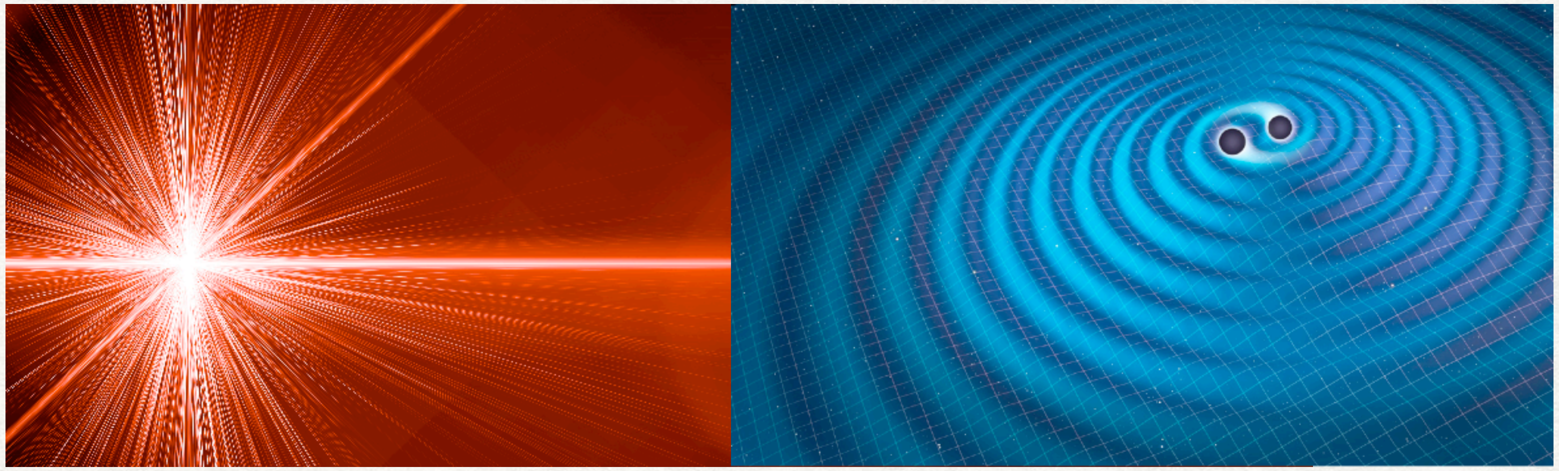
Ultra-high frequency gravitational waves from (twisted) light-beams: where do we stand?



Based on: 2309.04191 and 2501.11723

In collaboration with Ramy Aboushelbaya[1], Eduard Atonga[1], Aurélien Barrau[2], and Peter Norreys[1]

[1] : *Clarendon laboratory, Oxford*
[2] : *LPSC, Grenoble*

Question:

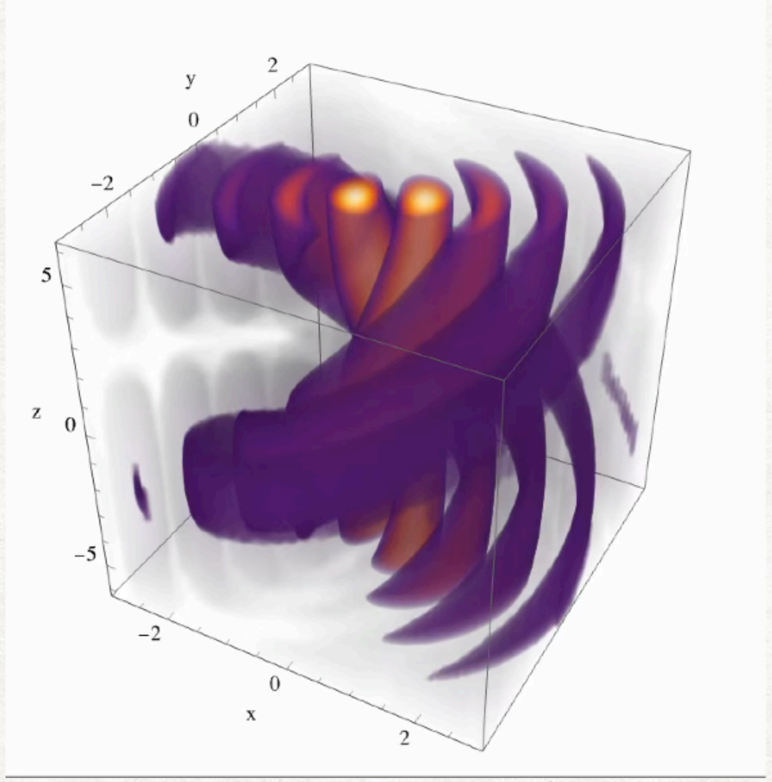
Can high-power laser systems open the path towards a generation/detection setup of laboratory-controlled gravitational waves?

Gravitational waves from twisted beams

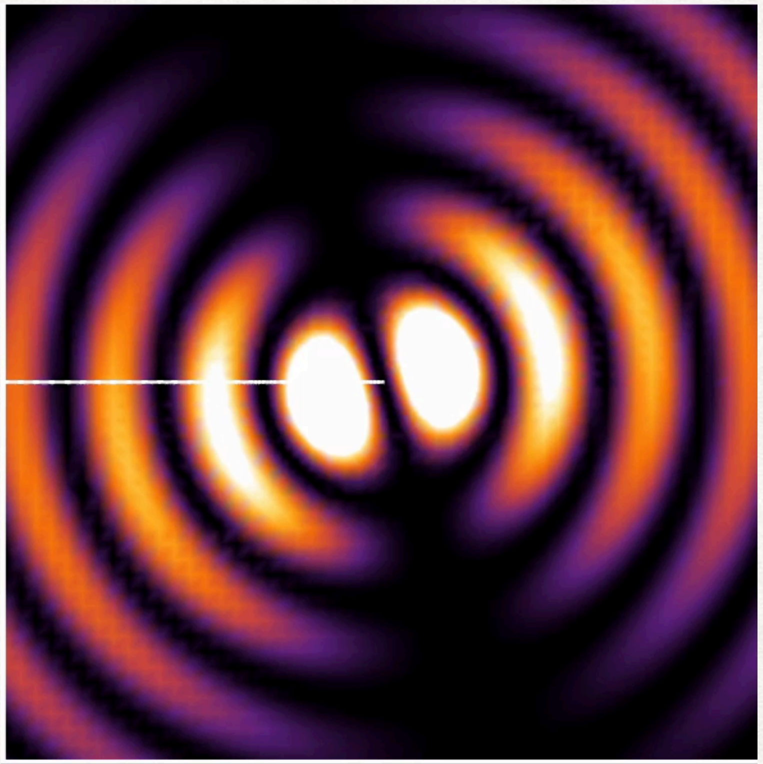
- **Light can carry orbital angular momentum (OAM)**

The OAM is characterized by an azimuthal phase dependance of the electromagnetic field.

$$E(x^\mu) \propto e^{il\phi}, l \in \mathbb{Z}$$



$l = 1$ twisted light propagating along the z-axis



Projection onto the azimuthal plane

Generates gravitational perturbations?

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

Reduced mass: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Solar mass: 10^{30} kg

MJ laser: 10^{-11} kg

⊖

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

Reduced mass: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Solar mass: 10^{30} kg

MJ laser: 10^{-11} kg

⊖

LVK black holes mergers events: 10^9 ly $\approx 10^{24}$ m

⊕

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

$\left.\begin{array}{r}\omega_s \approx 10^{15}\ \mathrm{Hz} \\ R \approx 10^{-8}\ \mathrm{m}\end{array}\right\}$ speed of light ⊕

Reduced mass: $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

Solar mass: 10^{30} kg

MJ laser: 10^{-11} kg ⊖

LVK black holes mergers events: 10^9 ly $\approx 10^{24}$ m ⊕

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

$\omega_s \approx 10^{15}$ Hz, $R \approx 10^{-8}$ m } speed of light ⊕

Reduced mass: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Solar mass: 10^{30} kg

MJ laser: 10^{-11} kg ⊖

LVK black holes mergers events: 10^9 ly $\approx 10^{24}$ m ⊕

- **The radiated power increases heavily with the GWs frequency**

$$\mathcal{F}_{GW} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \approx 10^{34}\omega_g^2 h^2 \text{ W.m}^{-2}$$

High-Power LASER	GW15094 LVK signal
$\mathcal{F}_{GW} = 1.4 \times 10^{-5}$ W.m^{-2} *Close to the pulse*	$\mathcal{F}_{GW} = 3.6 \times 10^{-6}$ W.m^{-2} *At the detector location*

