Topological perspective on bulk boundary thermodynamic equivalence

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Overview

Background

A BH is a thermodynamic system:

- Temperature $T \propto \kappa$;
- Entropy $S \propto A$.

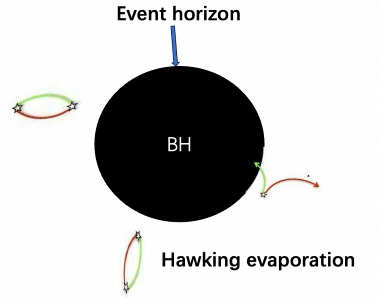
The 4 laws of BH thermodynamics

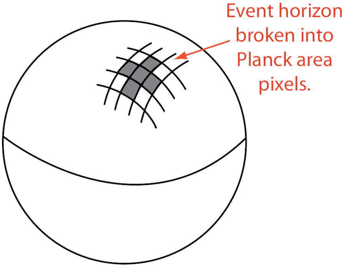
Bardeen, Carter and Hawking, Commun. Math. Phys. **31**, 161 (1973); Bekenstein, PRD **7**, 2333 (1973).:

- 0th law: $\kappa|_{\rm H} = \text{constant}$;
- 1st law:

$$dM = TdS + \Phi_{\rm h} dQ + \Omega_{\rm h} dJ; \quad (1)$$

- 2ed law: $dA \geq 0$;
- 3rd law: $\kappa_n > \kappa_{n+1} > 0, n < \infty$.





Background

Phase transitions of charged AdS BH

- Early work: van der Waals like phase

  Chamblin, Emparan, Johnson and Myers, PRD **60**, 104026 (1999);

  Chamblin, Emparan, Johnson and Myers, PRD **60**, 064018 (1999).

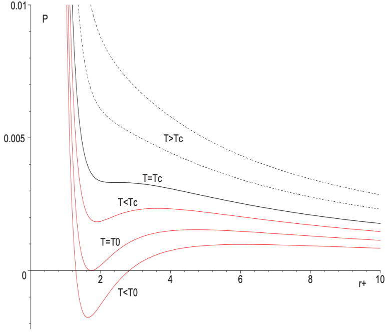
- Extended phase space thermodynamics:

  Kastor, Ray and Traschen, CQG **26**, 195011 (2009).

$$P = -\frac{\Lambda}{8\pi}. \qquad (2)$$

- $P-V$ criticality

  Kubiznak and Mann, JHEP **07**, 033 (2012).



Background

Foundations of a dynamic cosmological constant:

- The extended Iyer-Wald formalism;
  Xiao, Tian and Liu, PRL **132**, 2 (2024);

  Xiao, Liu, Tian and Zhang, [arXiv:2512.01916 [gr-qc]].

- The coupling constant of gauge field;

  Hajian and Tekin, PRL **132**, 191401 (2024).

- The higher-dimensional origin.

  Frassino, Pedraza, Svesko and Visser, PRL **130**, 161501 (2023).

Background & Motivation

Why holographic thermodynamics?

The holographic interpretation of extended thermodynamics remained unclear:

1. Variation of color N in $SU(N)$?
   The central charge is related to the AdS radius
   $$C = \frac{\Omega_{d-2} L^{d-2}}{16\pi G_{\mathrm{N}}}. \tag{3}$$
2. The corresponding CFT first law is degenerate as the $\mu\delta C$ and $-P\delta V$ terms are not truly independent.

Ways out of trouble:

- Varying Newton constant;

  Cong, Kubiznak and Mann, PRL **127**, 091301 (2021).

- New boundary and varying it[1]:

$$ds^2 = \omega^2 \left(-dt^2 + L^2 d\Omega^2\right) . \tag{4}$$

The first law and Euler relation

$$d\tilde{E} = \tilde{T} d\tilde{S} + \tilde{\phi} d\tilde{Q} + \tilde{\Omega} d\tilde{J} + \mu_C dC - \mathcal{P} d\mathcal{V}, \tag{5}$$

$$\tilde{E} = \tilde{T}\tilde{S} + \tilde{\phi}\tilde{Q} + \tilde{\Omega} d\tilde{J} + \mu_C C. \tag{6}$$

[1] Ahmed, Cong, Kubizňák, Mann and Visser, PRL **130**, 181401 (2023).

The central charge:

$$C = \frac{\Omega_{d-2} L^{d-2}}{16\pi G_{\mathrm{N}}}. \tag{7}$$

The problem

1. The number of central charge:
   - spacetime dimension.
2. The value and expression of central charge:
   - gravitational theories.

Our aim: extend the holographic discription to Gauss-Bonnet AdS gravitational theory.

