Gravitational waves simulations from past to future null infinity

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Outline of talk

- The global scattering problem in general relativity
- Our approach to the problem
- Non-linear semi-global scattering
- Future perspectives

The global scattering problem in general relativity

Basic concepts

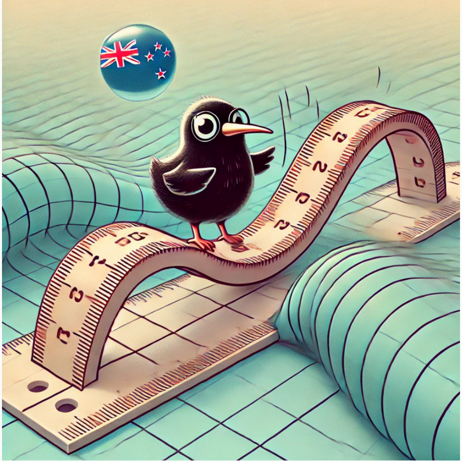
- We work with a manifold M equipped with a Lorentzian metric g that satisfies the Einstein equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \lambda g_{ab} = -\frac{8\pi G}{c^4}T_{ab}$$

- Project goal: To fully describe the scattering of gravitational waves through numerical simulations.
- Requires: Differential and conformal geometry, PDE theory, numerical methods for elliptic and hyperbolic systems, Lie groups and Lie algebras, asymptotic symmetries, HPC, lots of students, lots of coffee, lots of patience.

Gravitational waves are not locally well defined

- No background geometry as a reference. The wave *is* the geometry!



To infinity!

- For asymptotically flat space-times, a well defined prescription exists at *infinity*.
- This can be thought of as idealising *local* behaviour.

Penrose's great idea

- In the 1960's Roger Penrose published three seminal papers detailing a new technique to study asymptotic questions in general relativity:
  - Penrose, Roger. The light-cone at infinity. No. TN8. KING'S COLL LONDON (ENGLAND), 1962
  - Penrose, Roger. Asymptotic properties of fields and space-times. Physical Review Letters 1963
  - Penrose, Roger. Zero rest-mass fields including gravitation: asymptotic behaviour. Proceedings of the Royal Society of London. Series A. Mathematical and physical sciences 1965
- These papers introduced the idea of looking at the Minkowski metric $\widetilde{g}$ only up to arbitrary conformal rescalings $g = \Theta^2 \widetilde{g}$.
- The set of end points of null geodesics in the physical space-time can then be represented by a light-cone at infinity in the conformal space-time.

Asymptotic simplicity

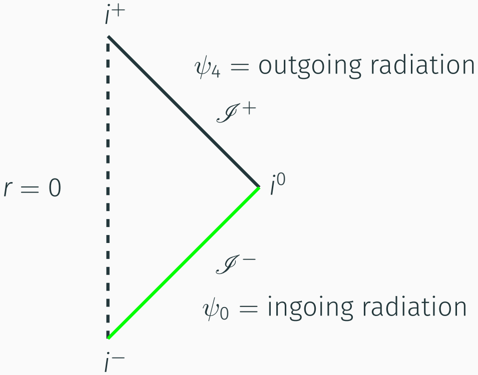
Definition

A smooth (time- and space-orientable) Lorentzian space-time $(\widetilde{M}, \widetilde{g}_{ab})$ is called asymptotically simple, if there exists another smooth Lorentzian space-time (M, g_{ab}) such that

- $\widetilde{M}$ is an open submanifold of M with smooth boundary $\partial\widetilde{M} = \mathscr{I}$;
- there exists a smooth scalar field Θ on M, such that $g_{ab} = \Theta^2 \widetilde{g}_{ab}$ on $\widetilde{M}$, and so that $\Theta = 0$, $d\Theta \neq 0$ on $\mathscr{I}$;
- every null geodesic in $\widetilde{M}$ acquires a future and a past endpoint on $\mathscr{I}$.

An asymptotically simple space-time is called asymptotically flat, if in addition $\widetilde{R}_{ab} = 0$ in a neighbourhood of $\mathscr{I}$.

The global scattering problem



Set the non-interacting initial data on $\mathscr{I}^-$ (in-states) and relate to the non-interacting final data on $\mathscr{I}^+$ (out-states).

Work done so far

- Analytical:
  - Conformal scattering (Nicolas, Mason, Friedlander, ...).
- Numerical:
  - Fully psuedo spectral methods for conformally invariant wave equation on Minkowski, Schwarzschild or Kerr space-times (Hennig, Frauendiener, Macedo)
  - Evolution of linearised spin-2 equation around Minkowski space-time (Frauendiener, Doulis)
  - No evolution of the Einstein equations had been performed from $\mathscr{I}^-$ to $\mathscr{I}^+$, even in the linearised case, until 2024!