E. Atonga et. al, arXiv:2501.11723

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Signal emitted by a binary system of masses in circular orbit:**

$$h \simeq \frac{4G}{c^4}\frac{\mu\omega_s^2 R^2}{r}$$

$\omega_s \approx 10^{15}$ Hz, $R \approx 10^{-8}$ m } speed of light ⊕

Reduced mass: $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Solar mass: 10^{30} kg

MJ laser: 10^{-11} kg ⊖

LVK black holes mergers events: 10^9 ly $\approx 10^{24}$ m ⊕

- **The radiated power increases heavily with the GWs frequency**

$$\mathcal{F}_{GW} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \approx 10^{34}\omega_g^2 h^2 \text{ W.m}^{-2}$$

High-Power LASER	GW15094 LVK signal
$\mathcal{F}_{GW} = 1.4 \times 10^{-5}$ W.m^{-2} *Close to the pulse*	$\mathcal{F}_{GW} = 3.6 \times 10^{-6}$ W.m^{-2} *At the detector location*

E. Atonga et. al, arXiv:2501.11723

- **Must deal with relativistic sources**

The formalism heavily relies on a v/c expansion:

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp} + \cdots\right]_{\mathrm{ret}} \rightarrow$$

Usual formulae are no longer valid:

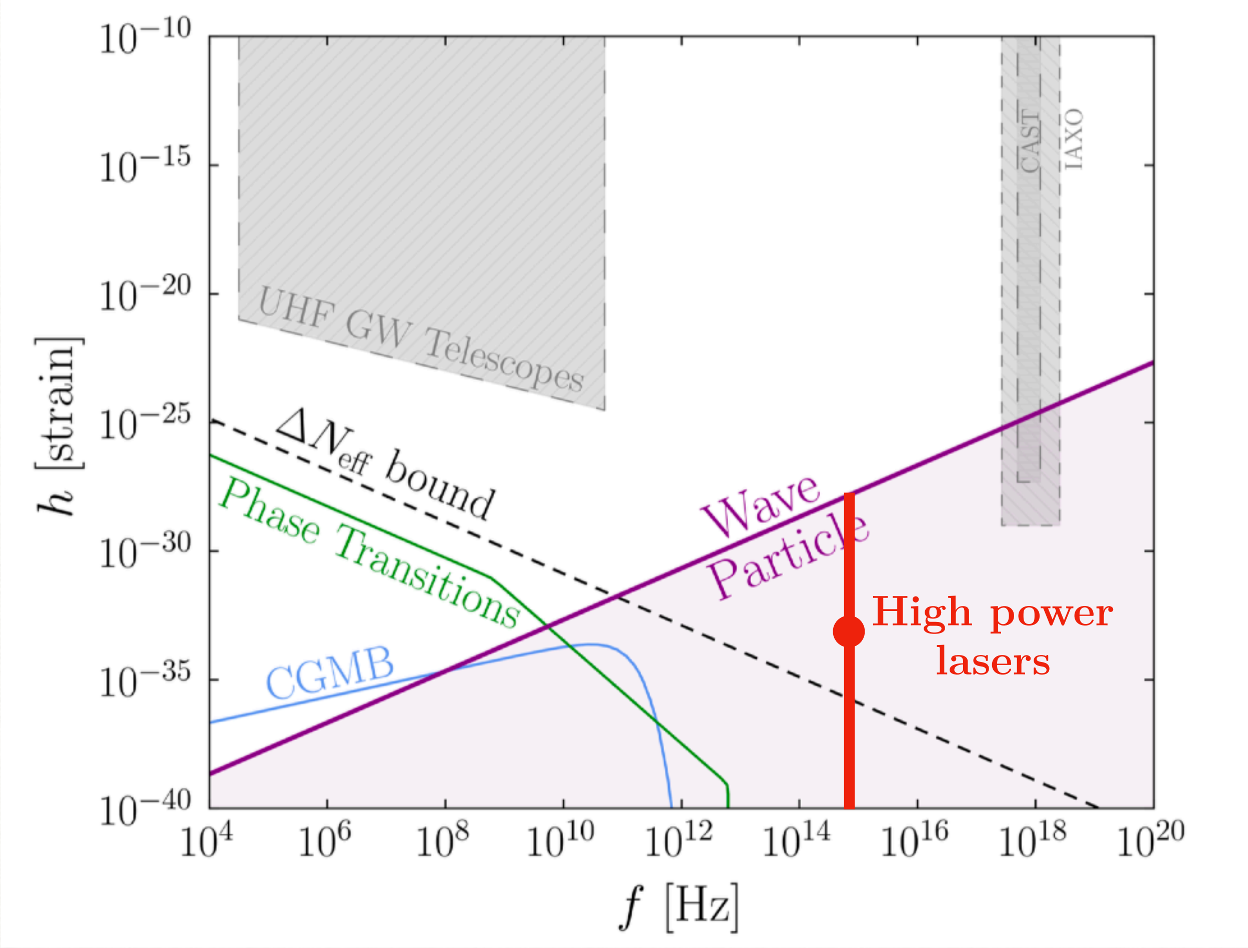
$$\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\mathrm{TT}}(t - r/c)\,.$$ (crossed out)

Gravitational waves from twisted beams

Pros and cons of using light as a source

- **Open the path towards quantum gravity?**

Quantumness of a gravitational signal: $$N = \frac{c^5}{32\pi G\hbar}\frac{h^2}{f_{GW}^2} \approx 10^{85}\frac{h^2}{f_{GW}^2}$$ **Dyson number**



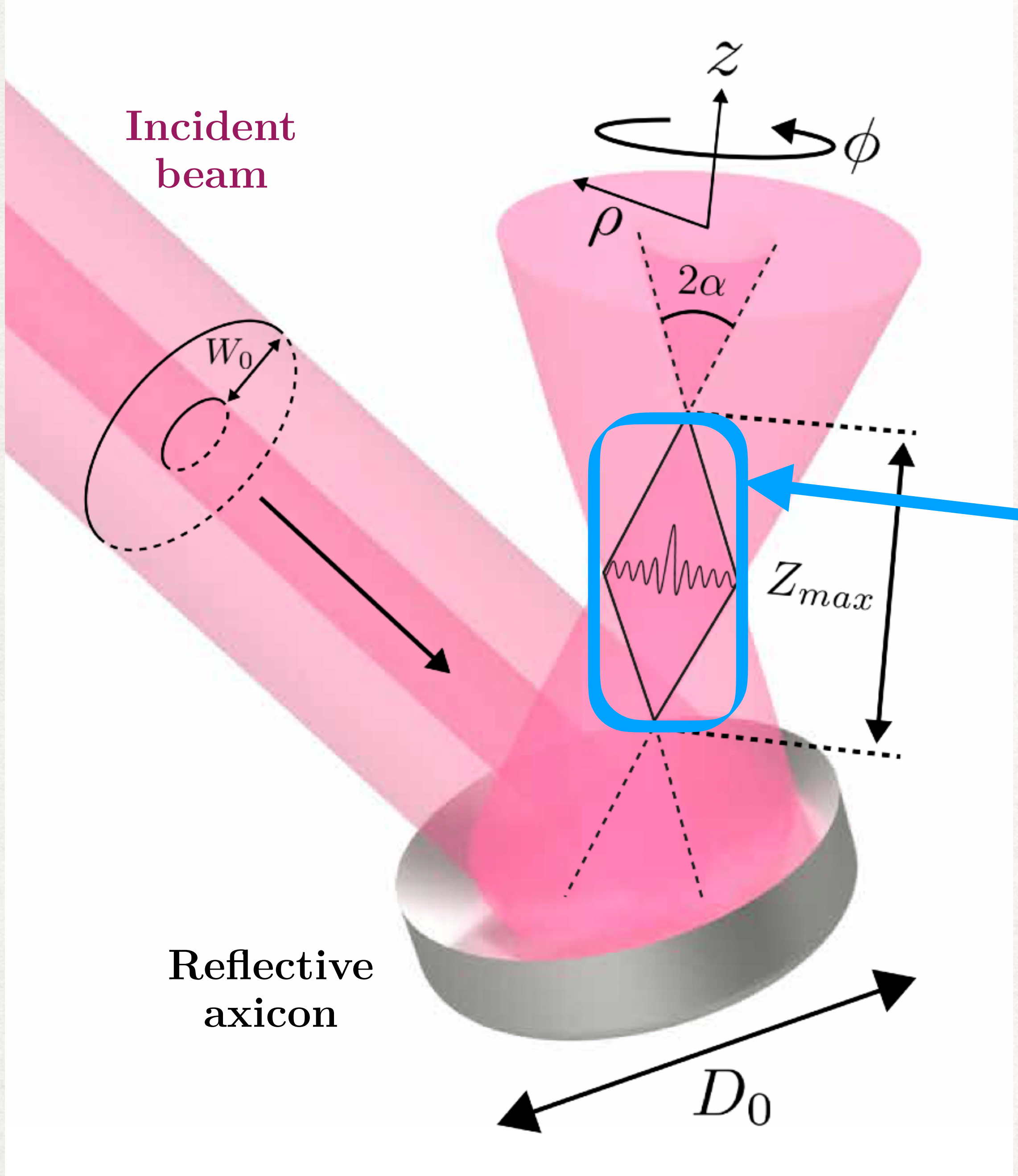
From D. Carney, V. Domcke, N. L. Rodd, arXiv:2308.12988

