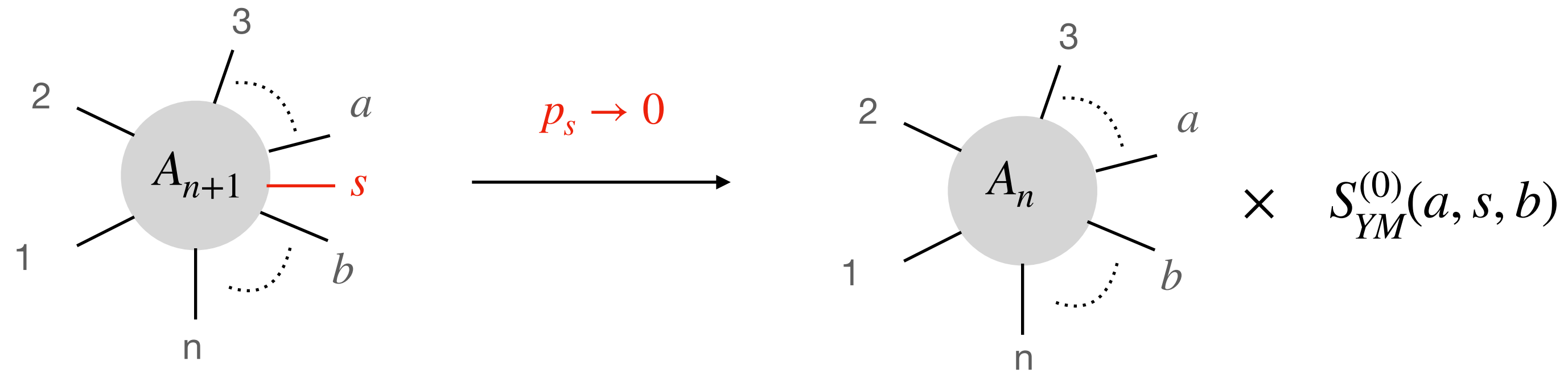
$$S_{YM}^{(0)}(a, s^+, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle}.$$

$$S_{YM}^{(0)}(a, s^-, b) = \frac{[ab]}{[as][sb]}.$$

Under the symmetric soft scaling,

$$S_{YM}^{(0)} = \mathcal{O}\left(\frac{1}{\varepsilon}\right), \quad \text{and therefore} \quad A_{n+1} \sim \frac{1}{\varepsilon} A_n.$$

Soft Gluon Theorem / Soft MHV Amplitude



Example: positive-helicity soft gluon from Parke–Taylor

Consider an MHV amplitude with a positive-helicity soft gluon s inserted between hard legs a and b :

$$\begin{aligned}
 A_{n+1}^{\text{MHV}}(1^-, 2^-, 3^+, \dots, a, s^+, b, \dots, n^+) &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle as \rangle \langle sb \rangle \cdots \langle n1 \rangle} \\
 &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle as \rangle \langle sb \rangle \cdots \langle n1 \rangle} \frac{\langle ab \rangle}{\langle ab \rangle} \\
 &= \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle ab \rangle \cdots \langle n1 \rangle} \\
 &= \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} A_n^{\text{MHV}}(1^-, 2^-, 3^+, \dots, a, b, \dots, n^+)
 \end{aligned}$$

$$\boxed{A_{n+1}^{\text{MHV}} \longrightarrow S_{\text{YM}}^{(0)}(a, s^+, b) A_n^{\text{MHV}}.}$$

Color-dressed Soft Gluon Theorem

For a full color-dressed Yang–Mills amplitude, the leading soft factor is not restricted to adjacent legs. Instead,

$$\mathcal{A}_{n+1}(1, \dots, n, s) \longrightarrow S_{\text{YM,full}}^{(0)}(s) \mathcal{A}_n(1, \dots, n).$$

The full leading soft-gluon factor is

$$S_{\text{YM,full}}^{(0)}(s) = \sum_{i=1}^n \mathbf{T}_i \frac{\varepsilon_s \cdot p_i}{p_s \cdot p_i}.$$

Here $\mathbf{T}_i$ is the color generator acting on the i -th hard external leg.