Our approach

Conformal Field Equations

- H. Friedrich devised a regular extension of the Einstein equations to the conformal space-time, The Conformal Field Equations.

$$e_{\mathsf{a}}(c^{\mu}_{\mathsf{b}}) - e_{\mathsf{b}}(c^{\mu}_{\mathsf{a}}) = \widehat{\Gamma}_{\mathsf{ab}}{}^{\mathsf{c}} c^{\mu}_{\mathsf{c}} - \widehat{\Gamma}_{\mathsf{ba}}{}^{\mathsf{c}} c^{\mu}_{\mathsf{c}},$$

$$e_{\mathsf{a}}(\widehat{\Gamma}_{\mathsf{bc}}{}^{\mathsf{d}}) - e_{\mathsf{b}}(\widehat{\Gamma}_{\mathsf{ac}}{}^{\mathsf{d}}) = \left(\widehat{\Gamma}_{\mathsf{ab}}{}^{\mathsf{e}} - \widehat{\Gamma}_{\mathsf{ba}}{}^{\mathsf{e}}\right)\widehat{\Gamma}_{\mathsf{ec}}{}^{\mathsf{d}} - \widehat{\Gamma}_{\mathsf{bc}}{}^{\mathsf{e}}\widehat{\Gamma}_{\mathsf{ae}}{}^{\mathsf{d}} + \widehat{\Gamma}_{\mathsf{ac}}{}^{\mathsf{e}}\widehat{\Gamma}_{\mathsf{be}}{}^{\mathsf{d}}$$
$$+ \Theta K_{\mathsf{abc}}{}^{\mathsf{d}} - 2\eta_{\mathsf{c[a}}\widehat{P}_{\mathsf{b]}}{}^{\mathsf{d}} + 2\delta_{\mathsf{[a}}{}^{\mathsf{d}}\widehat{P}_{\mathsf{b]c}} - 2\widehat{P}_{\mathsf{[ab]}}\delta_{\mathsf{c}}{}^{\mathsf{d}},$$

$$\widehat{\nabla}_a\widehat{P}_{bc} - \widehat{\nabla}_b\widehat{P}_{ac} = h_e K_{abc}{}^e,$$

$$\nabla_e K_{abc}{}^e = 0, \qquad [K_{abc}{}^d = \Theta^{-1}C_{abc}{}^d = \Theta^{-1}\tilde{C}_{abc}{}^d]$$

- From this, a first order, non-linear, symmetric hyperbolic evolution system for the frame components, connection and curvature can be extracted.
- We implement this as an Initial Boundary Value Problem (IBVP).

Numerical implementation of the IBVP

- Software: The Python package COFFEE[1]

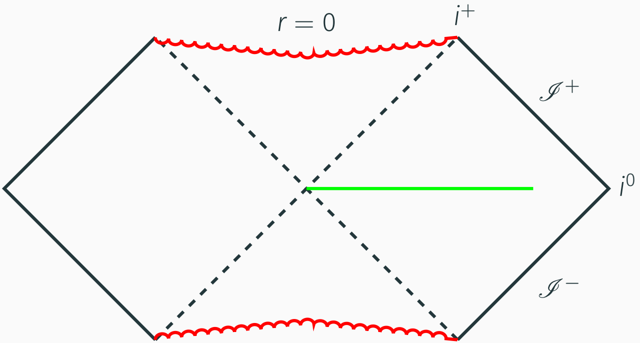
  pip install coffeegrinder

- Time integration: Explicit Runge-Kutta 4.
- Radial derivatives: Fourth order summation-by-parts finite difference operator of Strand.
- Angular derivatives: Pseudo-spectral implementation of $\eth$-derivatives.
- Boundary imposition: The Simultaneous Approximation Term (SAT) method to impose an ingoing gravitational wave profile.
- Optimisation: Regridding inside horizon and outside conformal boundary. Computationally intensive parts written in C. MPI and OpenMP parallelized.

[1] Doulis, G., Frauendiener, J., Stevens, C., & Whale, B. (2019). COFFEE–An MPI-parallelized Python package for the numerical evolution of differential equations. SoftwareX, 10, 100283.

Success of the numerical IBVP framework

Papers [2] [3] [4]



[2] Frauendiener, J., & Stevens, C. (2021). The non-linear perturbation of a black hole by gravitational waves. I. The Bondi–Sachs mass loss. Classical and Quantum Gravity, 38(19), 194002.

[3] Frauendiener, J. & Stevens, C. (2023). The non-linear perturbation of a black hole by gravitational waves. II. Quasinormal modes and the compactification problem. Classical and Quantum Gravity, 40(12), 125006.

[4] Goodenbour, A, Frauendiener, J. & Stevens, C. (2024). The non-linear perturbation of a black hole by gravitational waves. III. Newman-Penrose constants. Classical and Quantum Gravity 41(6), 065005.