Signals sensitive to quantum effects of gravity

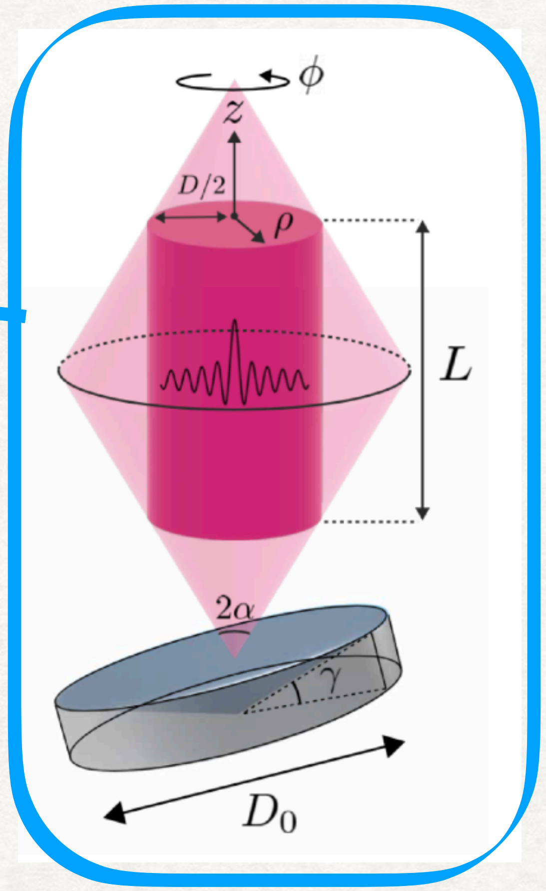
Gravitational waves from twisted beams

Generation of Bessel modes

E. Atonga, K.M, R. Aboushelbaya, A. Barrau et. al.. 2309.04191



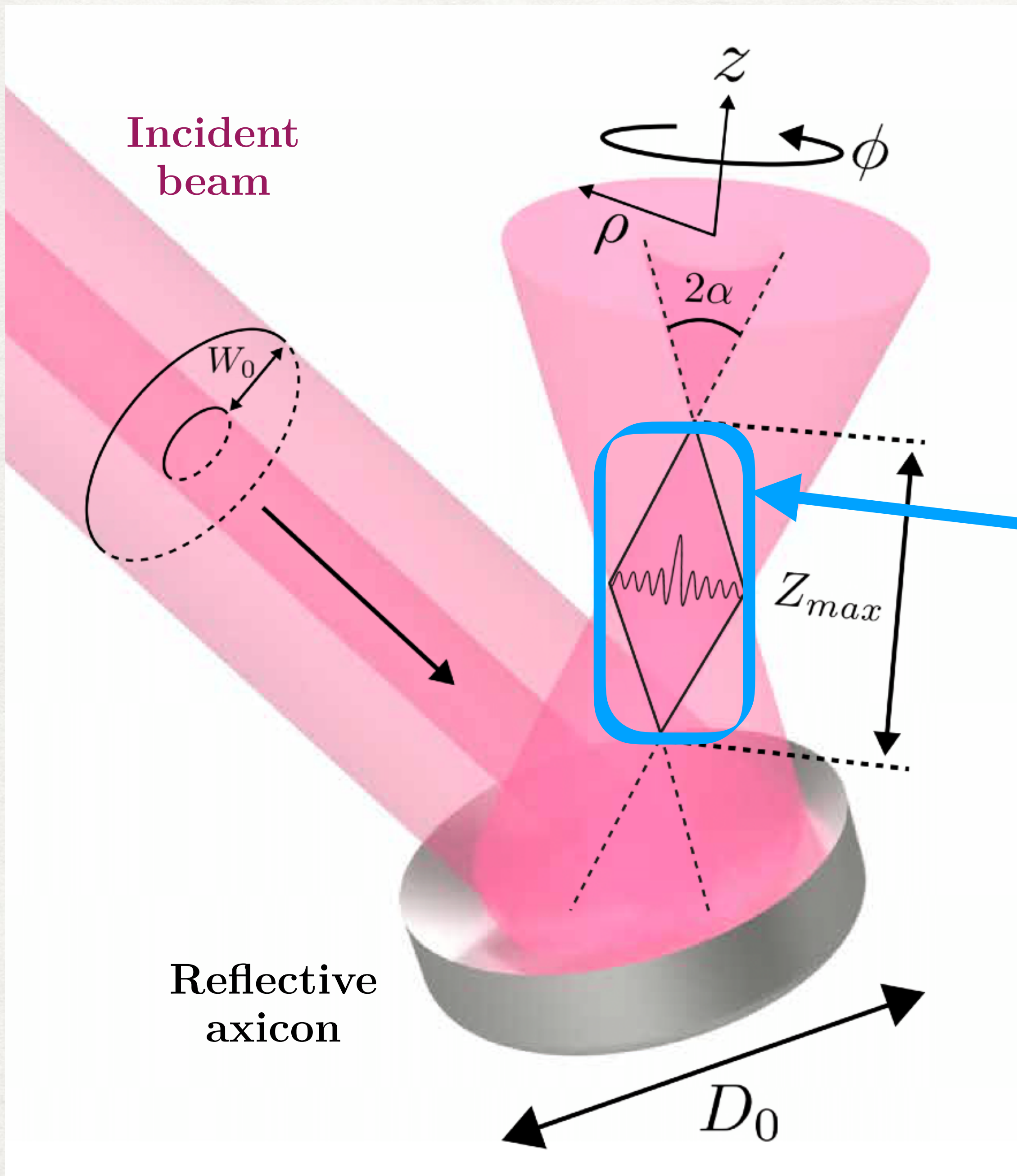
Region over which twisted light is generated



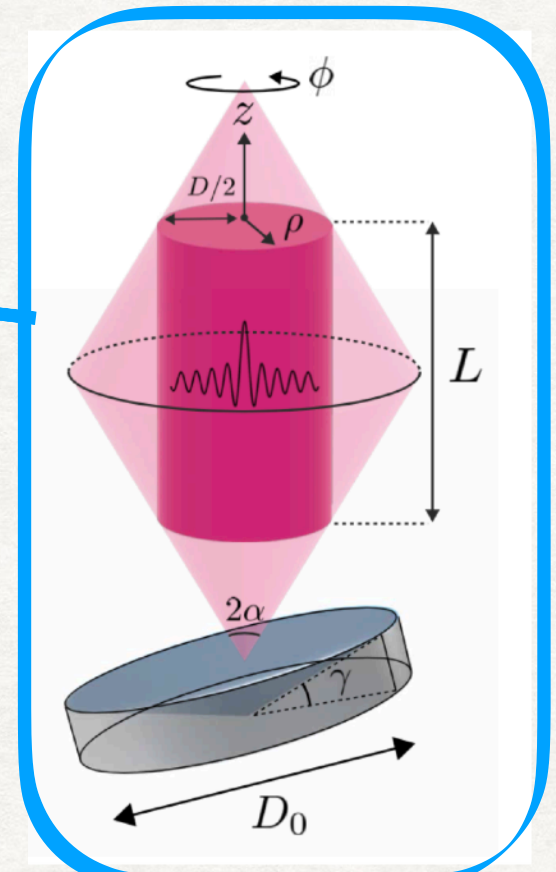
Gravitational waves from twisted beams

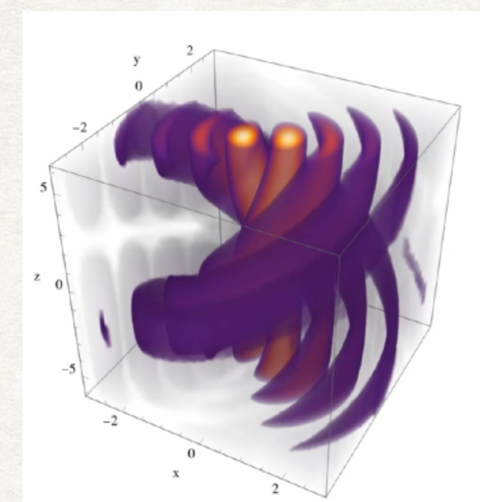
Generation of Bessel modes

E. Atonga, K.M, R. Aboushelbaya, A. Barrau et. al.. 2309.04191



Region over which twisted light is generated





Gravitational waves from twisted beams

- **Bessel modes can be modeled as an infinite superposition of plane waves:**

$$\vec{E}(\vec{x},t) = \frac{1}{2\pi}\int_0^{2\pi} i^l E_0 e^{i(k_\mu(\phi)x^\mu + l\phi)}\hat{n}(\phi)d\phi$$

