A Journey through Higher-Derivative Models in Contemporary Field Theory

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Summary

1. Historical Aspects
2. Higher-Order Klein-Gordon Equations
3. Pais-Uhlenbeck Oscillator
4. Bopp-Podolsky and Lee-Wick Theories
5. Reduction of Order
6. Conclusion and Final Remarks

1 - Historical Aspects

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2 - Generalizations of the Klein-Gordon Equation

Since Fock, Gordon, Klein and Schrödinger, the KG equation is a deep fundamental relation in QFT, concerning all elementary particles.
In modern notation, the KG equation may be written as

$$(\Box + m^2)\phi = 0\,, \tag{1}$$

or simply $K_m\phi = 0$ with

$$K_m \equiv \Box + m^2 \tag{2}$$

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As a second-order partial differential equation, it is in the essence of the mathematician and theoretical physicist to investigate consistent extensions and generalizations of the Klein-Gordon equation (1).

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RT, "Natural Higher-Derivatives Generalization for the Klein-Gordon Equation," Mod. Phys. Lett. A **36**, no.28, 2150205 (2021).

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The d'Alembertian $\Box \equiv \partial_\mu \partial^\mu$ is a regular local covariant 2nd-order differential operator, which can act recursively.
Introduce a length-dimensional multiplicative factor $a > 0$ and, for $n \in \mathbb{N}$, define

$$\mathcal{L}_n \equiv -\frac{a^{2(n-1)}}{2n!} \phi \Box^n \phi \tag{3}$$

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Construct the $2N$-th order Lagrangian density

$$\mathcal{L}^{(2N)} \equiv \sum_{n=0}^{n=N} \mathcal{L}_n = -\frac{1}{2}\phi\left(\sum_{n=0}^{N} \frac{a^{2(n-1)}\Box^n}{n!}\right)\phi\,, \tag{4}$$

for a fixed $N \in \mathbb{N}$.

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for a fixed $N \in \mathbb{N}$.
It is then natural to investigate the behavior of (4) in the limit of arbitrarily large N, for which we define further the complete Lagrangian density

$$\mathcal{L}_\phi \equiv -\frac{1}{2a^2}\phi e^{a^2\Box}\phi\,. \tag{5}$$

Integrating in space-time, we may define the natural actions

$$S^{(2N)} = -\frac{1}{2} \int d^D x \, \phi(x) \sum_{n=0}^{N} a^{2(n-1)} \frac{\Box^n \phi(x)}{n!} \tag{6}$$

and

$$S_\phi = -\frac{1}{2a^2} \int d^D x \phi(x) e^{a^2 \Box} \phi(x) \, . \tag{7}$$

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The field equation associated to (6) reads

$$\sum_{n=0}^{N}\frac{a^{2(n-1)}\Box^n}{n!}\phi = 0\,, \tag{8}$$

while in the limit of arbitralily large N, corresponding to (7), we have

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For even N or in the infinity limit, equations (8) or (9) do not have nontrivial real solutions.

For odd N, equation we may write a solution to (8) as

$$\phi_{(N)}(x) = \int \frac{d^{D-1}\mathbf{p}}{E_N(\mathbf{p}^2)} \left\{ \varphi_N(\mathbf{p}) e^{-i(E_N(\mathbf{p}^2)\,t - \mathbf{p}\cdot\mathbf{x})} + \varphi_N^*(\mathbf{p}) e^{i(E_N(\mathbf{p}^2)\,t - \mathbf{p}\cdot\mathbf{x})} \right\},$$

with

$$E_N(\mathbf{p}^2) \equiv \sqrt{\mathbf{p}^2 - q_N/a^2}\,, \tag{10}$$

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with

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where q_N represents the dimensionless real root of the algebraic equation

$$f_N(q) = 0 \tag{11}$$

with $f_N(q)$ defined as the N-th order polynomial in the dimensionless real variable q given by

$$f_N(q) \equiv \sum_{n=0}^{N} \frac{(-1)^n N!}{(N-n)!} q^{N-n}\,. \tag{12}$$

In terms of a given external current $J(x)$, we may write the functional generator associated to action (6) as

$$Z^{(2N)}[J] = \mathcal{N} \int [d\phi] \exp\left\{ iS^{(2N)} + i \int d^D x \, J(x)\phi(x) \right\}, \tag{13}$$

with

$$\mathcal{N}^{-1} \equiv \int [d\phi] \exp\left\{ iS^{(2N)} \right\}. \tag{14}$$

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The propagator for the scalar field $\phi(x)$ can be immediately computed as

$$D^{(2N)} = \frac{-ia^2}{\sum_{n=0}^{N} (-1)^n \frac{(a^2 p^2)^n}{n!}} \tag{15}$$

and has a real pole for odd N at $p^2 = q_N/a^2$ with q_N denoting the only real solution to the polynomial equation (11).