The 5D Gauss-Bonnet AdS BH

The metric for the 5D charged Gauss-Bonnet AdS BH [2]:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2[d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\psi^2)] \qquad (8)$$

with the metric function given by

$$f(r) = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{32\alpha M}{3\pi r^4} - \frac{\alpha Q^2}{3r^6} + \frac{2\alpha\Lambda}{3}}\right). \qquad (9)$$

The extended BH thermodynamics

- The 1st law:
  $dM = TdS + VdP + \Phi dQ + \mathcal{A}d\alpha,$
- Smarr relation:
  $2M = 3TS - 2PV + 2\Phi Q + 2\mathcal{A}\alpha.$

[2] R. G. Cai, Phys. Rev. D **65**, 084014 (2002) [arXiv:hep-th/0109133 [hep-th]]; R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, JHEP **09**, 005 (2013). [arXiv:1306.6233 [gr-qc]].

The boundary thermodynamics

- The boundary metric [3]:

$$ds^2 = \omega^2 \left(-dt^2 + L^2 d\Omega_3^2\right), \tag{10}$$

- The two central charge:

$$C = \frac{\pi}{8}\left(1 - 2\lambda f_\infty\right) L_{\text{eff}}^3, \tag{11}$$
$$A = \frac{\pi}{8}\left(1 - 6\lambda f_\infty\right) L_{\text{eff}}^3; \tag{12}$$

where

$$\lambda = \frac{\alpha}{L^2}, \quad f_\infty = \frac{L^2}{L_{\text{eff}}^2}, \quad L_{\text{eff}} = \sqrt{\frac{2\lambda}{1-\sqrt{1-4\lambda}}}L. \tag{13}$$

[3] A. Karch and B. Robinson, JHEP **12**, 073 (2015). [arXiv:1510.02472 [hep-th]].

The boundary thermodynamics

The first law of thermodynamics and Euler relation for the CFT

$$d\tilde{E} = \tilde{T}d\tilde{S} + \tilde{\phi}d\tilde{Q} + \mu_C dC + \mu_A dA - \mathcal{P}d\mathcal{V}, \tag{14}$$

$$\tilde{E} = \tilde{T}\tilde{S} + \tilde{\phi}\tilde{Q} + \mu_C C + \mu_A A. \tag{15}$$

The holographic dictionary:

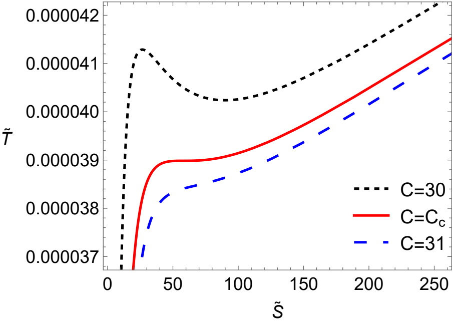
$$\begin{aligned} &\tilde{E} = \frac{M}{\omega}, \qquad \tilde{T} = \frac{T}{\omega}, \qquad \tilde{S} = S, \\ &\tilde{\Phi} = \frac{\Phi}{\omega L}, \qquad \tilde{Q} = QL. \end{aligned} \tag{16}$$

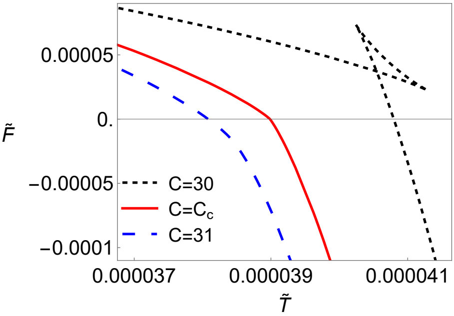
The main problem

Does it mirrors the thermodynamic properties of the bulk thermodynamics?

Phase transition

For $\tilde{Q} = 0.0867$, $R = 10,000$ and central charge $A = 27$.





Critical point

- In the fixed $(\tilde{Q}, C, A, \mathcal{V})$ ensemble, the critical point:

$$\left(\frac{\partial \tilde{T}}{\partial \tilde{S}}\right)_{\tilde{Q},C,A,\mathcal{V}} = 0, \qquad \left(\frac{\partial^2 \tilde{T}}{\partial^2 \tilde{S}}\right)_{\tilde{Q},C,A,\mathcal{V}} = 0. \tag{17}$$

- Is the critical point the same with that of bulk thermodynamics?
  - Neutral case:

$$r_\mathrm{h} = \sqrt{6\alpha}. \tag{18}$$

  - Charged case: it is the same!