$\phi \equiv \arctan(y/x)$
Azimuthal position on the conical surface

- **Bessel mode E-field orbital angular momentum**

$$\vec{J} = \iint \frac{\epsilon_0 E_0}{c}\frac{l}{k} J_l^2(\beta\rho)\hat{z}dS$$

l : OAM parameter
J_l : Bessel functions

Gravitational waves from twisted beams

- **Bessel modes can be modeled as an infinite superposition of plane waves:**

$$\vec{E}(\vec{x},t) = \frac{1}{2\pi}\int_0^{2\pi} i^l E_0 e^{i(k_\mu(\phi)x^\mu + l\phi)}\hat{n}(\phi)d\phi$$

$\phi \equiv \arctan(y/x)$
Azimuthal position on the conical surface

- **Bessel mode E-field orbital angular momentum**

$$\vec{J} = \iint \frac{\epsilon_0 E_0}{c}\frac{l}{k}J_l^2(\beta\rho)\hat{z}dS$$

l : OAM parameter
J_l : Bessel functions

- **Fields components**

$$E_x = \frac{E_0}{2}\left[J_{l+1}(\beta\rho)\sin(\omega t - k_z z + (l+1)\phi) - J_{l-1}(\beta\rho)\sin(\omega t - k_z z + (l-1)\phi)\right]$$
$$E_y = -\frac{E_0}{2}\left[J_{l+1}(\beta\rho)\cos(\omega t - k_z z + (l+1)\phi) + J_{l-1}(\beta\rho)\cos(\omega t - k_z z + (l-1)\phi)\right]$$
$$E_z = 0$$

ω : **Pulse frequency**

α : **Half-cone angle**

β : **Beam waist**

Gravitational waves from twisted beams

- **Bessel modes can be modeled as an infinite superposition of plane waves:**

$$\vec{E}(\vec{x},t) = \frac{1}{2\pi}\int_0^{2\pi} i^l E_0 e^{i(k_\mu(\phi)x^\mu + l\phi)}\hat{n}(\phi)d\phi$$

$\phi \equiv \arctan(y/x)$
Azimuthal position on the conical surface

- **Bessel mode E-field orbital angular momentum**

$$\vec{J} = \iint \frac{\epsilon_0 E_0}{c}\frac{l}{k} J_l^2(\beta\rho)\hat{z}dS$$

l : OAM parameter
J_l : Bessel functions

- **Fields components**

$$E_x = \frac{E_0}{2}\left[J_{l+1}(\beta\rho)\sin(\omega t - k_z z + (l+1)\phi) - J_{l-1}(\beta\rho)\sin(\omega t - k_z z + (l-1)\phi)\right]$$

$$E_y = -\frac{E_0}{2}\left[J_{l+1}(\beta\rho)\cos(\omega t - k_z z + (l+1)\phi) + J_{l-1}(\beta\rho)\cos(\omega t - k_z z + (l-1)\phi)\right]$$

$$E_z = 0$$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$B_x = \cos(\alpha)\frac{E_0}{2}\left[J_{l+1}(\beta\rho)\cos(\omega t - k_z z + (l+1)\phi) + J_{l-1}(\beta\rho)\cos(\omega t - k_z z + (l-1)\phi)\right]$$

$$B_y = \cos(\alpha)\frac{E_0}{2}\left[J_{l+1}(\beta\rho)\sin(\omega t - k_z z + (l+1)\phi) - J_{l-1}(\beta\rho)\sin(\omega t - k_z z + (l-1)\phi)\right]$$

$$B_z = \sin(\alpha)\frac{E_0}{c}J_l(\beta\rho)\cos(\omega t - k_z z + l\phi)$$

ω : **Pulse frequency**
α : **Half-cone angle**
β : **Beam waist**

Gravitational waves from twisted beams

- **Stress-energy tensor of the system** *E. Atonga, K.M, R. Aboushelbaya, A. Barrau et. al.. 2309.04191*

$$T^{\mu\nu} = \begin{pmatrix} u & \vec{N}/c \\ \vec{N}/c & -\sigma_{i,j} \end{pmatrix}$$

$$u = \frac{\epsilon_0 c}{2}(E^2 + c^2 B^2)$$

EM field energy density

$$\vec{N} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Poynting vector

$$\sigma_{ij} = \epsilon_0 c(E_i E_j + c^2 B_i B_j) - u\delta_{i,j}$$

Maxwell tensor

Gravitational waves from twisted beams

- **Stress-energy tensor of the system** *E. Atonga, K.M, R. Aboushelbaya, A. Barrau et. al.. 2309.04191*

$$T^{\mu\nu} = \begin{pmatrix} u & \vec{N}/c \\ \vec{N}/c & -\sigma_{i,j} \end{pmatrix}$$

$$u = \frac{\epsilon_0 c}{2}(E^2 + c^2 B^2)$$

EM field energy density

$$\vec{N} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Poynting vector

$$\sigma_{ij} = \epsilon_0 c(E_i E_j + c^2 B_i B_j) - u\delta_{i,j}$$

Maxwell tensor

- **Associated spacetime deformations**

$$\bar{h}^{\mu\nu}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\int_{\mathcal{V}} T^{\mu\nu}\left(t - \frac{r}{c} + \frac{\vec{x'}\cdot\hat{n}}{c}, \vec{x'}\right) d^3x'$$

$$\begin{aligned}
\bar{h}_D &\equiv \frac{1}{2}\bar{h}_0(r)\sin^2(\alpha)\Gamma_l(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\sin(\psi_q) \ , \\
\bar{h}_{ZZ} &\equiv 2\bar{h}_0(r)\left[1 + \cos^2(\alpha)\right]\Gamma_l(\theta)\mathrm{Sinc}\left[\eta(\theta)\right] \\
&\quad \times \cos(\psi_q) \ , \\
\bar{h}_+^{(Q)} &\equiv \bar{h}_0(r)\sin^2(\alpha)\Gamma_Q(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\cos(\psi_q) \ , \\
\bar{h}_\times^{(Q)} &\equiv \bar{h}_0(r)\sin^2(\alpha)\Gamma_Q(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\sin(\psi_q) \ , \\
\bar{h}_{XZ}^{\pm,(s)} &\equiv \frac{1}{4}\bar{h}_0(r)\sin(2\alpha)\Lambda_s^{\pm}(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\cos(\psi_q) \\
\bar{h}_{YZ}^{\pm,(s)} &\equiv \frac{1}{4}\bar{h}_0(r)\sin(2\alpha)\Lambda_s^{\pm}(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\sin(\psi_q) \\
\bar{h}_N &\equiv 2\bar{h}_0(r)\cos(\alpha)\Gamma_l(\theta)\mathrm{Sinc}\left[\eta(\theta)\right]\cos(\psi_q) \ .
\end{aligned}$$

$$\begin{aligned}
\bar{h}_0(r) &\equiv \frac{4\pi\epsilon_0 c E_0^2 G L}{\beta^2 c^5 r} \ , \\
\psi_q(t, r) &\equiv 2\omega(t - r/c) + 2q(\phi - \pi/2) \ , \\
\Gamma_q(\theta) &\equiv \int_0^{\frac{D\beta}{2}} \tau J_q^2(\tau) J_{2q}\left(\frac{\omega\tau}{c\beta}\sin(\theta)\right) d\tau \ , \\
\Lambda_s^{\pm}(\theta) &\equiv \int_0^{\frac{D\beta}{2}} \tau J_l(\tau) J_{l\pm1}(\tau) J_s\left(\frac{\omega\tau}{c\beta}\sin(\theta)\right) d\tau \\
\eta(\theta) &\equiv \frac{\omega L}{c}\left[\cos(\theta) - \cos(\alpha)\right] \ ,
\end{aligned}$$

