Measurement-induced time operator and its uncertainty relation with energy

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(PTEP(2025), 063A02 and)
ongoing work with Kin-ya Oda(KO)

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Take-Home message

- Time operator is induced by measurement in Lee-Tsutsui formalism:

$$\hat{T} := \widehat{\mathrm{M}^{\star}\mathrm{id}}$$

- It obeys **CCR-like relation** with $\widehat{H_{\mathrm{free}}}$ & time-energy **uncertainty relation**

- **Lee error of $\widehat{H_{\mathrm{free}}}$ diverges:**

  time-of-arrival measurement cannot see energy

Time in QM

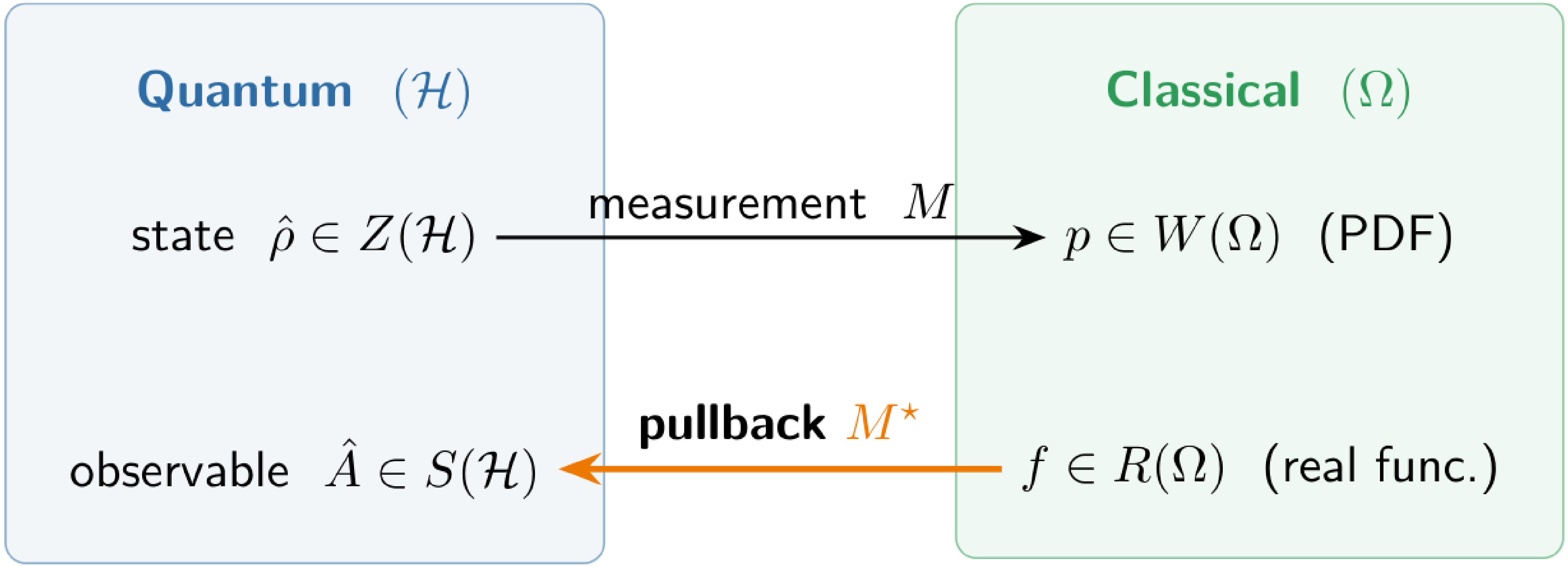
- Time is usually **parameter**, **not operator** (cf. **Pauli's theorem**)
- **Time-energy uncertainty relation** subject to long-standing debate
- **Time operator** by **Aharonov** and **Bohm**: **symmetric operator** ($\neq$ self-adjoint operator)

$$\widehat{T_{\mathrm{AB}}} = -\frac{m}{2}\left(\hat{p}^{-1}\hat{x} + \hat{x}\hat{p}^{-1}\right),\ \left[\widehat{T_{\mathrm{AB}}}, \widehat{H_{\mathrm{free}}}\right] = -i\hat{1}$$

- Many other approaches using measurement-theoretic machinery, especially **POVM**

(Kijowski 1974, Holevo 1978, etc.)

