Metric expansions close to an initial isotropic singularity in a Bianchi I space-time with kinetic matter

Ho, Lee, Ernesto Nungesser, John Stalker, Paul Tod

Universidad Politécnica de Madrid

em.nungesser@upm.es

Abstract

We study massless solutions to the Einstein equations coupled to different matter models with a magnetic field and a conformal gauge singularity assuming spatial homogeneity with three commuting spatial translations. We show that there are no solutions in the case that the matter model is a radiation fluid. If the matter is described via kinetic theory we obtain that there exist unique solutions to the Einstein-Vlasov system and the Einstein-Boltzmann system for a certain range of soft potentials. For both the Vlasov and the Boltzmann case we also obtain asymptotic expansions close to the initial conformal gauge singularity.

1 The equations

The Bianchi I physical metric is

$$\tilde{g} = -dt^2 + \tilde{a}_{ij}dx^i dx^j, \tag{1}$$

with $i, j, \ldots = 1, 2, 3$ and the t-coordinate labelled 0, and where $\tilde{a}_{ij}$ is a function only of $x^0 = t$. The metric is tilded so that the rescaled metric, with which we mainly work, can be untilded. Set $\tilde{b}^{ij}$ inverse to $\tilde{a}_{ij}$ so that

$$\tilde{b}^{ik}\tilde{a}_{kj} = \delta^i_j. \tag{2}$$

Introduce the rate of change, which is a constant multiple of the second fundamental form, and its trace as

$$\tilde{k}_{ij} := \frac{d}{dt}\tilde{a}_{ij}, \quad \tilde{k} := \tilde{b}^{ij}\tilde{k}_{ij}. \tag{3}$$

We obtain

$$\tilde{R}_{00} = -\frac{1}{2}\frac{d\tilde{k}}{dt} - \frac{1}{4}\tilde{b}^{im}\tilde{b}^{jn}\tilde{k}_{ij}\tilde{k}_{mn}, \tag{4}$$
$$\tilde{R}_{0i} = 0, \tag{5}$$
$$\tilde{R}_{ij} = \frac{1}{2}\frac{d}{dt}\tilde{k}_{ij} + \frac{1}{4}\tilde{k}\tilde{k}_{ij} - \frac{1}{2}\tilde{k}_{im}\tilde{b}^{mn}\tilde{k}_{nj}, \tag{6}$$

and

$$\tilde{G}_{00} = \tilde{R}_{00} - \frac{1}{2}\tilde{g}_{00}\tilde{R} = \frac{1}{8}\tilde{k}^2 - \frac{1}{8}\tilde{b}^{im}\tilde{b}^{jn}\tilde{k}_{mn}\tilde{k}_{ij}. \tag{7}$$

1.1 The various sources

We'll be interested in three kinds of source, all of which have a trace-free stress tensor and we will also allow a cosmological constant.

1.1.1 The radiation perfect fluid

The first kind of source, which is instructive but turns out to have no interesting solutions, is the radiation perfect fluid, for which ecomposing in the coordinate basis, we have

$$\tilde{T}^{\text{rad}}_{00} = \tilde{\rho}, \tilde{T}^{\text{rad}}_{0i} = 0, \tilde{T}^{\text{rad}}_{ij} = \frac{1}{3}\tilde{\rho}\tilde{a}_{ij}. \tag{8}$$

1.1.2 The source-free magnetic field

The second kind of source is source-free magnetic field. In the coordinate basis, a purely magnetic, homogeneous Maxwell field can be expressed in in terms of the magnetic field vector h as

$$\tilde{T}^{\text{max}}_{00} = \frac{1}{4\pi\tilde{a}}\tilde{a}_{ij}h^i h^j, \tilde{T}^{\text{max}}_{0i} = 0, \tilde{T}^{\text{max}}_{ij} = \frac{1}{4\pi\tilde{a}}(\tilde{a}_{ij}\tilde{a}_{mn} - 2\tilde{a}_{im}\tilde{a}_{jn})h^m h^n. \tag{9}$$

1.1.3 The kinetic part

The third source is matter satisfying the assumptions of kinetic theory. The distribution function $f = f(t, p_i)$ is independent of x^i, to be consistent with the Bianchi I symmetry. The distribution function on the cotangent bundle is supported on the null-cone bundle with fibre N_x. Thus the stress tensor is

$$\tilde{T}^{\text{kt}}_{\alpha\beta} = \int_{N_x} f p_\alpha p_\beta \tilde{\omega}_p, \tag{10}$$

where $\tilde{\omega}_p$ is the invariant volume measure and the distribution function satisfies the Boltzmann equation

$$\mathcal{L}_{\tilde{g}}(f) = \tilde{C}(f, f). \tag{11}$$

We shall here consider the scattering cross-sections

$$\tilde{\sigma} = \tilde{h}^{-\gamma}, \tag{12}$$

where $\tilde{h}$ is the physical relative momentum, which is

$$\tilde{h} = \sqrt{-2\tilde{g}^{\alpha\beta}p_\alpha q_\beta}, \tag{13}$$

for a collision between massless particles of momenta p_α, q_β, and γ is a real number in the range $1 < \gamma < 2$. It's worth noting here that for Bianchi I metrics there is a great simplification in the Liouville operator $\mathcal{L}$ which becomes just

