Canonical Quantum Gravity I

Classical Formalism and Quantization

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CONTENTS OF THE SIX LECTURES

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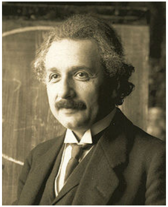
General relativity



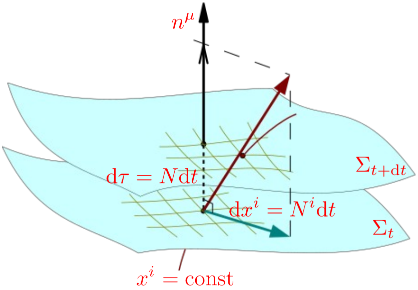
$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

(November 25, 1915)

- left: geometry (gravitational field)
- right: matter fields (Standard Model of particle physics)

Canonical Formalism

$$\dot{X}^\nu \equiv t^\nu = N n^\nu + N^a X^\nu_{,a}$$



$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t) \\ &= (h_{ab}N^aN^b - N^2)\mathrm{d}t^2 + 2h_{ab}N^a\mathrm{d}x^b\mathrm{d}t + h_{ab}\mathrm{d}x^a\mathrm{d}x^b \,. \end{aligned}$$

configuration variable: three-metric h_{ab}

More fundamental viewpoint:

three-manifold Σ is given; only after solving the dynamical equations can we construct spacetime and interpret the time dependence of the metric h_{ab} of Σ as being brought about by 'wafting' Σ through a four-manifold via a one-parameter family of embeddings

- Six evolution equations for h_{ab} and its canonical momentum π^{ab}
- Four constraints

Spacetime can then be interpreted as a 'trajectory of spaces'

Einstein's equations can be written as a dynamical system (for the three-metric h_{ab} and its canonical momentum π^{ab} on a spacelike hypersurface Σ) of evolution equations together with constraints:

$$\begin{aligned}\mathcal{H}_\perp &= 2\kappa\, G_{ab\,cd}\pi^{ab}\pi^{cd} - (2\kappa)^{-1}\sqrt{h}({}^{(3)}R - 2\Lambda) + \sqrt{h}\rho \approx 0\\ \mathcal{H}^a &= -2D_b\pi^{ab} + \sqrt{h}j^a \approx 0\,,\end{aligned}$$

with the DeWitt metric

$$G_{ab\,cd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd})$$

and

$$\kappa = 8\pi G/c^4$$

$H \approx 0$ is called "Hamiltonian constraint", $\mathcal{H}^a \approx 0$ are called "momentum (diffeomorphism) constraints".

Constraints and evolution

I

Constraints are preserved in time $\Longleftrightarrow$ energy–momentum tensor of matter has vanishing covariant divergence

compare with electrodynamics: Gauss constraint preserved in time $\Longleftrightarrow$ charge conservation

II

Einstein's equations are the unique propagation law consistent with the constraints

compare with electrodynamics: Maxwell's equations are the unique propagation law consistent with the Gauss constraint

Constraints: Laws of the Instant

Problem of time I

Restriction to *compact* three-spaces Σ:

- The total Hamiltonian is a combination of pure constraints; all of the evolution will be generated by constraints;
- no external time parameter exists;
- all physical time parameters are to be constructed from within our system, that is, as functional of the canonical variables; a priori there is no preferred choice of such an intrinsic time parameter.

The absence of an extrinsic time and the non-preference of an intrinsic one is known as the problem of time in (classical) canonical gravity. Still, spacetime exists.

Structure of configuration space

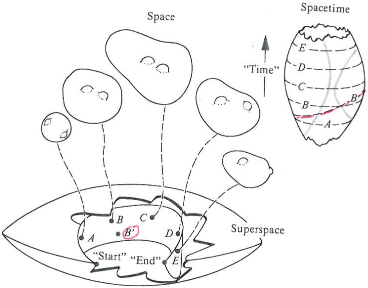
Superspace (Wheeler 1968):

$$\mathcal{S}(\Sigma) := \mathrm{Riem}\ \Sigma/\mathrm{Diff}\ \Sigma.$$

By going to superspace, the momentum constraints are automatically fulfilled. Whereas $\mathrm{Riem}\ \Sigma$ has a simple topological structure, the topological structure of $\mathcal{S}(\Sigma)$ is very complicated because it inherits (through $\mathrm{Diff}\ \Sigma$) some of the topological information contained in Σ.