Lee-Tsutsui(LT) formalism



- Recently(2020's) proposed by **Lee** and **Tsutsui**
- **Measurement**(M) **induces pullback**($\mathrm{M}^\star$):

$$\boxed{\langle \widehat{\mathrm{M}^\star \mathrm{f}} \rangle_{\hat{\rho}} = \langle \mathrm{f} \rangle_{\mathrm{M}\hat{\rho}}}$$

- **LT error** and **Lee error(partial inverse of pullback** $\mathrm{M}^\star|^{-1}$**)**

Position and momentum

Our previous work for **position and momentum** using **Gaussian wave-packet basis**(PTEP(2025), 063A02)

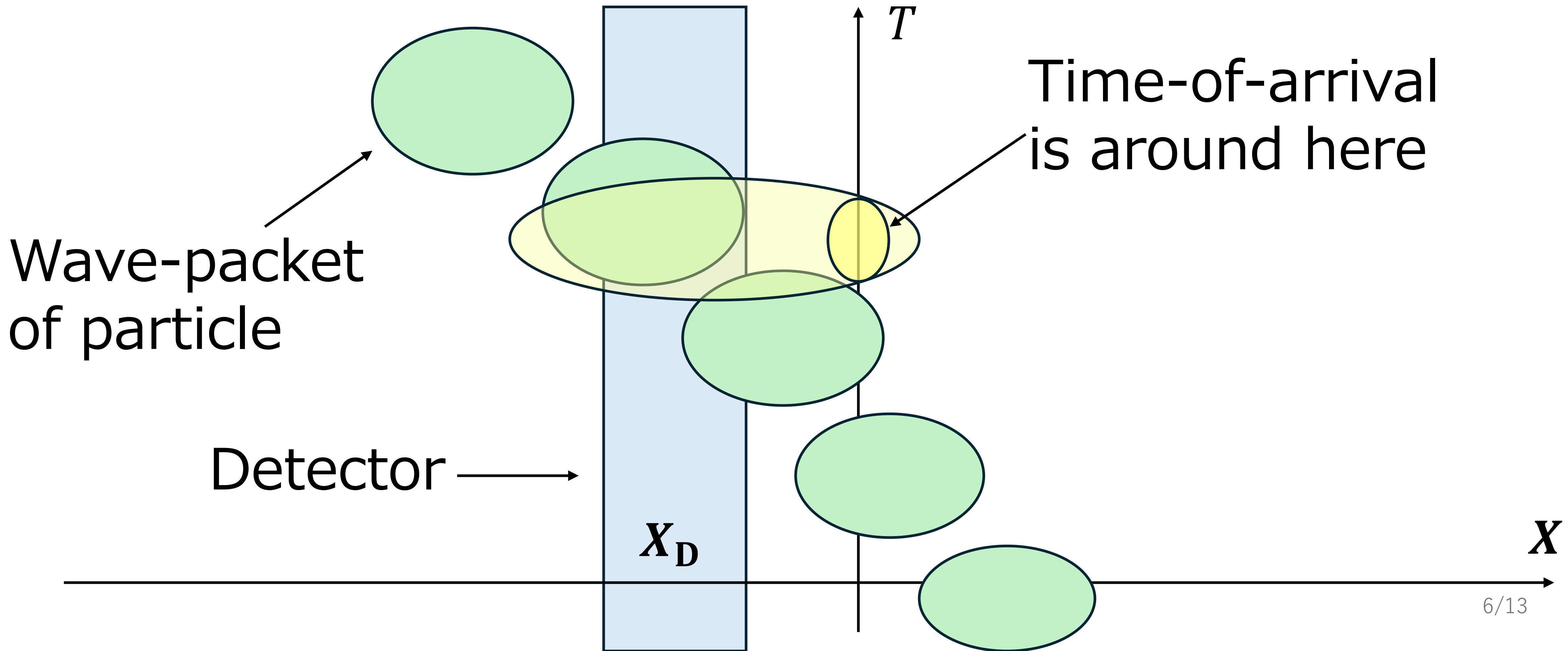
$$\boxed{\left(\tilde{\varepsilon}[\hat{x}]\ \tilde{\varepsilon}[\widehat{p_x}] =\right)\frac{1}{2} \geq \frac{1}{2}}$$

"It would be fascinating to explore the time–energy uncertainty relation on this ground."