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More details can be found in

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3 - Pais-Uhlenbeck Oscillator

Consider a generalized KG equation of the form

$$(\partial_0^2 - \nabla^2)(\partial_0^2 - \nabla^2 + M^2)\varphi(t, \mathbf{x}) = 0\,, \tag{16}$$

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Performing an expansion in Fourier modes

$$\varphi(t, \mathbf{x}) = \phi(t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{17}$$

and defining

$$\omega_1^2 + \omega_2^2 = 2\mathbf{k}^2 + M^2 \quad \text{and} \quad \omega_1^2\omega_2^2 = \mathbf{k}^2(\mathbf{k}^2 + M^2) \tag{18}$$

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we get

$$\ddddot{\phi} + (\omega_1^2 + \omega_2^2)\ddot{\phi} + \omega_1^2\omega_2^2\phi = 0 \tag{19}$$

which characterizes the famous Pais-Uhlenbeck oscillator

Pais and Uhlenbeck, "On field theories with nonlocalized action," Phys. Rev. **79**, 145 (1950).

Lagrangian and Hamiltonian Approaches

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$$L(\phi, \dot{\phi}, \ddot{\phi}) = \frac{\gamma}{2}[\ddot{\phi}^2 - ({\omega_1}^2 + {\omega_2}^2)\dot{\phi}^2 + {\omega_1}^2{\omega_2}^2\phi^2] \tag{20}$$

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$$L_c(\phi, x, \lambda, \dot{\phi}, \dot{x}, \dot{\lambda}) = \frac{\gamma}{2}[\dot{x}^2 - ({\omega_1}^2 + {\omega_2}^2)x^2 + {\omega_1}^2{\omega_2}^2\phi^2] + \lambda(\dot{\phi} - x)\,, \tag{21}$$

P. D. Mannheim and A. Davidson, Phys. Rev. A **71**, 042110 (2005).

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Dirac-Bergmann constraints approach

$$H_T = \frac{p_x^2}{2\gamma} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2\phi^2 + \pi x + \gamma\omega_1^2\omega_2^2\phi p_\lambda\,. \tag{22}$$

$$\chi_1 \equiv \phi - \lambda\,, \quad \text{and} \quad \chi_2 \equiv p_\lambda\,, \tag{23}$$

BFV/BRST Quantization for the PU oscillator

We may write a first-class Hamiltonian (22) as (BFFT conversion)

$$\mathcal{H} = \frac{p_x^2}{2\gamma} + \frac{\gamma(\omega_1^2+\omega_2^2)x^2}{2} - \frac{\gamma}{2}\omega_1^2\omega_2^2(q+\varphi)^2 + xp + \gamma\omega_1^2\omega_2^2(q+\varphi)(p_\lambda+\varphi)\,. \quad (24)$$

Following the usual BRST-BFV quantization formalism, we introduce a set of odd Grassmannian parity ghost fields $(\mathcal{C}^a, \bar{\mathcal{C}}_a)$ and momenta $(\bar{\mathcal{P}}_a, \mathcal{P}^a)$, satisfying

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We define the nillpotent BRST charge

$$\mathcal{Q} = \mathcal{C}^a\Phi_a - i\mathcal{P}^a w_a\,, \quad (26)$$

as the generator of the BRST symmetry

$$\begin{aligned} &sq = sp_\lambda = -s\varphi = \mathcal{C}^1\,, s\lambda = -s\pi = \mathcal{C}^2\,, \\ &sv^a = -i\mathcal{P}^a\,, s\bar{\mathcal{C}}_a = iw_a\,, s\bar{\mathcal{P}}_1 = -p+\lambda+\pi\,, s\bar{\mathcal{P}}_2 = -p_\lambda - \varphi\,. \end{aligned} \quad (27)$$

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As can be checked, the transformation above is nillpotent and represent a symmetry of the Hamiltonian. This allows a consistent BRST quantization.