L : Interaction length

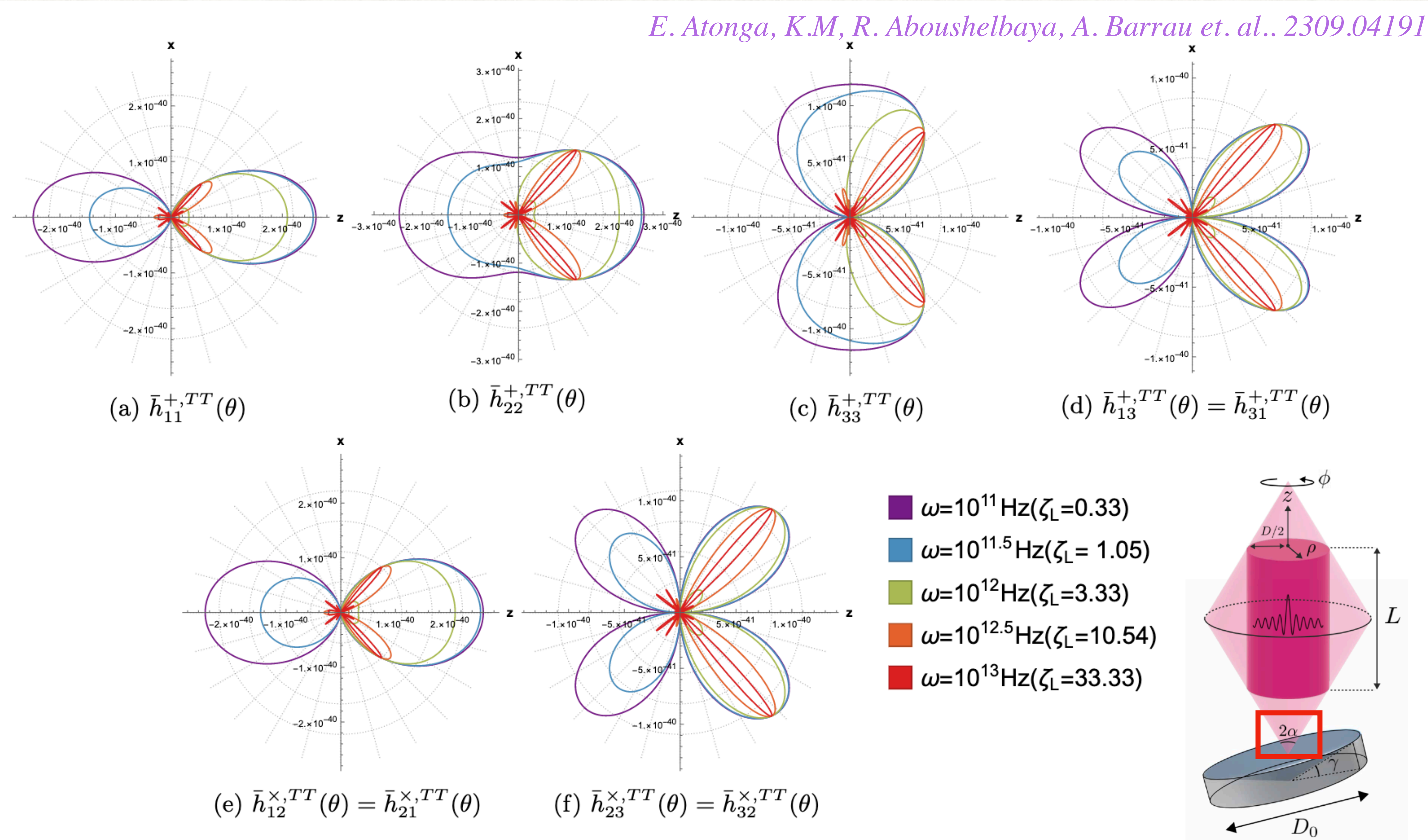
ω : Pulse frequency

α : Half-cone angle

β : Beam waist

D : Interaction region width

Gravitational waves from twisted beams



When $\zeta_L \equiv \omega L/c > 1$ $\longrightarrow$ **Beaming effect towards the half-cone angle α**

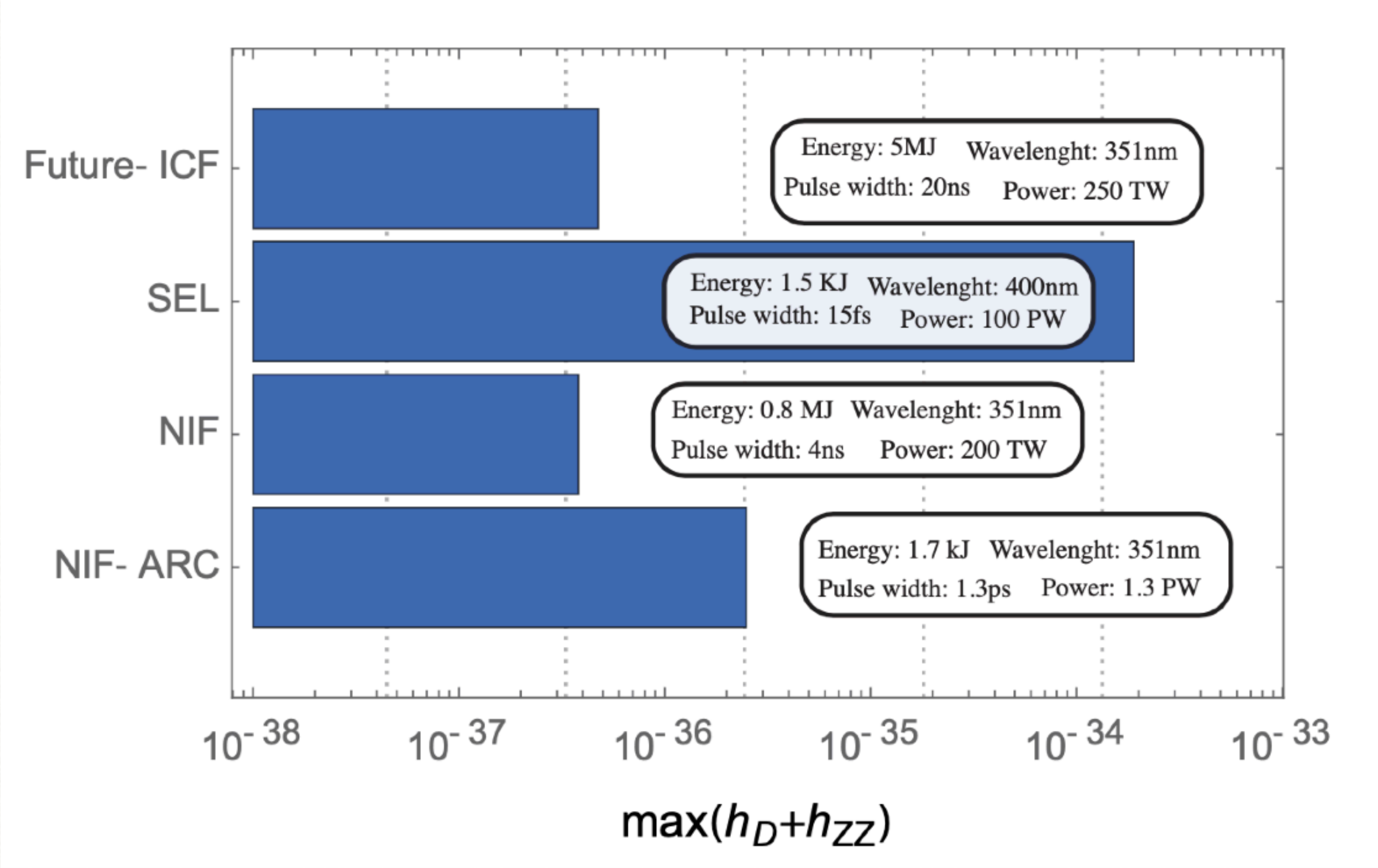
Gravitational waves from twisted beams

- **Frequency**

$$\omega_{GW} = 2\omega_{\text{pulse}} \simeq 8.55 \times 10^{14} Hz$$

For a NIF's 351 nm laser pulse

- **Strain**



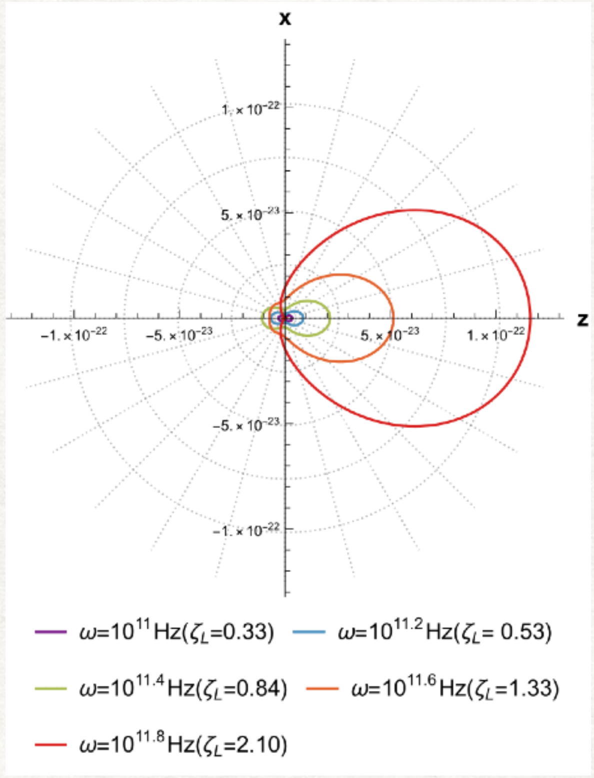
$$h \sim 1.90 \times 10^{-36} \left(\frac{P}{1\,\mathrm{PW}}\right)\left(\frac{r}{1.36\,\mu\mathrm{m}}\right)^{-1}$$