This talk does exactly that with time-of-arrival measurement

Concept

Detector by **time-translated Gaussian wave-packet basis** at X_D detects time of particle arriving there



Gaussian detector

Positive-Operator-Valued Measure(POVM):

$$\boxed{\hat{\Pi}(\omega) \geq 0, \int \mathrm{d}\omega\, \hat{\Pi}(\omega) = \hat{1}, \mathrm{Prob}(\omega) = \mathrm{Tr}[\hat{\Pi}(\omega)\hat{\rho}]}$$

Gaussian detector(POVM):

$\mathcal{J}$: temporal interval in which we perform time measurements

Outcome space $\Omega = \mathcal{J} \sqcup \mathrm{ND}$: **detection can fail !**

$$\hat{\Pi}(T_\mathrm{D}) := \gamma \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{P} \frac{\left|\overrightarrow{X_\mathrm{D}}, \boldsymbol{P}; \sigma_\mathrm{D}\right\rangle \left\langle \overrightarrow{X_\mathrm{D}}, \boldsymbol{P}; \sigma_\mathrm{D}\right|}{(2\pi)^d},$$

$$\hat{E}(\mathrm{ND}) := \hat{1} - \int_{\mathcal{J}} \mathrm{d}T_\mathrm{D}\, \hat{\Pi}(T_\mathrm{D})$$

Pullback time operator and its conditional expectation

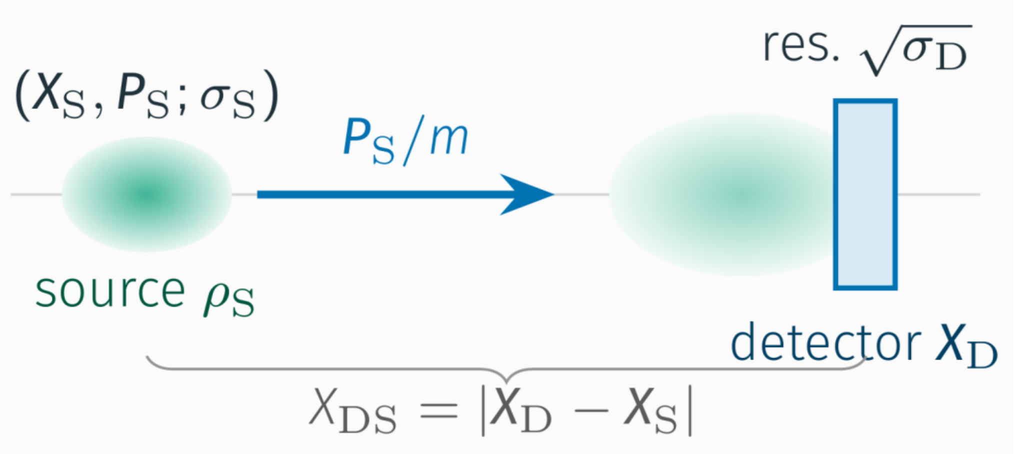
$$\boxed{\hat{T} \coloneqq \widehat{\mathrm{M}^{\star}\mathrm{id}}} \qquad \boxed{\hat{I} \coloneqq \widehat{\mathrm{M}^{\star}\chi}}$$

When particle is coming straight towards detector

$$\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}} \propto \boldsymbol{P}_{\mathrm{S}},\ (\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}}) \cdot \boldsymbol{P}_{\mathrm{S}} > 0$$

conditional expectation of $\hat{T}$ reproduces **classical arrival time** !

$$\frac{\langle \hat{T} \rangle_{\hat{\rho}}}{\langle \hat{I} \rangle_{\hat{\rho}}} \simeq T_{\mathrm{S}} + m\frac{X_{\mathrm{DS}}}{|\boldsymbol{P}_{\mathrm{S}}|} \eqqcolon T_{\mathrm{cl}}$$



This suggests $\hat{T}$ is legitimate operator corresponding to arrival time

Commutator with free Hamiltonian

Analogous to CCR!

$$\boxed{[\hat{T}, \widehat{H_{\mathrm{free}}}] = -i\hat{I}}$$

Sharing same structure as result written in Höhn et al., 2021 for n=1 case, whereas its conceptual background is different from ours

RHS involves $\hat{I}$, not $\hat{1}$, so this clearly evades Pauli's obstruction

Time–energy uncertainty relation

Conditioned on detection

$$\Delta_{\hat{\rho}}[\hat{T}; \mathrm{M}]\, \sigma_{\hat{\rho}}[\widehat{H_{\mathrm{free}}}] \geq \frac{1}{2} \mathrm{P_D}[\hat{\rho}; \mathrm{M}] \longrightarrow \frac{1}{2}$$

$\Delta_{\hat{\rho}}[\hat{T}; \mathrm{M}]$: standard deviation of **measured values** of arrival time

$\sigma_{\hat{\rho}}[\widehat{H_{\mathrm{free}}}]$: **quantum fluctuation** of free Hamiltonian

$\mathrm{P_D}[\hat{\rho}; \mathrm{M}]$: **total detection probability** given state $\hat{\rho}$ and measurement M