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B. P. Mandal, V. K. Pandey and RT, "BFV quantization and BRST symmetries of the gauge invariant fourth-order Pais-Uhlenbeck oscillator," Nucl. Phys. B **982**, 115905 (2022).

3 - Bopp-Podolsky and Lee-Wick Theories

$$\mathcal{L}_B = -\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} - a^2\partial_\rho F^{\mu\nu}\partial^\rho F_{\mu\nu}\right] \quad \text{Bopp (1940)}$$

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$$\mathcal{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \quad \text{Podolsky (1942)}$$

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A generalized electrodynamics

Generalized Coulomb Potential (electrostatics) $$V(r) = \frac{1 - e^{-r/a}}{r}$$

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$$(1 + a^2\Box)\nabla \cdot \mathbf{E} = j^0 \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{32}$$

$$(1 + a^2\Box)(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}) = \mathbf{j} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,. \tag{33}$$

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The general solution to the GPE (34) and some interesting particular solutions can be found in

C. R. Ji, A. T. Suzuki, J. H. O. Sales and RT, Eur. Phys. J. C **79**, no.10, 871 (2019).

Generalized Scalar Electrodynamics

Interaction with a charged bosonic field

$$\mathcal{L}_{int} = ieA_\mu \left[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] + e^2A^2|\phi|^2 \,, \tag{35}$$

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$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2 \qquad \text{and} \tag{36}$$

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \;\; \text{with} \;\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{37}$$

By demanding stationarity of the total gauge invariant action

$$S = \int d^4x \, \left[\mathcal{L}_\phi + \mathcal{L}_A + \mathcal{L}_{int}\right] \tag{38}$$

with respect to arbitrary variations of ϕ and A_μ we obtain the field equations:

Generalized Scalar Electrodynamics

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$$(\Box + m^2)\phi = -ieA_\mu\partial^\mu\phi - ie\partial^\mu(\phi A_\mu) + e^2A^2\phi\,, \qquad (39)$$

$$(\Box + m^2)\phi^* = ieA_\mu\partial^\mu\phi^* + ie\partial^\mu(\phi^* A_\mu) + e^2A^2\phi^* \qquad (40)$$

$$(1 + a^2\Box)\partial_\nu F^{\mu\nu} = ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + 2e^2A^\mu|\phi|^2 \qquad (41)$$

I. G. Oliveira, J. H. Sales and RT, Eur. Phys. J. Plus **135**, no.9, 713 (2020).

4 - Gauge-Fixing and Propagators

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we can invert the corresponding gauge field kinetic term and obtain

$$P_{\mu\nu}(k) = \frac{-i}{(1-a^2k^2)k^2}\left[\eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]. \tag{43}$$

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$$\begin{aligned} Z[j^\mu] \quad = \quad & N\int DA_\mu DCD\bar{C}DB \exp\{iS_0 \\ & +i\int d^4x\,[\bar{C}(1+a^2\Box)\Box C + B(1+a^2\Box)\partial^\mu A_\mu \\ & -\frac{a^2\xi}{2}\partial_\mu B\partial^\mu B + \frac{\xi B^2}{2} - j^\mu A_\mu]\} . \end{aligned} \tag{44}$$

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we have the propagator

$$P_{ab}(k) = \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} + \frac{(\alpha k^2 + n^2)}{(n\cdot k)^2}k_a k_b - \frac{1}{(n\cdot k)}(k_a n_b + k_a n_b)\right]\,.$$

For the usual light-front gauge we choose a light-like direction n, with $n^2 = 0$, and consider the limit $\alpha \to 0$. In this case

$$P_{ab} = \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} - \frac{1}{(n\cdot k)}(k_a n_b + k_b n_a)\right]\,. \tag{46}$$

Light-Front Gauges

In order to obtain the doubly transverse three-term propagator we may use the mixed gauge-fixing (A. T. Suzuki and J. H. O. Sales, Nucl. Phys. A **725**, 2003)

$$\mathcal{L}_3 = -\frac{1}{\beta}(n \cdot A)(\partial \cdot A)$$

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$$\begin{aligned} P_{ab}(k) &= \frac{-i}{k^2(1-a^2k^2)}\left[\eta_{ab} + \frac{\beta^2 k^2 + n^2}{(n\cdot k)^2 - n^2 k^2} k_a k_b - \frac{n\cdot k + i\beta k^2}{(n\cdot k)^2 - n^2k^2} k_a n_b + \right. \\ & \left. - \frac{n\cdot k + i\beta k^2}{(n\cdot k)^2 - n^2 k^2} k_b n_a + \frac{k^2}{(n\cdot k)^2 - n^2k^2} n_a n_b \right] \end{aligned} \tag{48}$$

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As can be directly checked, the three-term propagator satisfies

$$k^a P_{ab} = 0 \tag{49}$$

and

$$n^a P_{ab} = 0 \tag{50}$$

being in this sense doubly transverse.