Important: DeWitt metric and its projection on superspace

Superspace



From: Misner, Thorne, Wheeler, Gravitation

Erwin Schrödinger 1926:

We know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is in strict analogy with the failure of geometrical optics . . . that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wavelength. . . . Then it becomes a question of searching for an undulatory mechanics, and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.[1]

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[1] *wir wissen doch heute, daß unsere klassische Mechanik bei sehr kleinen Bahndimensionen und sehr starken Bahnkrümmungen versagt*. Vielleicht ist dieses Versagen eine volle Analogie zum Versagen der geometrischen Optik . . . , das bekanntlich eintritt, sobald die 'Hindernisse' oder 'Öffnungen' nicht mehr groß sind gegen die wirkliche, endliche Wellenlänge. . . . Dann gilt es, eine 'undulatorische Mechanik' zu suchen – und der nächstliegende Weg dazu ist wohl die wellentheoretische Ausgestaltung des Hamiltonschen Bildes.

Hamilton–Jacobi equation

$\boxed{\text{Hamilton–Jacobi equation} \longrightarrow \text{guess a wave equation}}$

In the vacuum case, one has

$$\begin{aligned} 16\pi G\, G_{abcd}\frac{\delta S}{\delta h_{ab}}\frac{\delta S}{\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G}({}^{(3)}R - 2\Lambda) &= 0\,, \\ D_a\frac{\delta S}{\delta h_{ab}} &= 0 \end{aligned}$$

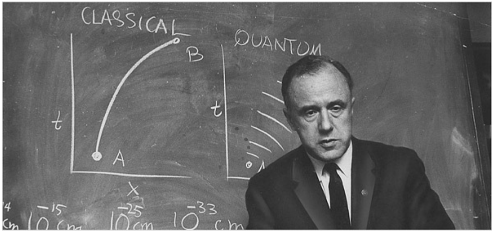
(Peres 1962)

Find wave equation which yields the Hamilton–Jacobi equation in the semiclassical limit:

$$\text{Ansatz}: \quad \Psi[h_{ab}] = C[h_{ab}]\exp\left(\frac{\mathrm{i}}{\hbar}S[h_{ab}]\right)$$

The dynamical gravitational variable is the three-metric h_{ab}! It is the argument of the wave functional.

Quantum geometrodynamics



(a) John Archibald Wheeler

(b) Bryce DeWitt

Application of Schrödinger's procedure to general relativity leads to

$$\hat{\mathcal{H}}_\perp\Psi \equiv \left(-16\pi G\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - (16\pi G)^{-1}\sqrt{h}\big({}^{(3)}R - 2\Lambda\big) + \sqrt{h}\hat{\rho}\right)\Psi = 0$$

Wheeler–DeWitt equation

$$\hat{\mathcal{H}}^a\Psi \equiv -2D_b\frac{\hbar}{\mathrm{i}}\frac{\delta\Psi}{\delta h_{ab}} + \sqrt{h}\hat{j}^a\Psi = 0$$

quantum diffeomorphism (momentum) constraint

Comparison between geometrodynamics and particle dynamics		
underlying notion	geometrodynamics	particle dynamics
dynamical object	space	particle
configuration	h_{ij} 3-geometry	x, t events
classical description	$h_{ij}(t)$ 4-geometry	$x(t)$ trajectory
dynamical arena	superspace	spacetime
wave function(al)	$\Psi[h_{ij}]$	$\psi(x, t)$

Solutions?

- Formal solutions can be obtained in simple situations, e.g. when only a cosmological constant is present (Isham 1976)
- Solutions to a lattice version of the Wheeler–DeWitt equation can be obtained (e.g. Hamber, Toriumi, Williams 2012)
- Some solutions for spherically-symmetric models (e.g. black holes) and many solutions in quantum cosmology (see Lecture V)
- Proposal of regularization methods (e.g. J. C. Feng 2018)

Problem of time II

- Spacetime has vanished from the formalism; only space remains.
- This holds also for loop quantum gravity and probably for string theory;
- it holds for *any* theory that has no absolute time at the classical level.
- The Wheeler–DeWitt equation has the structure of a wave equation and may therefore allow the introduction of an intrinsic time.
- The Hilbert-space structure in quantum mechanics is connected with the probability interpretation, in particular with probability conservation *in time* t; what happens with this structure in a timeless situation? (Schrödinger inner product or Klein–Gordon inner product or no inner product?)

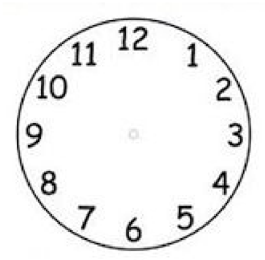


Figure: Absence of time in full quantum gravity

Figure from: C.K, B. Nikolić, *J.Phys.Conf.Ser.* **880**, 012002 (2017).