$$\mathcal{L}_{\tilde{g}}f = \tilde{p}^0\frac{\partial f}{\partial t}, \tag{14}$$

where

$$\tilde{p}^0 = (\tilde{b}^{mn}p_m p_n)^{1/2}. \tag{15}$$

The components of the kinetic theory stress tensor are

$$\tilde{T}^{\text{kt}}_{00} = \frac{1}{\sqrt{\tilde{a}}}\int f(\tilde{b}^{mn}p_m p_n)^{1/2}d^3p, \tag{16}$$
$$\tilde{T}^{\text{kt}}_{0i} = -\frac{1}{\sqrt{\tilde{a}}}\int f p_i d^3p, \tag{17}$$
$$\tilde{T}^{\text{kt}}_{ij} = \frac{1}{\sqrt{\tilde{a}}}\int f p_i p_j(\tilde{b}^{mn}p_m p_n)^{-1/2}d^3p. \tag{18}$$

Main results

Theorem 1. Let $f_0 \geq 0$ be a smooth function with compact support in $\mathbb{R}^3 \setminus \{0\}$ and $h \in \mathbb{R}^3$. Suppose that f_0 is not identically zero and satisfies the constraints. Then, there exists a unique Bianchi I solution $\tilde{a}_{ij}, \tilde{k}_{ij} \in C^0((0, T]; S_2(\mathbb{R}^3))$ and $f \in C^1((0, T]; L^1(\mathbb{R}^3))$ to the massless (unrescaled) Einstein-Vlasov system with a magnetic field with an initial conformal gauge singularity . Furthermore, the solutions have the following asymptotics as $t \to 0^+$:

$$\tilde{a}_{ij} = 2ta_{0ij} + o(t^{\frac{3}{2}}), \tag{19}$$
$$\tilde{k}_{ij} = 2a_{0ij} + o(t^{\frac{1}{2}}). \tag{20}$$

where a_{0ij} is a constant which only depends on f_0 and h. If in addition $a_{0ij}h^i h^j < \frac{1}{6}$, then we even have unique differentiable solutions $\tilde{a}_{ij}, \tilde{k}_{ij} \in C^1((0, T]; S_2(\mathbb{R}^3))$ and $f \in C^1((0, T]; L^1(\mathbb{R}^3))$ with the following asymptotics as $t \to 0^+$:

$$\tilde{a}_{ij} = 2ta_{0ij} + o(t^2), \tag{21}$$
$$\tilde{k}_{ij} = 2a_{0ij} + o(t). \tag{22}$$

The proof will be obtained using the conformally rescaled variables which will be done in the next section. Concerning solutions of the Einstein-Boltzmann system with a magnetic field our main result is as follows:

Theorem 2. Let $f_0 \geq 0$ be a smooth function with compact support in $\mathbb{R}^3 \setminus \{0\}$. Suppose that f_0 is not identically zero and satisfies the constraints. Then, there exists a unique Bianchi I solution $\tilde{a}_{ij}, \tilde{k}_{ij} \in C^0((0, T]; S_2(\mathbb{R}^3))$ and $0 \leq f \in C^1((0, T]; L^1(\mathbb{R}^3))$ to the massless (unrescaled) Einstein-Boltzmann system with a magnetic field and an initial conformal gauge singularity for the scattering cross-sections in (12) such that f converges to f_0 in L^1 as $t \to 0^+$. Furthermore, the solutions have the following asymptotics as $t \to 0^+$:

$$\tilde{a}_{ij} = \mathcal{A}_{ij}t + \mathcal{B}_{ij}t^{\frac{\gamma+1}{2}} + o(t^{\frac{\gamma+1}{2}}), \tag{23}$$
$$\tilde{k}_{ij} = \mathcal{A}_{ij} + \frac{\gamma+1}{2}\mathcal{B}_{ij}t^{\frac{\gamma-1}{2}} + o(t^{\frac{\gamma-1}{2}}), \tag{24}$$

with

$$\mathcal{A}_{ij} = 2a_{0ij}, \quad \mathcal{B}_{ij} = \frac{1}{\gamma-1}2^{\frac{\gamma+1}{2}}a_{1ij}, \tag{25}$$

where a_{0ij} and a_{1ij} are constants which only depend on f_0 and h. If in addition $a_{0ij}h^i h^j < \frac{1}{6}$, there even exists a unique differentiable Bianchi I solution $\tilde{a}_{ij}, \tilde{k}_{ij} \in C^1((0, T]; S_2(\mathbb{R}^3))$ with asymptotics as $t \to 0^+$:

$$\tilde{a}_{ij} = \mathcal{A}_{ij}t + \mathcal{B}_{ij}t^{\frac{\gamma+1}{2}} + \mathcal{C}_{ij}t^\gamma + o(t^\gamma), \tag{26}$$
$$\tilde{k}_{ij} = \mathcal{A}_{ij} + \frac{\gamma+1}{2}\mathcal{B}_{ij}t^{\frac{\gamma-1}{2}} + \gamma\mathcal{C}_{ij}t^{\gamma-1} + o(t^{\gamma-1}), \tag{27}$$

with

$$\mathcal{C}_{ij} = \frac{1}{(\gamma-1)^2}2^\gamma a_{2ij}, \tag{28}$$

where a_{2ij} is a constant which depends only on f_0 and h.