The color-ordered soft factor is the planar version of this statement. In a fixed cyclic ordering, the soft gluon only attaches to its two adjacent color neighbours, giving

$$S_{\text{YM}}^{(0)}(a, s, b) = \frac{\varepsilon_s \cdot p_a}{p_s \cdot p_a} - \frac{\varepsilon_s \cdot p_b}{p_s \cdot p_b}.$$

Soft Graviton Theorem

Now consider a pure gravity tree amplitude

$$M_{n+1}(1, \dots, n, s).$$

When the graviton s becomes soft,

$$p_s \rightarrow \varepsilon p_s,$$

the leading behaviour is

$$M_{n+1}(1, \dots, n, s^{h_s}) \longrightarrow S_{\text{grav}}^{(0)}(s^{h_s}) M_n(1, \dots, n).$$

The leading gravity soft factor is Weinberg's soft factor,

$$S_{\text{grav}}^{(0)}(s) = \sum_{i=1}^n \frac{\varepsilon_{\mu\nu}^{(s)} p_i^\mu p_i^\nu}{p_s \cdot p_i}.$$

For a graviton polarization tensor written as a product of two gluon polarizations,

$$\varepsilon_{\mu\nu}^{(s)} = \varepsilon_\mu^{(s)} \tilde{\varepsilon}_\nu^{(s)},$$

this becomes

$$S_{\text{grav}}^{(0)}(s) = \sum_{i=1}^n \frac{(\varepsilon_s \cdot p_i)(\tilde{\varepsilon}_s \cdot p_i)}{p_s \cdot p_i}.$$

Unlike the color-ordered gluon soft factor, the graviton soft factor is a sum over all hard external particles.

Soft Graviton Theorem

For a positive-helicity soft graviton s^+ , introduce two reference spinors x and y . The polarization tensor may be viewed as

$$\varepsilon_{\mu\nu}^+(s; x, y) = \varepsilon_{\mu}^+(s; x) \varepsilon_{\nu}^+(s; y).$$

Then the leading soft graviton factor is

$$S_{\text{grav}}^{(0)}(s^+) = \sum_{i=1}^n \frac{[si]}{\langle si \rangle} \frac{\langle xi \rangle}{\langle xs \rangle} \frac{\langle yi \rangle}{\langle ys \rangle}.$$

Similarly, for a negative-helicity soft graviton,

$$S_{\text{grav}}^{(0)}(s^-) = \sum_{i=1}^n \frac{\langle si \rangle}{[si]} \frac{[xi]}{[xs]} \frac{[yi]}{[ys]}.$$

Under the symmetric soft scaling,

$$S_{\text{grav}}^{(0)} = \mathcal{O}\left(\frac{1}{\varepsilon}\right),$$

and hence

$$M_{n+1} \sim \frac{1}{\varepsilon} M_n.$$

Soft Graviton Theorem

The leading color-ordered Yang–Mills soft limits are

$$A_{n+1}(\dots, a, s^+, b, \dots) \longrightarrow \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} A_n(\dots, a, b, \dots),$$

$$A_{n+1}(\dots, a, s^-, b, \dots) \longrightarrow \frac{[ab]}{[as][sb]} A_n(\dots, a, b, \dots).$$

For gravity,

$$M_{n+1}(1, \dots, n, s^+) \longrightarrow \left[\sum_{i=1}^n \frac{[si]}{\langle si \rangle} \frac{\langle xi \rangle}{\langle xs \rangle} \frac{\langle yi \rangle}{\langle ys \rangle} \right] M_n(1, \dots, n),$$

and

$$M_{n+1}(1, \dots, n, s^-) \longrightarrow \left[\sum_{i=1}^n \frac{\langle si \rangle}{[si]} \frac{[xi]}{[xs]} \frac{[yi]}{[ys]} \right] M_n(1, \dots, n).$$

These are the leading Low–Weinberg soft factors written in spinor-helicity variables.