Diffeomorphism constraints

Under

$$x^a \mapsto \bar{x}^a = x^a + \delta N^a(\mathbf{x}),$$

the three-metric transforms as

$$h_{ab}(\mathbf{x}) \mapsto \bar{h}_{ab}(\mathbf{x}) = h_{ab}(\mathbf{x}) - D_a \delta N_b(\mathbf{x}) - D_b \delta N_a(\mathbf{x}).$$

The wave functional then transforms according to

$$\Psi[h_{ab}] \mapsto \Psi[h_{ab}] - 2 \int \mathrm{d}^3 x \, \frac{\delta \Psi}{\delta h_{ab}(\mathbf{x})} D_a \delta N_b(\mathbf{x}).$$

Assuming the invariance of the wave functional under this transformation, one is led to

$$D_a \frac{\delta \Psi}{\delta h_{ab}} = 0.$$

Under large diffeomorphisms, the wave functional can acquire a phase.

A simple analogy is Gauss's law in QED (or its generalizations to the non-Abelian case). The quantized version of the constraint $\nabla \mathbf{E} \approx 0$ reads

$$\frac{\hbar}{\mathrm{i}} \nabla \frac{\delta \Psi[\mathbf{A}]}{\delta \mathbf{A}} = 0,$$

from which invariance of Ψ with respect to gauge transformations of the form $\mathbf{A} \to \mathbf{A} + \nabla \lambda$ follows.

Constraint algebra

$$
\begin{aligned}
\{\mathcal{H}_\perp(\mathbf{x}), \mathcal{H}_\perp(\mathbf{y})\} &= -\sigma\delta_{,a}(\mathbf{x},\mathbf{y}) \left(h^{ab}(\mathbf{x})\mathcal{H}_b(\mathbf{x}) + h^{ab}(\mathbf{y})\mathcal{H}_b(\mathbf{y})\right) \\
\{\mathcal{H}_a(\mathbf{x}), \mathcal{H}_\perp(\mathbf{y})\} &= \mathcal{H}_\perp(\mathbf{x})\delta_{,a}(\mathbf{x},\mathbf{y}) \\
\{\mathcal{H}_a(\mathbf{x}), \mathcal{H}_b(\mathbf{y})\} &= \mathcal{H}_b(\mathbf{x})\delta_{,a}(\mathbf{x},\mathbf{y}) + \mathcal{H}_a(\mathbf{y})\delta_{,b}(\mathbf{x},\mathbf{y})
\end{aligned}
$$

Relation to four-dimensional diffeomorphism invariance of general relativity?

Important: are there central terms (anomalies) $\propto \hbar^n$ in the quantum theory? If yes, not all of the quantum constraints would hold (compare string theory).

Dirac (1968)

The problem has been investigated by J. Schwinger (Phys. Rev. 132 (1962) 1317). Schwinger uses a different notation, based on the dynamical co-ordinates $q^{rs} = K^2 e^{rs}$, with appropriate momenta conjugate to them, but his equations are equivalent to the ones given here.

Schwinger makes his attack more powerful by considering the quantum analogue, not of ϕ_L, but of $K^n\phi_L$, where n is some number at our disposal. Classically, $K^n\phi_L$ satisfies, as the conditions replacing Eqs (14) and (15)

$$[K^n\phi_L, \phi'_u] = (K^n\phi_L)_{,u}\,\delta(x-x') + (n+1)K^n\phi_L\,\delta_{,u}(x-x') \tag{18}$$

$$[K^n\phi_L, K'^n\phi'_L] = -K^n K'^n\{e^{rs}\phi_r + e'^{rs}\phi'_r\}\,\delta_{,s}(x-x') \tag{19}$$

Schwinger assumes for the quantum $K^n\phi_L$ the Hermitian expression

$$(K^n\phi_L)_Q = p^{rs}K^{n-1}(g_{rs}\,g_{ab} - g_{ra}\,g_{sb})p^{ab} + K^{n+1}R_3 \tag{20}$$

It satisfies Eq.(18) for any n, since this equation holds for any quantity that transforms suitably under transformations of the co-ordinates x^1, x^2, x^3 of the surface.

