Canonical Quantum Gravity II

Semiclassical Limit and the Recovery of Time

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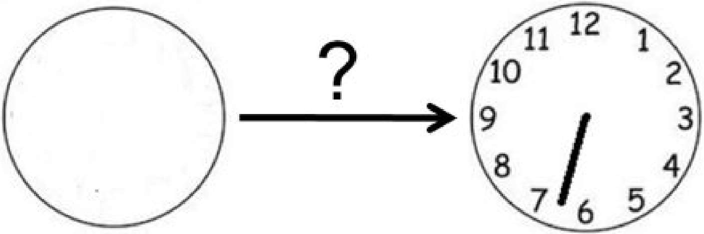

Figure: Emergence of time from a fundamentally timeless quantum theory of gravity?

Time in quantum mechanics

- Time-dependent Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi;$$

- can be obtained from the classical constraint

$$p_t + H \approx 0.$$

What is more fundamental: the time-dependent or the time-independent Schrödinger equation?

Quantizing time?

To quantize is a verb – suggests to turn classical time t into an operator $\hat{t}$; but does this make sense?

Wolfgang Pauli (1933):

We thus conclude that one must completely go without the introduction of an operator t and that the time t in wave mechanics must necessarily be considered as an ordinary number ('c-number').

Unruh and Wald (1989):

. . . in ordinary Schrödinger quantum mechanics for a system with a Hamiltonian bounded from below, no dynamical variable can correlate monotonically with the Schrödinger time t, and thus the role of t in the interpretation of Schrödinger quantum mechanics cannot be replaced by that of a dynamical variable.

Recovery of the time-dependent Schrödinger equation

Neville Mott (1931):

...we shall confine ourselves ...to the somewhat idealised case of the collision between an α particle and an atom consisting of a single electron bound in the field of an infinitely heavy nucleus. We have to show that the same probabilities of excitation result whether we consider the problem as a one-body problem ...or as a two-body problem ...

Making the ansatz

$$\Psi(\mathbf{r}, \mathbf{R}) = f(\mathbf{r}, \mathbf{R})\mathrm{e}^{\mathrm{i}kZ},$$

where $\mathbf{r}$ and $\mathbf{R} \equiv (X, Y, Z)$ are the positions of the electron and α-particle, respectively, Mott derives the time-dependent Schrödinger equation (with t related to Z) from the (more fundamental) time-independent Schrödinger equation $H\Psi = 0$; he explicitly refers to the Born–Oppenheimer approximation (1927).

The semiclassical approximation

Wheeler–DeWitt equation and momentum constraints in the presence of matter (e.g. a scalar field):

$$\begin{aligned}
\left\{-\frac{1}{2m_{\mathrm{P}}^2}G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}}-2m_{\mathrm{P}}^2\sqrt{h}\,{}^{(3)}R+\hat{\mathcal{H}}_\perp^{\mathrm{m}}\right\}|\Psi[h_{ab}]\rangle &= 0\,,\\
\left\{-\frac{2}{\mathrm{i}}h_{ab}D_c\frac{\delta}{\delta h_{bc}}+\hat{\mathcal{H}}_a^{\mathrm{m}}\right\}|\Psi[h_{ab}]\rangle &= 0
\end{aligned}$$

(bra and ket notation refers to non-gravitational fields)

Make comparison with a quantum-mechanical model:

$$\begin{aligned}
-\frac{1}{2M}\frac{\partial^2}{\partial Q^2} &\quad\leftrightarrow\quad -\frac{1}{2m_{\mathrm{P}}^2}G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}}\,,\\
V(Q) &\quad\leftrightarrow\quad -2m_{\mathrm{P}}^2\sqrt{h}\,{}^{(3)}R\,,\\
h(q,Q) &\quad\leftrightarrow\quad \hat{\mathcal{H}}_\perp^{\mathrm{m}}\,,\\
\Psi(q,Q) &\quad\leftrightarrow\quad |\Psi[h_{ab}]\rangle\,.
\end{aligned}$$

A quantum-mechanical model

Divide the total system into a 'heavy part' described by the variable Q and a 'light part' described by the variable q; full system be described by a stationary Schrödinger equation:

$$\boxed{H\Psi(q,Q) = E\Psi(q,Q)}$$

with

$$H = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial Q^2} + V(Q) + h(q,Q)$$

$$\text{Ansatz}: \quad \Psi(q,Q) = \sum_n \chi_n(Q)\psi_n(q,Q)$$

(assume that $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ for each Q)

Get an effective (exact) equation for the "heavy" part:

$$\sum_n \left(\frac{\mathcal{P}_{mn}{}^2}{2M} + \epsilon_{mn}(Q) \right) \chi_n(Q) + V(Q)\chi_m(Q) = E\chi_m(Q) \ ,$$

$\epsilon_{mn}(Q) \equiv \langle \psi_m | h | \psi_n \rangle$: "Born–Oppenheimer potential"

$$\mathcal{P}_{mn} \equiv \frac{\hbar}{\mathrm{i}} \left(\delta_{mn} \frac{\partial}{\partial Q} - \frac{\mathrm{i}}{\hbar} A_{mn} \right),$$

$A_{mn}(Q) \equiv \mathrm{i}\hbar \left\langle \psi_m \,\middle|\, \frac{\partial \psi_n}{\partial Q} \right\rangle$: "connection"

Born–Oppenheimer approximation

First approximation: neglect off-diagonal terms in the effective equation

$$\left[\frac{1}{2M}\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q} - A_{nn}(Q)\right)^2 + V(Q) + E_n(Q)\right]\chi_n(Q) = E\chi_n(Q),$$