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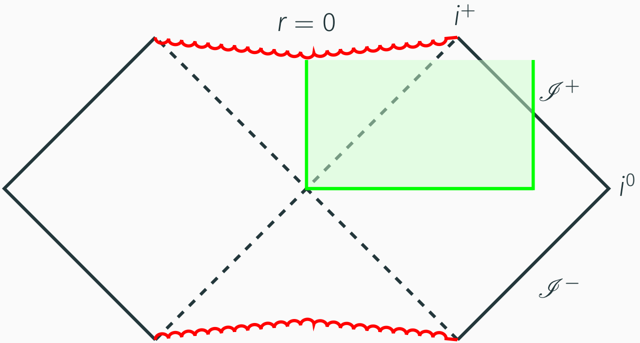

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Fully non-linear results for scattering with a black hole

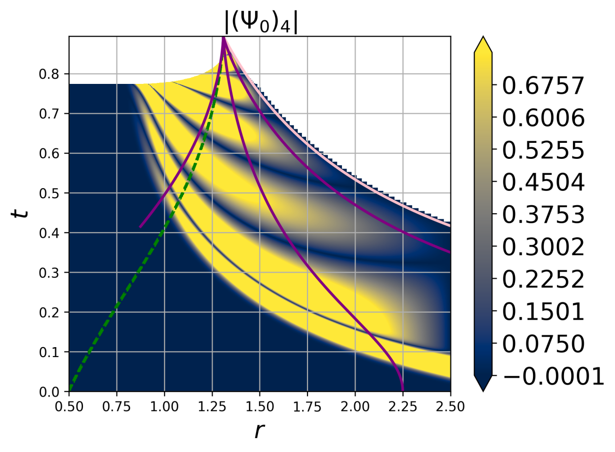

Figure 1: From Frauendiener, J. & Stevens, C. (2023). The non-linear perturbation of a black hole by gravitational waves. II. Quasinormal modes and the compactification problem. Classical and Quantum Gravity, 40(12), 125006.

Wellposedness of the IBVP

Wellposedness proof (with Juan Valiente-Kroon)

- Proving wellposedness helps guarantee that numerical techniques used for approximating solutions will behave.
- In the 90s Friedrich proved wellposedness of the IBVP for Anti-de Sitter space-time.
- This space-time has a timelike conformal boundary – a geometrically special surface – which is used in the IBVP.
- Project result: A proof of wellposedness of the IBVP for the CFE where the boundary is an arbitrary surface ruled by conformal geodesics.

Theorem (Well-posedness of the IBVP for the CFE)

Let (M, g, Ω) be a conformally regular spacetime region with timelike boundary $\mathcal{T}$ ruled by a congruence of conformal geodesics, and write the extended conformal Einstein equations in the conformal Gaussian gauge. Let S_0 be the initial spacelike hypersurface at $t = 0$ with boundary $S_0 \cap \mathcal{T}$.

Assume:

1. ***Initial data:*** *Smooth CFE data on S_0 satisfying the conformal constraints and the required compatibility conditions at $S_0 \cap \mathcal{T}$.*
2. ***Boundary condition:*** *Along $\mathcal{T}$ impose the maximally dissipative condition*

   $$\widetilde{\phi}_4 - c\,\widetilde{\phi}_0 = q, \qquad |c| \leq 1,$$

   where c and q are smooth.

Then there exists $t_* > 0$ such that the initial–boundary value problem has a unique solution on $[0, t_*]$. The reduced CFE system is symmetric hyperbolic with maximally dissipative boundary conditions, solutions depend continuously on the data, and all conformal constraints propagate. Hence the IBVP for the CFE is locally well-posed.

Non-linear semi-global scattering

Step toward global non-linear scattering problem

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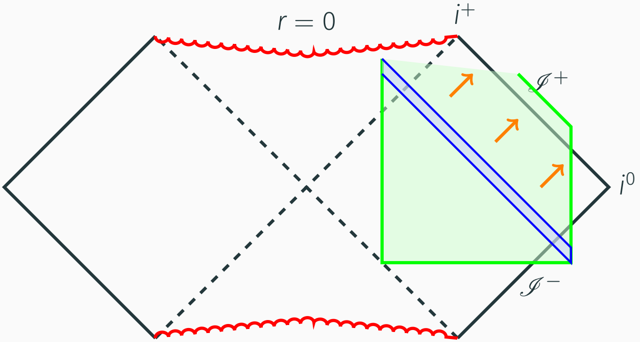


[5] Frauendiener, J., Stevens, C. & Thwala, S.(2025). Fully Nonlinear Gravitational Wave Simulations from Past to Future Null Infinity. Physical Review Letters 134, 161401 (PRL Collection 2025)