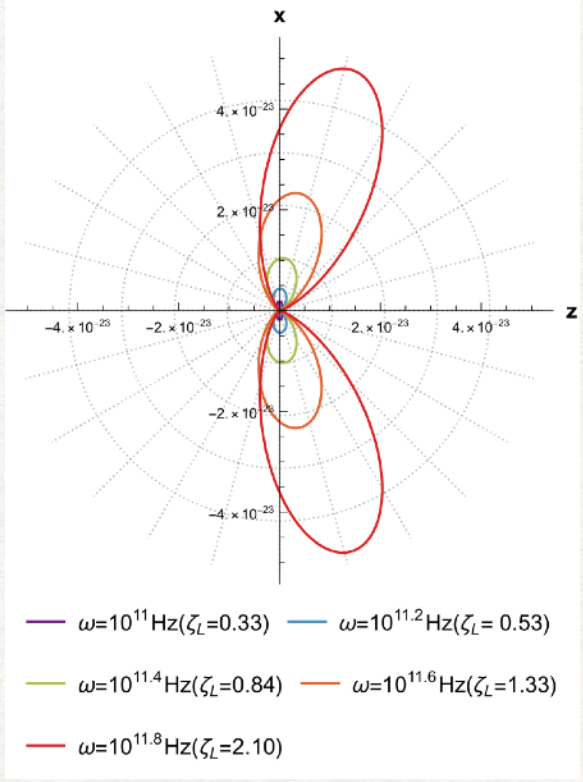
Gravitational waves from twisted beams

- **Power radiated away by the system:**

$$\frac{dP}{d\Omega} = \frac{r^2 c^3}{32\pi G} \left\langle \frac{\partial h_{ij}^{TT}}{\partial t} \frac{\partial h^{ij,TT}}{\partial t} \right\rangle ,$$

Emission mainly along the optical axis





No emission along the optical axis

At the SEL facility: $P \sim 9.4 \times 10^{-5}$ W.m^{-2} $> P_{GW15094} \sim 3.6 \times 10^{-6}$ W.m^{-2}

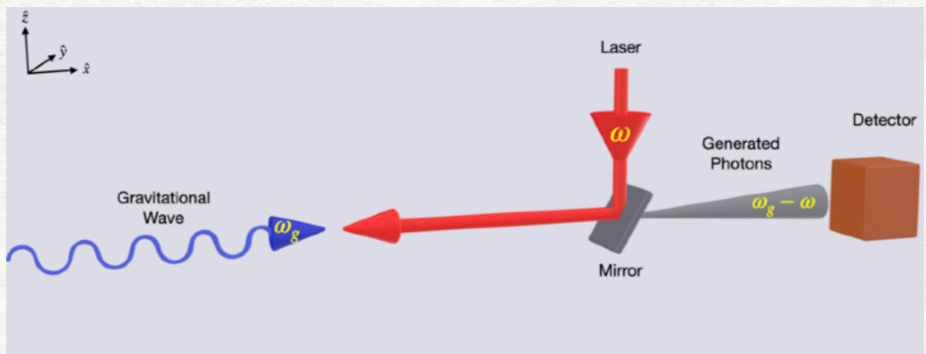
Close to the source — *At the detector location*

Gravitational waves from twisted beams

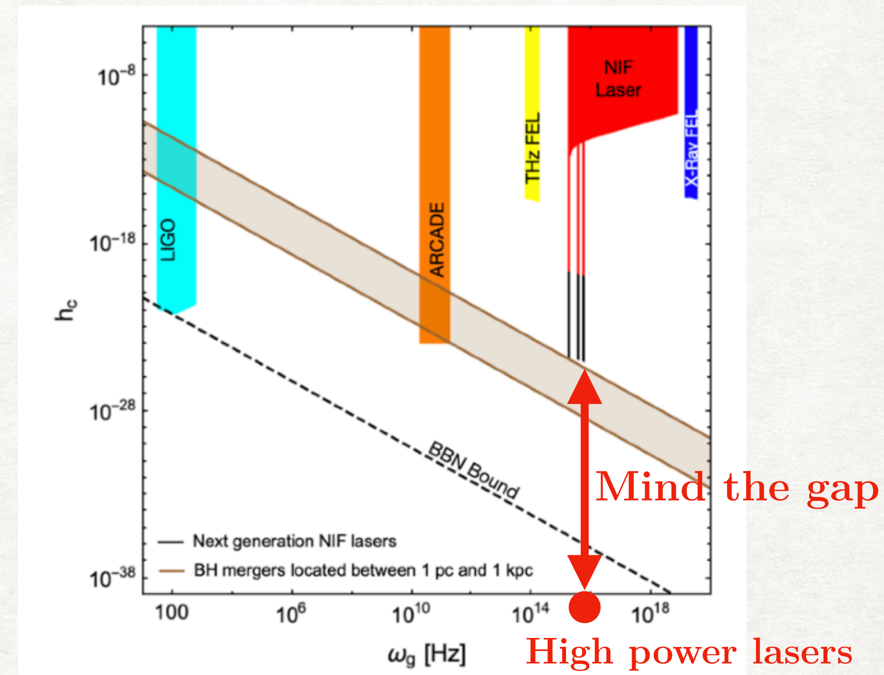
Detection scheme proposals at optical frequencies

- **With high-energy pulsed lasers**

  G. Vacalis, et al., arXiv:2301.08163, Class. Quantum Grav. 40 (2023)



$$h_{\min}^{\text{opt,fut}} = 6.3 \times 10^{-26} \left(\frac{8.6 \times 10^{8}}{n_s}\right)^{1/2} \left(\frac{8.5 \times 10^{14}\,\text{Hz}}{\omega_g/2\pi}\right)^{1/2} \left(\frac{20\,\text{ns}}{\tau}\right) \left(\frac{1.8\,\text{mJ}}{E_{\text{las}}}\right)^{1/2}$$

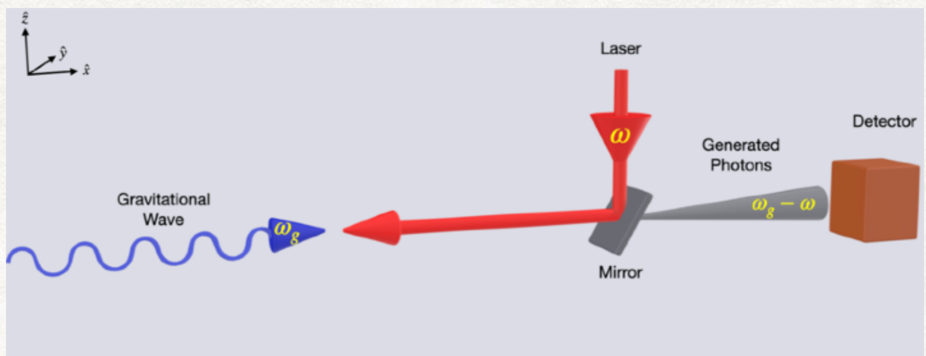


Gravitational waves from twisted beams

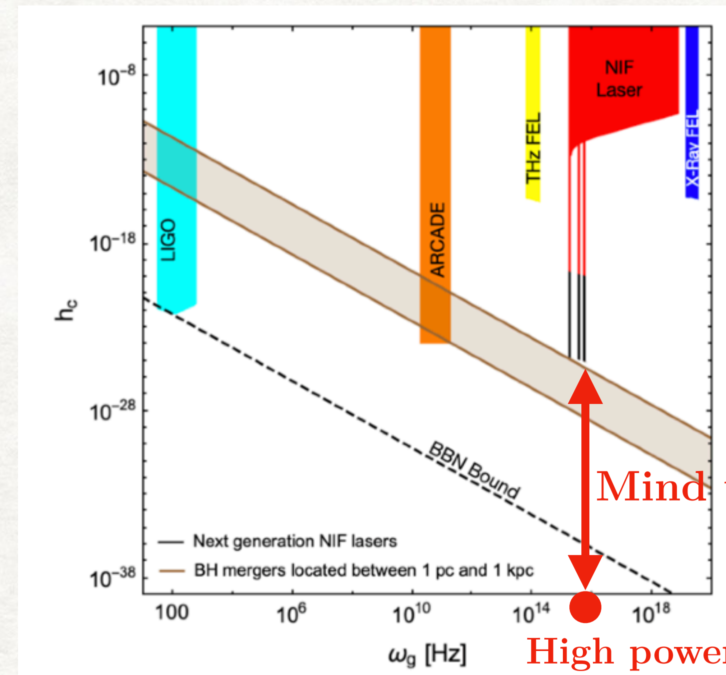
Detection scheme proposals at optical frequencies