Thermodynamic topology

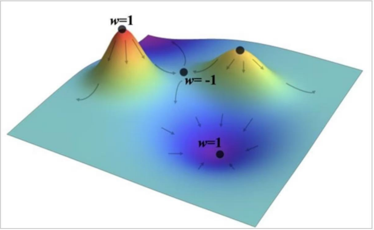
Thermodynamic topology

- Critical point;

  Wei and Liu, PRD **105**, 104003 (2022).

- Phase transition;

  Wei, Liu and Mann, PRL **129**, 191101 (2022).



Further development

- Residual method;
  Fang, Jiang and Zhang, JHEP **01**, 102 (2023).

  Zhang and Jiang, JHEP **06**, 115 (2023).

- Unified treatment:

  Wu, SJY and Wei, EPJC **85**, 1372 (2025).

Thermodynamic charge

Duan's ϕ-mapping topological current theory [4]

- Topological current:

$$j^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\alpha}\epsilon_{ab}\partial_{\nu}n^{a}\partial_{\alpha}n^{b}, \qquad \mu,\nu,\alpha = 0,1,2. \tag{19}$$

Two important properties:

1. Nonvanishing only at the zero points of the vector field

$$j^{0} = \sum_{i}\beta_{i}\eta_{i}\delta^{2}(\mathbf{x}-\mathbf{z}_{i}). \tag{20}$$

2. Topological charge: winding number

$$\mathcal{Q} = \int_{\Sigma} j^{0} d^{2}x = \sum_{i}\beta_{i}\eta_{i} = \sum_{i} w_{i}. \tag{21}$$

[4] Y.-S. Duan and M.-L. Ge, SU(2) Gauge Theory and Electrodynamics with N Magnetic Monopoles, Sci. Sin. 9 (1979) .

Topology of phase transition

The generalized free energy

$$\mathcal{F} = \tilde{E} - \frac{\tilde{S}}{\tau}. \qquad (22)$$

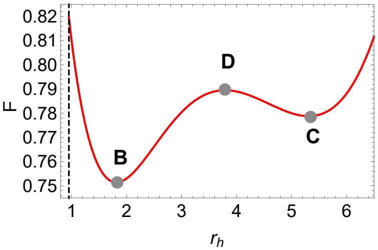


Figure: SJY, Zhou, Wei and Liu, PRD **105**, 084030 (2022).

- The vector field

$$\phi^{\tilde{S}} = \left(\frac{\partial \mathcal{F}}{\partial \tilde{S}}\right)_{\tilde{Q},\mathcal{V},C,A}, \qquad \phi^{\Theta} = -\frac{\cot\Theta\csc\Theta}{R}. \qquad (23)$$

Topology of phase transition

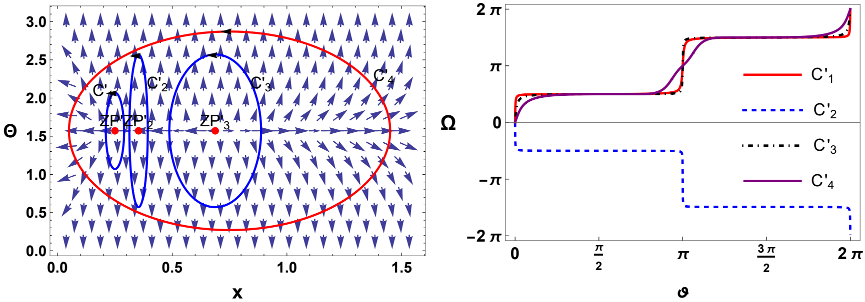


Figure: The unit vector field $\mathbf{n}$ on a portion of the $\Theta - x$ plane for charged CFT, with the zero points indicated by red dots. (a) The vector field distrubution; (b) The winding number.

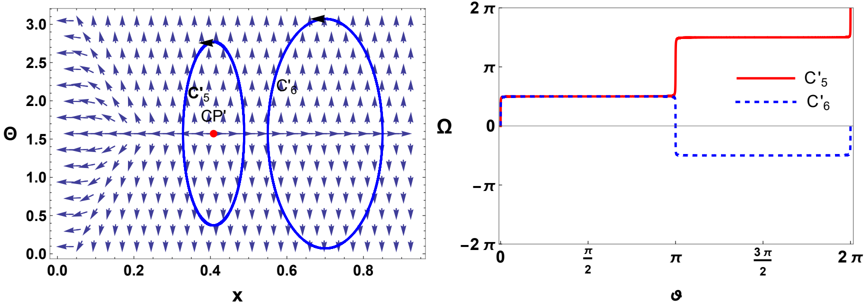


Figure: The vector field and the winding number for the thermodynamics of the charged CFT.

Summary

1. Two central charge: C and A;

2. Established an exact duality;

3. Demonstrate a precise correspondence:
   - Phase transition;
   - Critical point;
   - Thermodynamic topology.

Thank you!