Schwinger proceeds to examine what value of n, if any, would make $(K^n\phi_L)_Q$ satisfy Eq.(19) with the quantum expressions (17) for the ϕ_u's on the right-hand side, with their coefficients all on the left. The calculation involves quantities of the form $\delta(x-x')\delta_{,r}(x-x')$. There does not exist any general method for handling such quadratic quantities in the δ-function, free from inconsistencies. However, by using simple and plausible but non-rigorous methods, Schwinger is able to show that the condition is satisfied with n = 3. Since the right-hand side of Eq.(19) must be Hermitian, it must then be equally possible to have all the coefficients of the ϕ_u's on the right.

The problem of the quantization of the gravitational field is thus left in a rather uncertain state. If one accepts Schwinger's plausible methods, the problem is solved. But one cannot be happy with such methods without having a reliable procedure for handling quadratic expressions in the δ-function.

The sort of inconsistency one must prevent is illustrated by the following calculation with $\delta(y)$ for one variable y. We have

$$y\,\delta(y) = 0 \tag{21}$$

Differentiating, we get

$$y\,\delta'(y) = -\delta(y) \tag{22}$$

Multiplying Eq.(21) by $\delta'(y)$, we get

$$y\,\delta(y)\,\delta'(y) = 0$$

and multiplying Eq.(22) by $\delta(y)$, we get

$$y\,\delta(y)\,\delta'(y) = -\{\delta(y)\}^2$$

end of paper!

Path Integral satisfies constraints

- Quantum mechanics: path integral satisfies Schrödinger equation
- Quantum gravity: path integral satisfies Wheeler–DeWitt equation and diffeomorphism constraints

The full path integral with the Einstein–Hilbert action (if defined rigorously) should be equivalent to the constraint equations of canonical quantum gravity

A brief history of geometrodynamics

- F. Klein, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, **1918**, 171–189:
  first four Einstein equations are "Hamiltonian"and "momentum density" equations
- L. Rosenfeld, *Annalen der Physik*, 5. Folge, **5**, 113–152 (1930):
  general constraint formalism; first four Einstein equations are constraints; consistency conditions in the quantum theory ("Dirac consistency")

- P. Bergmann and collaborators (from 1949 on): general formalism (mostly classical); notion of observables
  Bergmann (1966): $H\psi = 0$, $\partial\psi/\partial t = 0$
  ("To this extent the Heisenberg and Schrödinger pictures are indistinguishable in any theory whose Hamiltonian is a constraint.")
- P. Dirac (1951): general formalism; Dirac brackets
- P. Dirac (1958/59): application to the gravitational field; reduced quantization
  ("I am inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world.")
- ADM (1959–1962): lapse and shift; rigorous definition of gravitational energy and radiation by canonical methods

- B. S. DeWitt, Quantum theory of gravity. I. The canonical theory. *Phys. Rev.*, **160**, 1113–48 (1967): general Wheeler–DeWitt equation; configuration space; quantum cosmology; semiclassical limit; conceptual issues, ...
- J. A. Wheeler, Superspace and the nature of quantum geometrodynamics. In *Battelle rencontres* (ed. C. M. DeWitt and J. A. Wheeler), pp. 242–307 (1968): general Wheeler–DeWitt equation; superspace; semiclassical limit; conceptual issues; ...

Ashtekar's new variables (1986)

- new momentum variable: densitized version of triad,

  $E_i^a(x) := \sqrt{h(x)}e_i^a(x)$;
- new configuration variable: 'connection' ,

  $GA_a^i(x) := \Gamma_a^i(x) + \beta K_a^i(x)$

$$\boxed{\{A_a^i(x), E_j^b(y)\} = 8\pi\beta\delta_j^i\delta_a^b\delta(x,y)}$$

New free parameter: Barbero–Immirzi parameter β

Loop quantum gravity

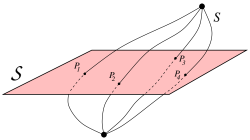
- new configuration variable: holonomy,

  $U[A, \alpha] := \mathcal{P} \exp \left(G \int_\alpha A \right) \;\; ;$

- new momentum variable: densitized triad flux

  $E_i[\mathcal{S}] := \int_{\mathcal{S}} \mathrm{d}\sigma_a \; E_i^a$

Under some mild assumptions, the holonomy–flux representation is unique



Quantization of area:

$$\hat{A}(\mathcal{S})\Psi_S[A] = 8\pi\beta l_{\mathrm{P}}^2 \sum_{P \in S \cap \mathcal{S}} \sqrt{j_P(j_P+1)}\,\Psi_S[A]$$

Open questions

- Hamiltonian constraint not yet fully understood
- Physically interesting solutions to all constraints?
- Is the quantum constraint algebra anomaly-free off-shell?
- Semiclassical limit?

Alternatives: spin-foam models, group-field theory