Vacuum energy density and strong gravity with running gravitational constant

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The scale dependence of the gravitational action ⟶ very large fluctuations of the metric on length scales of order the Planck length l_P, with high curvatures ⟶ spacetime foam.
However, everyday observations indicate that spacetime is nearly flat when viewed on standard scales (S.W. Hawking et al., Phys. Lett. 86 B (2), 175 (1979); Nucl. Phys. B170, 283 (1980))
Motivated by the above Refs., we are looking for a different recipe to get an almost flat background line-element for a foamlike-dominated geometry.

Static spacetime

Take the following static metric

$$ds^2 = -dt^2 + \frac{a^2 r^2}{a^2 r^2 - 1} dr^2 + r^2 d\Omega^2,$$

with $r > 1/a$, a being a positive constant, interpreted as an acceleration. The above geometry is curved, becoming Minkowskian for $ar \gg 1$. One finds that

$$R^b_{\ b} = -\frac{2}{a^2 r^4}, \quad K = \frac{12}{a^4 r^8}.$$

It is clear that an observer with $r =$ const. and 4-velocity $u^a = (1,0,0,0)$ is geodesic because $g_{tt} = -1$.

Anisotropic stress tensor

The stress tensor to be inserted on the r.h.s. of Einstein's equation and represented the source of curvature has the components:

$$T^t_{\ t} = -T^r_{\ r} = T^\theta_{\ \theta} = T^\varphi_{\ \varphi} = \frac{1}{8\pi a^2 r^4},$$

the others are vanishing.

$T^a{}_b$ can be generally written as

$$T^a{}_b = (\rho + p_t) u^a u_b + p_t g^a_b + (p_r - p_t) n^a n_b,$$

where ρ - the energy density, p_r - the radial pressure of the fluid, p_t - the transversal pressures, and

$$n^a = \left(0, \frac{\sqrt{a^2 r^2 - 1}}{ar}, 0, 0\right),$$

with $u^a n_a = 0$, $n_a n^a = 1$, $u^a u_a = -1$.

One obtains

$$\rho = T^a{}_b u^b u_a = -T^t{}_t = -\frac{1}{8\pi a^2 r^4} = -p_t$$

and $p_r = T^r{}_r = -\frac{1}{8\pi a^2 r^4}$

with $T^a{}_a = -2\rho = \frac{1}{4\pi a^2 r^4}$.

Radial geodesics

Using the Lagrangean method, we have

$$\mathcal{L} = -\left(\frac{ds}{d\lambda}\right)^2 = \frac{1}{2}\left(\dot{t}^2 - \frac{a^2 r^2}{a^2 r^2 - 1}\dot{r}^2\right), \ldots$$

with θ, φ - const.

The Euler-Lagrange equation are given by:

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0; \quad \frac{\partial \mathcal{L}}{\partial r} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0,$$

whence

$$\dot{t} = E, \quad \dot{r}^2 - r(a^2r^2-1)\ddot{r} = 0,$$

where E is the energy per unit mass of the test particle.

i) Null radial geodesics

For this case we have $\mathcal{L} = 0$, such that

$$\dot{t}^2 - \frac{a^2r^2}{a^2r^2-1}\dot{r}^2 = 0,$$

leading to

$$r(t) = \frac{1}{a}\sqrt{1+\frac{a^2t^2}{c^2}}, \quad r(0) = \frac{1}{a},$$

the well-known hyperbolic motion.

With the above solution, the equation containing $\ddot{r}$ is verified.

The test particle velocity is given by

$$v(t) = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}} < 1$$

ii) Timelike radial geodesics

We have now the additional equation

$$\dot{t}^2 - \frac{a^2r^2}{a^2r^2-1}\dot{r}^2 = 1,$$

which leads to

$$\frac{ar}{\sqrt{a^2r^2-1}}\dot{r} = \sqrt{E^2-1}, \quad \dot{r} = \frac{dr}{d\tau}$$

where τ is the proper time.

Finally one finds that

$$r(t) = \frac{1}{a}\sqrt{1+\frac{a^2(E^2-1)t^2}{E^2}},$$

where $a\sqrt{E^2-1}/E$ plays the role of an effective acceleration. (we have in fact $E^2=m^2$, where m is the particle rest mass. As a check, when $m=0$, we get exactly the geodesic for a massless particle.

Note that the ~~both~~ type of geodesics are given by a hyperbola, with a smaller acceleration for the time-like ones.

Applications in Microphysics

We had that the null geodesics $x(t)=(1/a)\sqrt{1+a^2t^2}$ become straight lines $x=t$ when a is very large. For a proton, the maximum acceleration is $a_{max}=c^2/\lambda_p \approx 10^{34}\,cm/s^2$, with $\lambda_p=\hbar/m_pc$.