Time measurement cannot see energy

By definition, the following must hold:

$$\mathrm{Tr}[\widehat{H_{\mathrm{free}}}\,\hat{\rho}] = \int \mathrm{d}T_{\mathrm{D}}\, [\mathrm{M}\hat{\rho}](T_{\mathrm{D}})\mathrm{M}^{\star}|^{-1}\widehat{H_{\mathrm{free}}}\,(T_{\mathrm{D}})$$

In reality, there is **no solution**, so

$$\widehat{H_{\mathrm{free}}} \notin \mathrm{ran}\,\mathrm{M}^{\star}$$

Lee error of $\widehat{H_{\mathrm{free}}}$ for considered measurement **blows up** by definition:

$$\boxed{\tilde{\varepsilon}_{\hat{\rho}}[\widehat{H_{\mathrm{free}}}\,; \mathrm{M}\,] = +\infty}$$

Summary(Take-Home message)

- Time operator is induced by measurement in Lee-Tsutsui formalism:

$$\hat{T} := \widehat{\mathrm{M}^{\star}\mathrm{id}}$$

- It obeys **CCR-like relation** with $\widehat{H_{\mathrm{free}}}$ & time-energy **uncertainty relation**

- **Lee error of $\widehat{H_{\mathrm{free}}}$ diverges:**

  time-of-arrival measurement cannot see energy

Thank you!

↑ QR code for PTEP(2025),063A02 (position momentum paper)

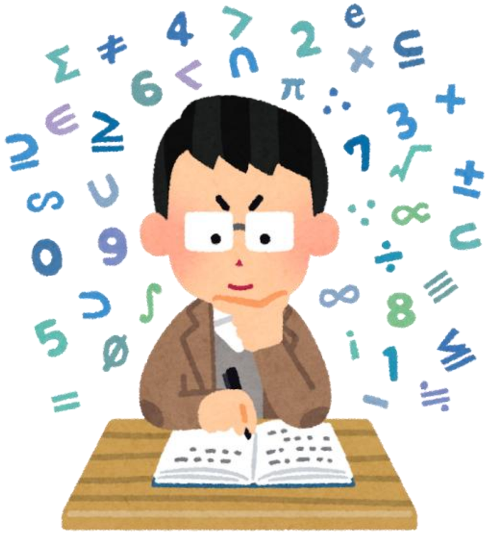
Backup

Main Theme

Time as

operator induced by

measurement



Lee-Tsutsui(LT) formalism

- Recently(2020's) proposed by **Lee** and **Tsutsui**
- **Universal** and **operational**
- Measurement (M) induces **pullback** $\mathrm{M}^{\star}$:
  Map from classical to quantum observable
  Critical for constructing time operator
- **LT error** and **Lee error(partial inverse)**
- We applied LT formalism to **Gaussian Wave-packet basis**: plausible uncertainty relation for **position and momentum (previous work)**

$\mathcal{H}$: system Hilbert space, Ω: outcome space

	Observable	**State**
Classical	Real function $\mathrm{f}(\omega) \in R(\Omega)$	Probability distribution $\mathrm{p}(\omega) \in W(\Omega)$
Quantum	Self-adjoint operator $\hat{A} \in S(\mathcal{H})$	Density operator $\hat{\rho} \in Z(\mathcal{H})$

	Expectation value	**Seminorm**	**Semi-inner product**
Classical	$\langle \mathrm{f} \rangle_{\mathrm{p}} := \int_{\Omega} \mathrm{d}\omega\, \mathrm{f}(\omega)\mathrm{p}(\omega)$	$\parallel \mathrm{f} \parallel_{\mathrm{p}}^{2} := \langle \mathrm{f}^{2} \rangle_{\mathrm{p}}$	$\langle \mathrm{f}, \mathrm{g} \rangle_{\mathrm{p}} := \langle \mathrm{fg} \rangle_{\mathrm{p}}$
Quantum	$\langle \hat{A} \rangle_{\hat{\rho}} := \mathrm{Tr}[\hat{A}\hat{\rho}]$	$\parallel \hat{A} \parallel_{\hat{\rho}}^{2} := \langle \hat{A}^{2} \rangle_{\hat{\rho}}$	$\langle \hat{A}, \hat{B} \rangle_{\hat{\rho}} := \langle \frac{\{\hat{A}, \hat{B}\}}{2} \rangle_{\hat{\rho}}$