- **With high-energy pulsed lasers**

  G. Vacalis, et al., arXiv:2301.08163, Class. Quantum Grav. 40 (2023)

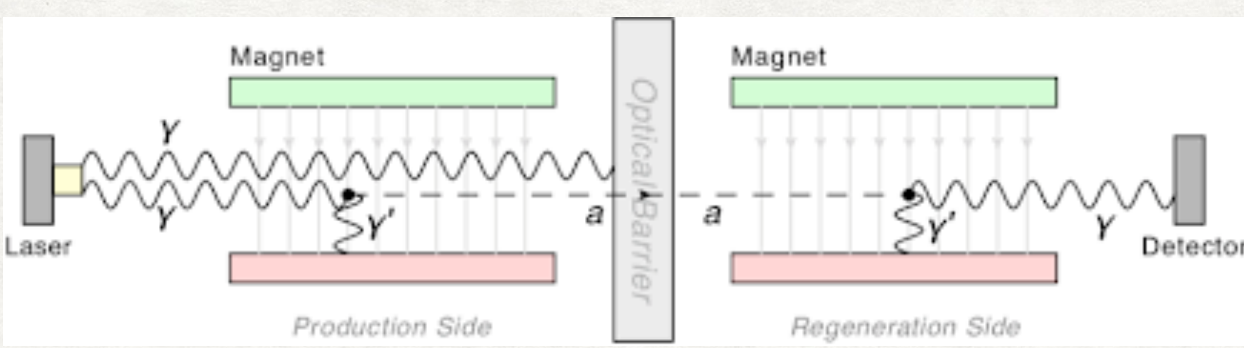


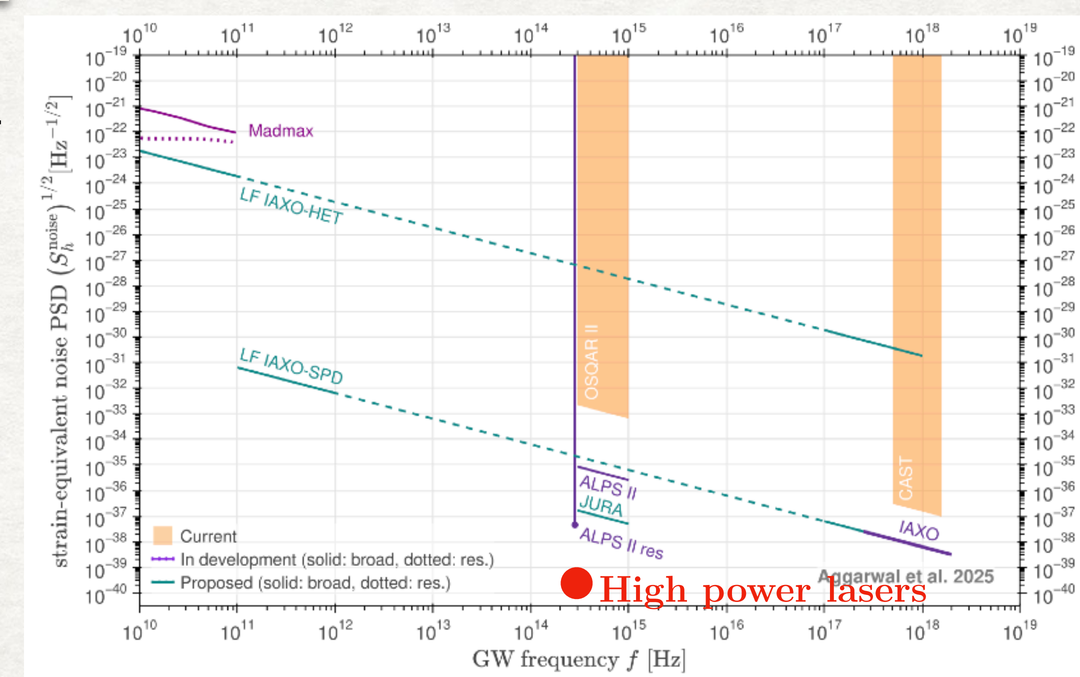
$$h_{\min}^{\text{opt,fut}} = 6.3 \times 10^{-26} \left(\frac{8.6 \times 10^{8}}{n_s}\right)^{1/2} \left(\frac{8.5 \times 10^{14}\,\text{Hz}}{\omega_g/2\pi}\right)^{1/2} \left(\frac{20\,\text{ns}}{\tau}\right) \left(\frac{1.8\,\text{mJ}}{E_{\text{las}}}\right)^{1/2}$$



- **Using regeneration cavities in LSW experiments**

  Review paper: arXiv:2501.11723





Credits: R. Battesti et. al., High magnetic fields for fundamental physics. Physics reports, 765-766 (2018)

Gravitational waves from high-power twisted beams

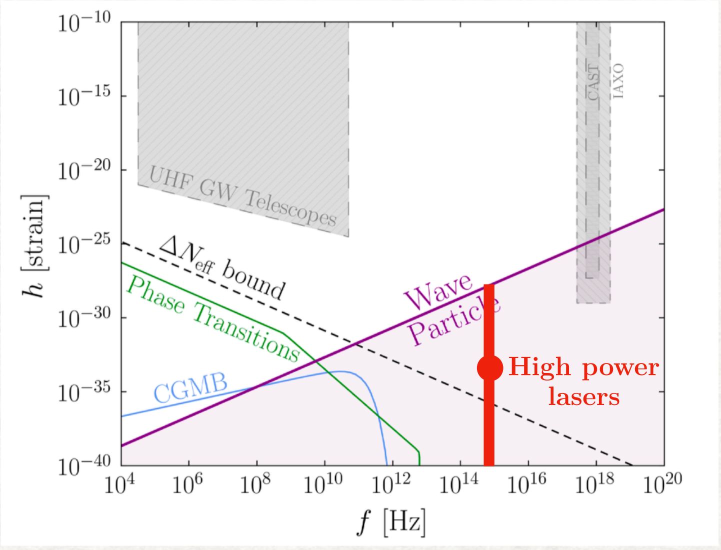
Take-away messages

- **Twisted beams of light such as Bessel beams do generate gravitational waves**

✓ {
- **The optical setup provides a very good control over the GWs properties**
  (frequency, direction of emission, polarisation states)
- **Subtle and unexpected effects do appear**
  (Beaming effect towards the half-cone angle alpha, ...)