Convergence test on $\mathscr{I}^-$

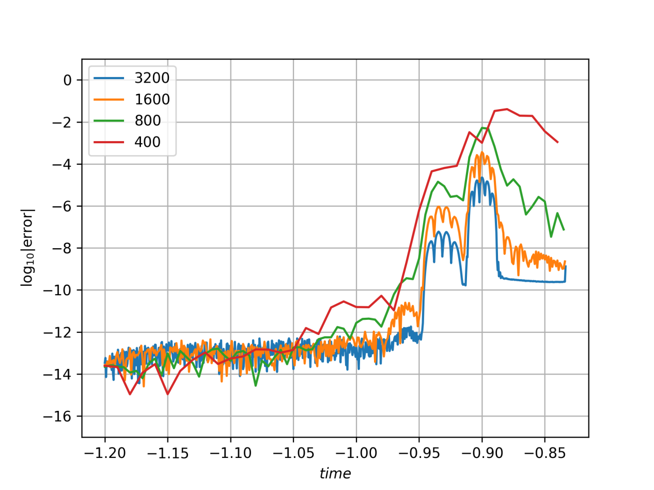


Figure 2: A constraint from $\nabla^a K_{abcd} = 0$ along $\mathscr{I}^-$

Convergence test on $\mathscr{I}^+$

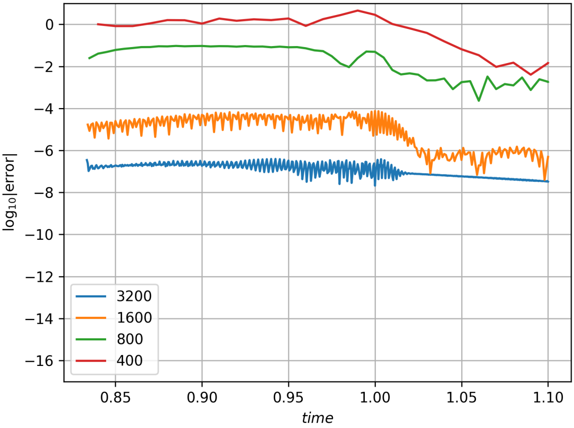
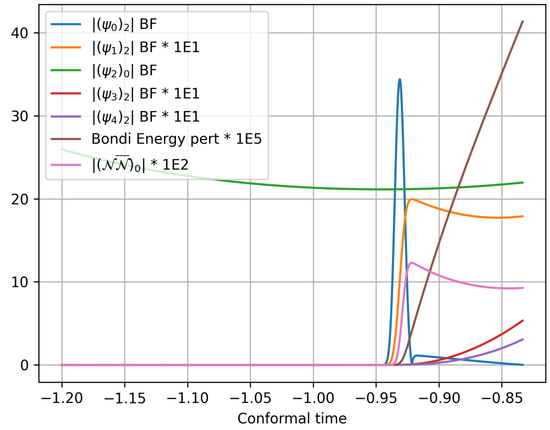


Figure 3: A constraint from $\nabla^a K_{abcd} = 0$ along $\mathscr{I}^+$

Bondi energy and gravitational data on $\mathscr{I}^-$



Bondi energy and gravitational data on $\mathscr{I}^+$

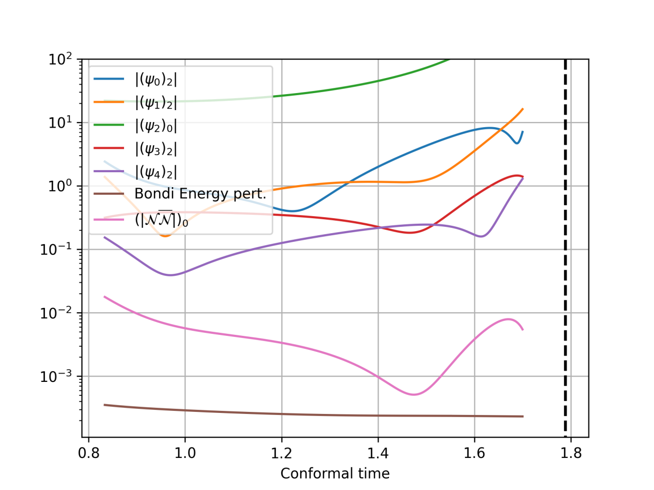


Figure 4: The vertical dashed line corresponds to future timelike infinity

Future work

- Include the origin (Wayne Pullan, Sebenele Thwala).
- Implement a scri-fixing gauge (Thomas Hopgood).
- Generalise linearised initial data solver on $\mathscr{I}^-$ (Merlyn Barrer) to nonlinear case.
- Generalise linearised evolution through cylinder (Ollie Markwell) to nonlinear case.
- Try to postpone compactification to resolve more of the physical space-time.

UC Gravity and Cosmology group 2025