LT-measurement process

Abstract map representing
quantum measurement

- LT-measurement process M :
  - Map from $Z(\mathcal{H})$ ($\ni \hat{\rho}$) to $W(\Omega)$ ($\ni \mathrm{p}$)
  - (if we consider probabilistic mixture) **Affinity** :

$$\mathrm{M}(\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2) = \lambda\mathrm{M}(\hat{\rho}_1) + (1-\lambda)\mathrm{M}(\hat{\rho}_2)$$

$$\lambda \in [0,1]$$

Hereafter denote M($\hat{\rho}$) as $\mathrm{M}\hat{\rho}$

Lee-Tsutsui(LT) adjoint（Pullback)

Measurement process M induces **Lee-Tsutsui(LT) adjoint (pullback)** for given classical observable f and any quantum state $\hat{\rho}$:

$$\boxed{\langle \widehat{\mathrm{M}^{\star}\mathrm{f}} \rangle_{\hat{\rho}} = \langle \mathrm{f} \rangle_{\mathrm{M}\hat{\rho}}}$$

Defined as operator by mapping classical observable to quantum side via quantum measurement (critical for considering time)

Pushforward $\mathrm{M}_\star \hat{A}$

$$\boxed{\langle \hat{A}, \widehat{\mathrm{M}^\star \mathrm{f}} \rangle_{\hat{\rho}} = \langle \mathrm{M}_\star \hat{A}, \mathrm{f} \rangle_{\mathrm{M}\hat{\rho}}}$$

Lee-Tsutsui(LT) error

$$\boxed{\| \hat{A} \|_{\hat{\rho}}^2 := \langle \hat{A}^2 \rangle_{\hat{\rho}}} \qquad \boxed{\| \mathrm{f} \|_{\mathrm{p}}^2 := \langle \mathrm{f}^2 \rangle_{\mathrm{p}}}$$

$$\boxed{\varepsilon_{\hat{\rho}}[\hat{A}; \mathrm{M}] := \sqrt{\| \hat{A} \|_{\hat{\rho}}^2 - \| \mathrm{M}_\star \hat{A} \|_{\mathrm{M}\hat{\rho}}^2}}$$

Pre-measurement **Post-measurement**

Clear operational meaning

Lee error

$$\| f \|_p^2 := \int_\Omega d\omega\, f^2(\omega) p(\omega)$$

$$\| \hat{A} \|_{\hat{\rho}}^2 := \langle \hat{A}^2 \rangle_{\hat{\rho}}$$

$$\tilde{\varepsilon}_{\hat{\rho}}[\hat{A}; M] := \begin{cases} \sqrt{\| M^\star|^{-1}\hat{A} \|_{M\hat{\rho}}^2 - \| \hat{A} \|_{\hat{\rho}}^2} & (\hat{A} \in \operatorname{ran} M^\star) \\ +\infty & (\hat{A} \notin \operatorname{ran} M^\star) \end{cases}$$

Post-measurement **Pre-measurement**

where **(standard) partial inverse of pullback :**

$$M^\star|^{-1}\hat{A} := \operatorname{argmin}_{f:\, \widehat{M^\star f} = \hat{A}} \| f \|_{M\hat{\rho}}$$

Clear operational meaning

Lee-Tsutsui(LT) inequality

$$\boxed{\varepsilon[\hat{A}]\varepsilon[\hat{B}] \geq \sqrt{\mathcal{J}^2_{\hat{\rho}}[\hat{A},\hat{B}] + \mathcal{R}^2_{\hat{\rho}}[\hat{A},\hat{B}]}}$$

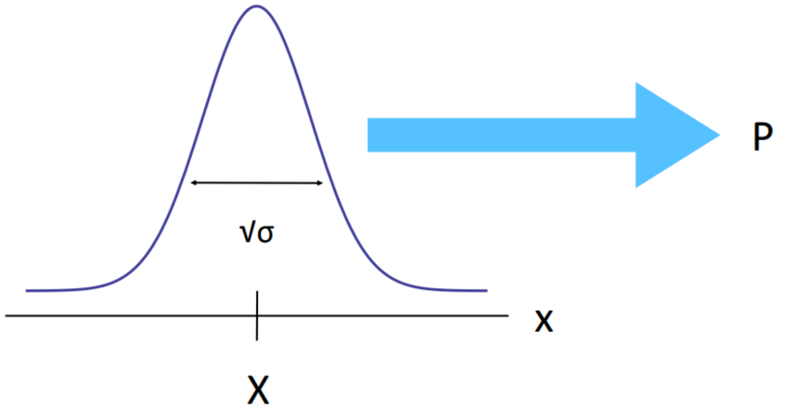
Lee inequality

$$\boxed{\tilde{\varepsilon}[\hat{A}]\tilde{\varepsilon}[\hat{B}] \geq \sqrt{\mathcal{J}^2_{0\hat{\rho}}[\hat{A},\hat{B}] + \tilde{\mathcal{R}}^2_{\hat{\rho}}[\hat{A},\hat{B}]}}$$