✗
- **Many orders of magnitude are missing for any detection to be expected**

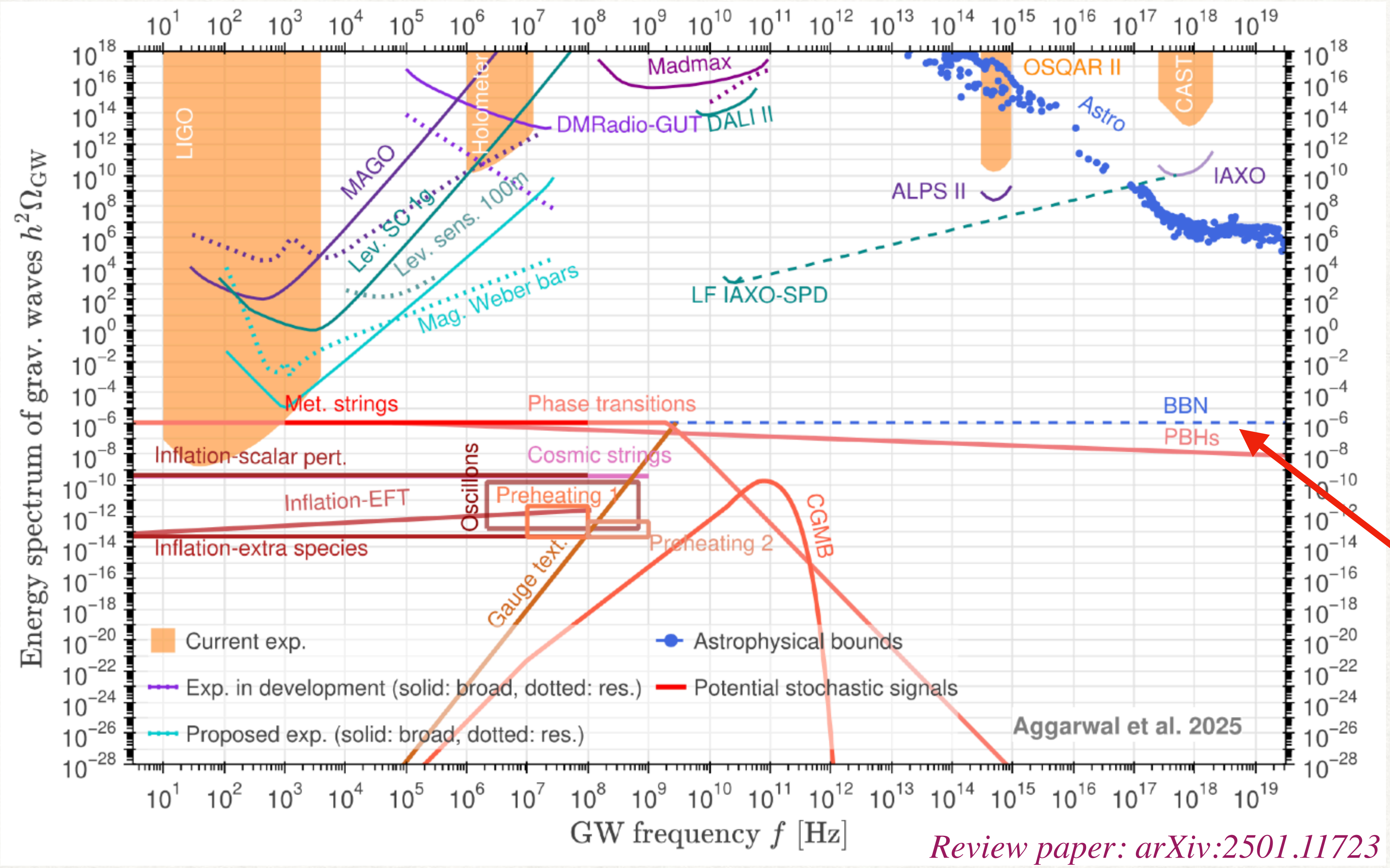
- **Quantum description of the emitted signal?**



From D. Carney, V. Domcke, N. L. Rodd, arXiv:2308.12988

Thank you!

Few words about stochastic GW backgrounds



Review paper: arXiv:2501.11723

The total energy density of a GW background decays as $\rho_{GW} \propto a^{-4}$.

↓

Additional source of radiation in the Universe

↓

Stringent constraints coming from CMB and BBN

↓

When combined:

$$\Omega_{GW,0}h^2 < 1.1 \times 10^{-6}$$

with $h \equiv \frac{H_0}{100\text{km.s}^{-1}.\text{Mpc}^{-1}} \approx 0.7$

No experiment operating at high frequencies gets even close to the BBN/CMB bound.

Gravitational waves from twisted beams

$$\bar{h}^{TT}_{\mu\nu} = \bar{h}^{D,TT}_{\mu\nu} + \bar{h}^{ZZ,TT}_{\mu\nu} + \bar{h}^{+,TT}_{\mu\nu} + \bar{h}^{\times,TT}_{\mu\nu}$$

$$\bar{h}^{D,TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2(\theta)\,[1-\cos(2\theta)] & 0 & \cos(\theta)\sin(\theta)\,[\cos(2\theta)-1] \\ 0 & 0 & \cos(2\theta)-1 & 0 \\ 0 & \cos(\theta)\sin(\theta)\,[\cos(2\theta)-1] & 0 & \sin^2(\theta)\,[1-cos(2\theta)] \end{pmatrix} \bar{h}_D$$

$$\bar{h}^{ZZ,TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2(\theta)\,[1-\cos(2\theta)] & 0 & \cos(\theta)\sin(\theta)\,[\cos(2\theta)-1] \\ 0 & 0 & \cos(2\theta)-1 & 0 \\ 0 & \cos(\theta)\sin(\theta)\,[\cos(2\theta)-1] & 0 & \sin^2(\theta)\,[1-cos(2\theta)] \end{pmatrix} \bar{h}_{ZZ}$$

$$\bar{h}^{+,TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\cos^2(\theta)\,[\cos^2(\theta)+1] & 0 & -\frac{1}{2}\sin(\theta)\cos(\theta)\,[\cos^2(\theta)+1] \\ 0 & 0 & -\frac{1}{2}\,[1+\cos^2(\theta)] & 0 \\ 0 & -\frac{1}{2}\sin(\theta)\cos(\theta)\,[\cos^2(\theta)+1] & 0 & \frac{1}{2}\sin^2(\theta)\,[\cos^2(\theta)+1] \end{pmatrix} \bar{h}_+$$

$$\bar{h}^{\times,TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \cos^2(\theta) & 0 \\ 0 & \cos^2(\theta) & 0 & -\sin(\theta)\cos(\theta) \\ 0 & 0 & -\sin(\theta)\cos(\theta) & 0 \end{pmatrix} \bar{h}_\times .$$

$$\begin{aligned}
\bar{h}_+ &\equiv 2\bar{h}^{(2)}_+ + \bar{h}^{(l+1)}_+ + \bar{h}^{(l-1)}_+ , \\
\bar{h}_\times &\equiv 2\bar{h}^{(2)}_\times + \bar{h}^{(l+1)}_\times + \bar{h}^{(l-1)}_\times , \\
\bar{h}_{XZ} &\equiv \bar{h}^{+,(1)}_{XZ} + \bar{h}^{-,(1)}_{XZ} + \bar{h}^{+,(2l+1)}_{XZ} + \bar{h}^{-,(2l-1)}_{XZ} \\
\bar{h}_{YZ} &\equiv \bar{h}^{+,(1)}_{YZ} - \bar{h}^{-,(1)}_{YZ} + \bar{h}^{+,(2l+1)}_{YZ} - \bar{h}^{-,(2l-1)}_{YZ}
\end{aligned}$$

$$\begin{aligned}
\bar{h}_D &\equiv \frac{1}{2}\bar{h}_0(r)\sin^2(\alpha)\Gamma_l(\theta)\mathrm{Sinc}\,[\eta(\theta)]\sin(\psi_q) , \\
\bar{h}_{ZZ} &\equiv 2\bar{h}_0(r)\left[1+\cos^2(\alpha)\right]\Gamma_l(\theta)\mathrm{Sinc}\,[\eta(\theta)] \\
&\quad \times \cos(\psi_q) , \\
\bar{h}^{(Q)}_+ &\equiv \bar{h}_0(r)\sin^2(\alpha)\Gamma_Q(\theta)\mathrm{Sinc}\,[\eta(\theta)]\cos(\psi_q) , \\
\bar{h}^{(Q)}_\times &\equiv \bar{h}_0(r)\sin^2(\alpha)\Gamma_Q(\theta)\mathrm{Sinc}\,[\eta(\theta)]\sin(\psi_q) , \\
\bar{h}^{\pm,(s)}_{XZ} &\equiv \frac{1}{4}\bar{h}_0(r)\sin(2\alpha)\Lambda^\pm_s(\theta)\mathrm{Sinc}\,[\eta(\theta)]\cos(\psi_q) \\
\bar{h}^{\pm,(s)}_{YZ} &\equiv \frac{1}{4}\bar{h}_0(r)\sin(2\alpha)\Lambda^\pm_s(\theta)\mathrm{Sinc}\,[\eta(\theta)]\sin(\psi_q) \\
\bar{h}_N &\equiv 2\bar{h}_0(r)\cos(\alpha)\Gamma_l(\theta)\mathrm{Sinc}\,[\eta(\theta)]\cos(\psi_q) .
\end{aligned}$$

$$\begin{aligned}
\bar{h}_0(r) &\equiv \frac{4\pi\epsilon_0 c E_0^2 G L}{\beta^2 c^5 r} , \\
\psi_q(t,r) &\equiv 2\omega(t-r/c) + 2q(\phi-\pi/2) , \\
\Gamma_q(\theta) &\equiv \int_0^{\frac{D\beta}{2}} \tau J_q^2(\tau) J_{2q}\left(\frac{\omega\tau}{c\beta}\sin(\theta)\right) d\tau , \\
\Lambda^\pm_s(\theta) &\equiv \int_0^{\frac{D\beta}{2}} \tau J_l(\tau) J_{l\pm1}(\tau) J_s\left(\frac{\omega\tau}{c\beta}\sin(\theta)\right) d\tau \\
\eta(\theta) &\equiv \frac{\omega L}{c}[\cos(\theta)-\cos(\alpha)] ,
\end{aligned}$$

L : Interaction length

ω : Pulse frequency

α : Half-cone angle

β : Beam waist

D : Interaction region width