5 - Reduction of Order

Back to Bopp-Podolsky, we may try a similar derivative order reduction

$$\mathcal{L}_{int} + \mathcal{L}_\phi = ieA_\mu \left[\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right] + e^2A^2|\phi|^2 + \partial_\mu\phi^*\partial^\mu\phi - m^2|\phi|^2 \qquad (51)$$

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\rho F_{\mu\rho} \qquad (52)$$

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Introducing an auxiliary field B_{μ}, we decouple the massive and massless modes

$$\mathcal{L}_{AB} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{a^{2}}{2}B_{\mu}B^{\mu} + a^{2}\partial_{\mu}B_{\nu}F^{\mu\nu}\,, \tag{53}$$

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and obtain the equivalent reduced-order model

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I. G. Oliveira, J. H. Sales and RT, Eur. Phys. J. Plus **135**, no.9, 713 (2020).

Canonical Quantization

$$\mathcal{L} = \frac{1}{2} F_{0i} F_{0i} - a^2 (\partial_0 B_i - \partial_i B_0) F_{0i} - \mathcal{H}_{sp} \,, \tag{58}$$

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$$H_c = \int d^3x \left[-\frac{\Pi^i \Pi^i_B}{a^2} - \frac{\Pi^i_B \Pi^i_B}{2a^4} + \mathcal{H}_{sp} - A_0 \partial_i \Pi^i - B_0 \partial_i \Pi^i_B \right]. \tag{60}$$

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Constraints in phase space:

$$\chi_1 = \Pi^0_B \approx 0\,, \tag{61}$$
$$\chi_2 = \partial_i \Pi^i_B - a^2 B_0 \approx 0\,, \tag{62}$$
$$\chi_3 = \Pi^0 \approx 0\,, \tag{63}$$
$$\chi_4 = \partial_i \Pi^i \approx 0\,. \tag{64}$$

Constraints χ_1 and χ_2 are second-class while χ_3 and χ_4 are first-class.

Gauge fixing:

$$\chi_5 = A_0\,, \quad \chi_6 = \partial_i A_i\,. \tag{65}$$

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Table: Dirac Brackets

	A_j	B_0	B_j	Π^j	Π_B^j
A_i	.	.	.	$(\delta_i^j - \frac{\partial_i\partial_j}{\nabla^2})$	.
B_0	.	.	$-\frac{1}{a^2}\partial_j$	.	.
B_i	.	$-\frac{1}{a^2}\partial_i$	.	.	δ_i^j
Π^i	$(-\delta_j^i + \frac{\partial_i\partial_j}{\nabla^2})$	.	.	.	.
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$$[A_i(\mathbf{x}), \Pi^j(\mathbf{y})]^* = (\delta_i^j - \frac{\partial_i\partial_j}{\nabla^2})\delta(\mathbf{x}-\mathbf{y}) \qquad [B_i(\mathbf{x}), \Pi_B^j(\mathbf{y})]^* = \delta_i^j\delta(\mathbf{x}-\mathbf{y}) \tag{66}$$

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More details can be found in

M. C. Bertin and RT, Eur. Phys. J. Plus **140**, no.4, 341 (2025).

Conclusion and Final Remarks

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- We have obtained a natural class of higher-order gauge-fixing terms for Bopp-Podolsky and Lee-Wick like theories.

Conclusion and Final Remarks

- The quest for understanding higher-derivative models in QFT is a longterm one - an important challenging still open problem.
- We have proposed a natural higher-order generalization for the KG equation and investigated its classical solutions.
- We have obtained a natural class of higher-order gauge-fixing terms for Bopp-Podolsky and Lee-Wick like theories.
- The reduction of order technique can be very helpful and handy for higher-derivative theories.

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- The reduction of order technique can be very helpful and handy for higher-derivative theories.
- We have not discussed here the important open interelated issues of unitarity, causality, positiveness and propagating ghost modes.

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