LT / Lee inequalities are CS inequalities

Error-disturbance relation is similar

Gaussian wave-packet basis $\{|X,P;\sigma\rangle\}$(Husimi distribution)



- Width $\sqrt{\sigma}$ for position, $\sqrt{1/\sigma}$ for momentum
- Set of states localizing in the way of Gaussian distribution
- Forms basis
- Nonorthogonality, overcomplete

Previous results with KO for position and momentum

model

- Pure Gaussian initial state:

$$\hat{\rho}_{\mathrm{in}} := |X_{\mathrm{in}}, P_{\mathrm{in}}; \sigma_{\mathrm{in}}\rangle\langle X_{\mathrm{in}}, P_{\mathrm{in}}; \sigma_{\mathrm{in}}|$$

- Joint measurement of position and momentum by POVM of another Gaussian wave-packet M

- Calculate LT/Lee inequalities

LT error

$$\varepsilon^2[\hat{x}] = \frac{\sigma_{\mathrm{red}}}{2}, \quad \varepsilon^2[\hat{p}] = \frac{1}{2\sigma_{\mathrm{sum}}}$$

Dependent on width of Gaussian wave-packet of detector and initial state

LT inequality

$$(\varepsilon[\hat{x}]\varepsilon[\widehat{p_x}] =)\, \frac{1}{2}\sqrt{\frac{\sigma_{\mathrm{red}}}{\sigma_{\mathrm{sum}}}} \geq 0$$

Lower bound saturated when $\sigma \to 0$ or $\sigma \to \infty$, measurement becomes projective

Lee error

$$\tilde{\varepsilon}^2[\hat{x}] = \frac{\sigma}{2},\ \tilde{\varepsilon}^2[\widehat{p_x}] = \frac{1}{2\sigma}$$

Dependent only on width of Gaussian wave-packet of detector

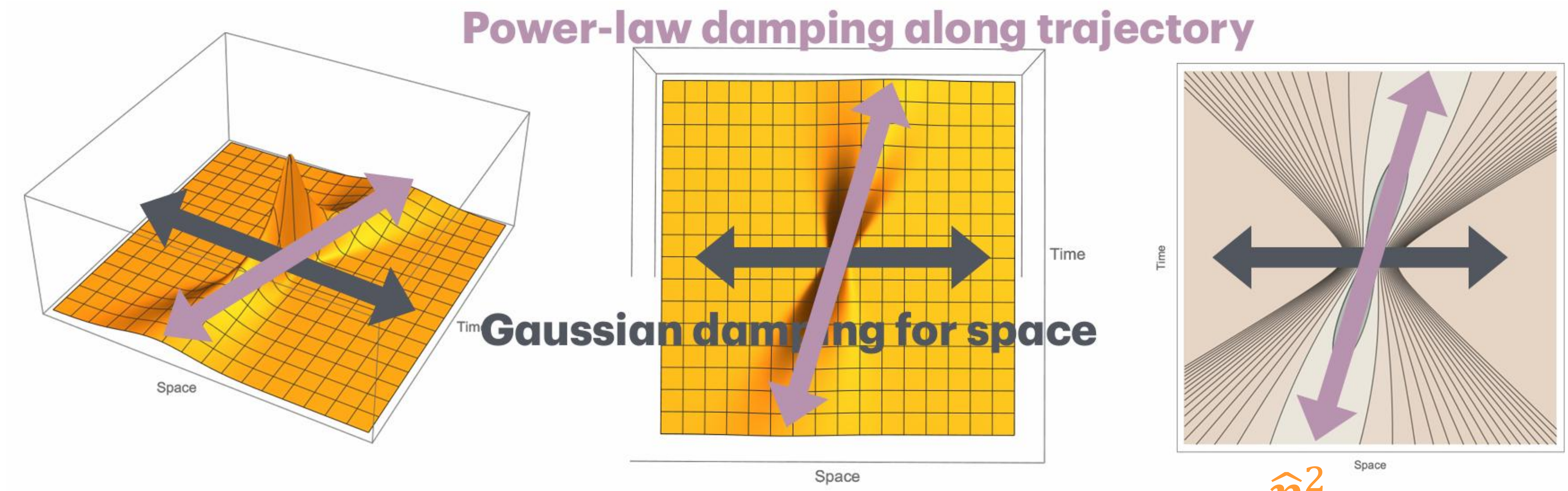
Lee inequality

$$\left(\tilde{\varepsilon}[\hat{x}]\ \tilde{\varepsilon}[\widehat{p_x}] =\right) \frac{1}{2} \geq \frac{1}{2}$$