$E_n(Q) := \epsilon_{nn}(Q) = \langle\psi_n|h|\psi_n\rangle$

Second approximation: WKB-ansatz for the "heavy" part

$$\chi_n(Q) = C_n(Q)\mathrm{e}^{\mathrm{i}MS_n(Q)/\hbar}$$

Neglecting the connection, the above equation then becomes the Hamilton–Jacobi equation:

$$H_{\mathrm{cl}} \equiv \frac{P_n^2}{2M} + V(Q) + E_n(Q) = E$$

(in gravity: semiclassical Einstein equations)

One can now introduce a time coordinate t_n ("WKB time") via the Hamilton equations of motion for the "heavy" part,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_n} P_n &= -\frac{\partial}{\partial Q} H_{\mathrm{cl}} = -\frac{\partial}{\partial Q}\left(V(Q) + E_n(Q)\right) , \\ \frac{\mathrm{d}}{\mathrm{d}t_n} Q &= \frac{\partial}{\partial P_n} H_{\mathrm{cl}} = \frac{P_n}{M} \end{aligned}$$

Use the WKB time in the effective equation for the "light" part

Get an effective (exact) equation for the “light” part:

$$\sum_n \chi_n(Q)\left[h(q,Q)-\left(E-V(Q)+\frac{\hbar^2}{2M\chi_n}\frac{\partial^2\chi_n}{\partial Q^2}\right)\right.$$
$$\left.-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial Q^2}-\frac{\hbar^2}{M\chi_n}\frac{\partial\chi_n}{\partial Q}\frac{\partial}{\partial Q}\right]\psi_n(q,Q)=0$$

- Neglect $\frac{\partial^2\chi_n}{\partial Q^2}$
  (assume slow variation of ψ_n with respect to Q);
- use the definition of WKB time in the last term:

$$-\frac{\hbar^2}{M\chi_n}\frac{\partial\chi_n}{\partial Q}\frac{\partial\psi_n}{\partial Q}\approx -\mathrm{i}\hbar\frac{\partial S_n}{\partial Q}\frac{\partial\psi_n}{\partial Q}\equiv -\mathrm{i}\hbar\frac{\partial\psi_n}{\partial t_n}$$

ψ_n is thus evaluated along a particular classical trajectory of the “heavy” variable, $\psi_n(Q(t_n),q)\equiv\psi_n(t_n,q)$.

Further algebra leads to

$$\sum_n \chi_n \left[h(q, t_n) - E_n(t_n) - \mathrm{i}\hbar \frac{\partial}{\partial t_n} \right] \psi_n(t_n, q) = 0$$

Restriction to one component and absorption of $E_n(t)$ into ψ would yield

$$\boxed{\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = h\psi}$$

cf. Mott (1931)

Back to quantum gravity

Ansatz:

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]\mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]}|\psi[h_{ab}]\rangle$$

One evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x},t)$, corresponding to a solution, $S[h_{ab}]$, of the Hamilton–Jacobi equations; this solution is obtained from

$$\dot{h}_{ab} = NG_{abcd}\frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}$$

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle = \int \mathrm{d}^3x\,\dot{h}_{ab}(\mathbf{x},t)\,\frac{\delta}{\delta h_{ab}(\mathbf{x})}|\psi[h_{ab}]\rangle$$

This leads to a functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field:

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle &= \hat{H}^{\mathrm{m}}|\psi(t)\rangle \\
\hat{H}^{\mathrm{m}} &\equiv \int \mathrm{d}^3x\,\Big\{N(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{\perp}(\mathbf{x}) + N^a(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_a(\mathbf{x})\Big\}
\end{aligned}$$

$\hat{H}^{\mathrm{m}}$: matter-field Hamiltonian in the Schrödinger picture, parametrically depending on (generally non-static) metric coefficients of the curved space–time background.

WKB time t controls the dynamics in this approximation

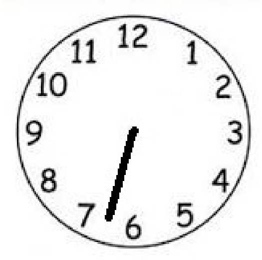

Figure: Emergence of semiclassical time

Figure from: C.K, B. Nikolić, *J.Phys.Conf.Ser.* **880**, 012002 (2017).

Quantum-gravitational corrections

Next order in the Born–Oppenheimer approximation gives

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2}\,(\text{various terms})$$

(*C.K. and Singh* (1991); *Barvinsky and C.K.* (1998))

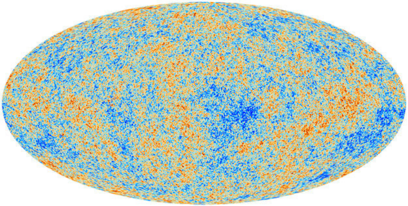

Figure credit: ESA/PLANCK Collaboration

See Lecture V for an example

Conformally invariant gravity

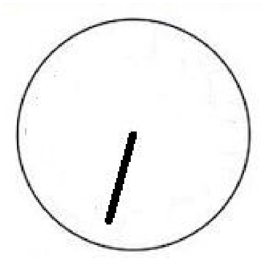

Figure: Emergence of scale-less time

Figure from: C.K, B. Nikolić, *J.Phys.Conf.Ser.* **880**, 012002 (2017).

Three central questions

- In quantum cosmology, arbitrary superpositions of the gravitational field and matter states can occur. How can we understand the emergence of an (approximate) classical Universe? Answer: by decoherence (see last lecture)
- Does the probability interpretation of quantum theory make sense at the most fundamental level of quantum gravity?
- Is there a sensible notion of time beyond the semiclassical level?