From now on we assume the previous $a = a_{max}$ and any massless particle moves hyperbolically but after a

very short time interval taken from the moment of emission its motion becomes practically uniform.
In other words, a massless particle cannot acquire the velocity c instantaneously, just like a massive particle.
The origin of acceleration is supposed to be given by the quantum vacuum fluctuation (the so called spacetime foam), which generates the previous curved geometry, that becomes flat for $ar \gg 1$.
The Zeldovich vacuum energy density is given by

$$\varepsilon_{vac} = \frac{G m_p^2}{\lambda_p \cdot V} \simeq \frac{G m_p^6 c^4}{\hbar^4} \sim m_p^6$$

where V is the proton volume and $\varepsilon_{vac} \simeq 10^{-2}\,\mathrm{erg/cm^3}$. In the strong gravity regime, with $G_S = \hbar c / m_p^2$, one obtains

$$\varepsilon_{vac}^{(S)} = \frac{G_S m_p^2}{\lambda_p \cdot V} \simeq \frac{m_p^4 c^5}{\hbar^3} \sim m_p^4, \ldots$$

and we get $\varepsilon_{vac}^{(S)} \simeq 10^{37}\,\mathrm{erg/cm^3}$, a more realistic value. A similar dependence of $\varepsilon_{vac}^{(S)}$ on the particle mass has been recently obtained by

André LeClair (arXiv: 2509.02636),

for his vacuum particle "zeton".

Note also that Zeldovich (Soviet Phys. Uspekhi 11, 381, 1968) and M. Ali et al. (2507.22096) proposed a connection between Cosmology (Universe radius) and the proton radius.

$$\left(\frac{R_U}{r_p}\right)^3 = \left(\frac{R_U}{l_p}\right)^2 \simeq 10^{122},$$

where l_p is the Planck length.

The energy scale dependence of G was also proposed by LeClair.

If one computes now the energy in the strong gravity regime, obtained from $\rho(r)$, we get

$$W = \int_{1/a}^{\infty} \rho\sqrt{-g}\, d^3r = -\frac{1}{2a}\int_{1/a}^{\infty}\frac{dr}{r\sqrt{a^2r^2-1}}$$

changing to $y \equiv \sqrt{a^2r^2-1}$, we have

$$W = -\frac{1}{2a}\int_0^{\infty}\frac{dy}{1+y^2} = -\frac{\pi}{4}m_pc^2;$$

with $|W|$ of the order the proton rest energy.

Let us consider a numerical example for the velocity $at/\sqrt{1+a^2t^2}$.

With $a = 10^{34}\,\mathrm{cm/s^2}$ and $t \simeq 10^{-23}\,\mathrm{s}$, we have $v = 3 \cdot 10^{11}/\sqrt{109} \simeq 2{,}87 \cdot 10^{10} < c$, but, however, close to c, even though the time interval was very short.

(a consequence of a_{max})

Another example refers to the radial pressure $p_r = -c^8/8\pi G_s a^2 r^4$, with G_s (H.C., arXiv:2512.02077) the strong gravitational constant.

If we compute p_r outside a proton but very close to it (at $r \simeq 10^{-13}\,\mathrm{cm}$) and with $a = a_{max}$, we get

$$p_r \simeq -10^{35}\,\mathrm{erg/cm^3}$$

which is (minus) the bag constant B, well known from QCD.

In other words, the interior and exterior pressures match at the proton surface.

Traversable wormhole

We will show now that the previous geometry represents a traversible WH, with some specific conditions satisfied.

- the redshift function $\phi(r) = 0$

- the shape function $b(r) = 1/a^2 r$.

We have $r_{min} = 1/a$, and we take the throat radius $r = 1/a \equiv r_0$.

- the flare out condition $r b'(r) - b(r) = = -\frac{1}{a^2 r} < 0$ is ~~fut~~ fulfilled for our metric.

- $b(r_0) = r_0$, $b'(r_0) = -1 < 1$ are valid.

- $\rho + p_r = -\frac{1}{4\pi a^2 r^4} < 0 \longrightarrow$ exotic matter (to maintain the throat open) $\longrightarrow$ NEC violated.

- the metric is asimptotically flat.

Our goal now is to express the stress tensor generating curvature in terms of an antiscalar field (ghost field), M. Makukov et. al., Found. Phys. 50, 1346 (2020), arXiv: 2009.08655; H. Yilmaz, Am. J. Phys. 43, 319 (1975).

The only nonzero component of the Ricci tensor is

$$R_{rr} = -\frac{2}{r^2(a^2 r^2 - 1)},$$

which can be written as

$$R_{rr} = -2\psi_{,r}\psi_{,r},$$

where $\psi(r)$ is the scalar field, with

$$\psi_{,r} \equiv \nabla_r \psi = \frac{1}{r\sqrt{a^2r^2-1}},$$

and by integration

$$\psi(r) = \arctan\sqrt{a^2r^2-1}$$

with

$$\Box\psi(r) \equiv g^{ab}\nabla_a\nabla_b\psi(r) = 0,$$

that means $\psi(r)$ is massless.

From above we get that

$$\kappa T_{ab} = -2\left(\nabla_a\psi\,\nabla_b\psi - \frac{1}{2}g_{ab}\,g^{cd}\,\nabla_c\psi\,\nabla_d\psi\right)$$

Because of the minus sign on the r.h.s. of the above equation, ψ is an antiscalar field (P. Boonserm et al., arXiv: 1805.03781; J.A. Gonzales et al., arXiv: gr-qc/0806.0608).

One may conclude that the null geodesics are hyperbolae due to the repulsive properties of the exotic matter generating our geometry.