Giving lower bound of usual uncertainty relation, and always saturating

Ongoing work with KO on time

Time-translated Gaussian wavepacket basis $\{|\vec{X},\boldsymbol{P};\sigma\rangle\}$



$$\left|\vec{X},\boldsymbol{P};\sigma\right\rangle = e^{i\widehat{H_{\mathrm{free}}}\,t}\,|\mathrm{X,P};\sigma\rangle \qquad \widehat{H_{\mathrm{free}}} := \frac{\hat{\boldsymbol{p}}^2}{2m}$$

Localizing around $\boldsymbol{X}(t) := \boldsymbol{X} + \frac{\boldsymbol{P}}{m}(t-T)$ with width $\sigma + \frac{(t-T)^2}{\sigma m^2}$

Momentum representation of time-translated Gaussian wave-packet basis:

$$\langle \boldsymbol{p} | \vec{X}, \boldsymbol{P}; \sigma \rangle \coloneqq \left(\frac{\sigma}{\pi}\right)^{\frac{1}{4}} \exp\left(\mathrm{iE}(\boldsymbol{p})T - \mathrm{i}\boldsymbol{p} \cdot \boldsymbol{X} - \frac{\sigma}{2}(\boldsymbol{p} - \boldsymbol{P})^2 \right)$$

$$\mathrm{E}(\boldsymbol{p}) \coloneqq \frac{\boldsymbol{p}^2}{2m}$$

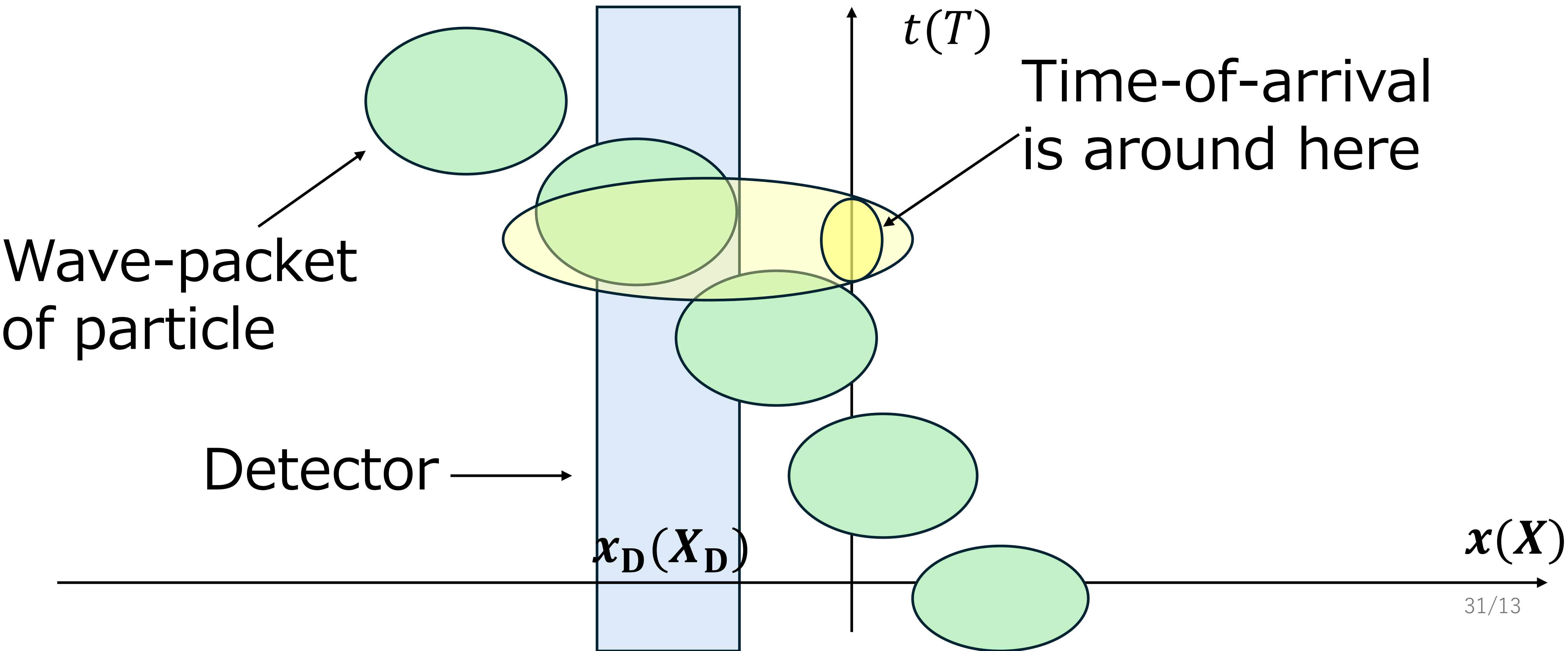
Nonorthogonality:

$$\langle \mathrm{T}, \mathbf{X}', \mathrm{P}'; \sigma | \mathrm{T}, \mathbf{X}, \mathrm{P}; \sigma \rangle$$

$$= \exp\left(i \frac{\mathbf{P}+\mathbf{P}'}{2} \cdot (\mathbf{X} - \mathbf{X}') - \frac{(\mathbf{X}-\mathbf{X}')^2}{4\sigma} - \frac{\sigma}{4}(\mathbf{P} - \mathbf{P}')^2 \right)$$

Concept

Detector at $x_{\mathbf{D}}(X_{\mathbf{D}})$ detects time of particle arriving there



Gaussian detector $\mathrm{M}_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}$

γ: coupling constant dependent on details of detector

outcome space :

$\Omega = \{\mathcal{J}, \mathrm{ND}\}$

$[\gamma] = -(d-1)$

POVM density of detection at time T_D :

$$\hat{\Pi}(T_{\mathrm{D}}) \coloneqq \gamma \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{P} \frac{\left|\overrightarrow{X_{\mathrm{D}}}, \boldsymbol{P}; \sigma_{\mathrm{D}}\right\rangle \left\langle \overrightarrow{X_{\mathrm{D}}}, \boldsymbol{P}; \sigma_{\mathrm{D}}\right|}{(2\pi)^d},$$

$[\hat{\Pi}(T_{\mathrm{D}})] = 1$

POVM element of detection in interval $\mathcal{J} = [T_{\mathrm{i}}, T_{\mathrm{f}}]$

$$\hat{\Pi}(\mathcal{J}) \coloneqq \int_{\mathcal{J}} \mathrm{d}T_{\mathrm{D}}\, \hat{\Pi}(T_{\mathrm{D}}),$$

$[\hat{\Pi}(\mathcal{J})] = 0$

POVM element of **no-detection** event:

$$\hat{\Pi}(\mathrm{ND}) \coloneqq \hat{1} - \int_{\mathcal{J}} \mathrm{d}T_{\mathrm{D}}\, \hat{\Pi}(T_{\mathrm{D}}),$$

$[\hat{\Pi}(\mathrm{ND})] = 0$

Perturbative condition

$$\frac{\langle\psi|\hat{\Pi}(\mathcal{J})|\psi\rangle}{\langle\psi|\psi\rangle} \ll 1, \mathrm{Tr}[\hat{\Pi}(\mathcal{J})\hat{\rho}] \ll 1$$

Under this condition, measurement-theoretic description of our approach is valid

Pullback time operator for Gaussian detector

$$\widehat{T_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}} := \mathrm{M}^\star_{\widehat{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}\mathrm{id}_\mathcal{J} \qquad \widehat{I_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}} := \mathrm{M}^\star_{\widehat{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}\chi_\mathcal{J}$$

$$\langle \boldsymbol{p}|\, \widehat{T_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}\, |\boldsymbol{p}'\rangle$$

$$= \frac{\gamma}{2\pi}\int_\mathcal{J} \mathrm{d}T\, T\exp\left[-i\left(\frac{\boldsymbol{p}'^2}{2m} - \frac{\boldsymbol{p}^2}{2m}\right) + i(\boldsymbol{p}' - \boldsymbol{p})\cdot \boldsymbol{X}_\mathbf{D} - \frac{\sigma_\mathrm{D}}{4}(\boldsymbol{p}' - \boldsymbol{p})^2\right]$$

$$\langle \boldsymbol{p}|\, \widehat{I_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}\, |\boldsymbol{p}'\rangle$$

$$= \frac{\gamma}{2\pi}\int_\mathcal{J} \mathrm{d}T\, \exp\left[-i\left(\frac{\boldsymbol{p}'^2}{2m} - \frac{\boldsymbol{p}^2}{2m}\right) + i(\boldsymbol{p}' - \boldsymbol{p})\cdot \boldsymbol{X}_\mathbf{D} - \frac{\sigma_\mathrm{D}}{4}(\boldsymbol{p}' - \boldsymbol{p})^2\right]$$

$$[\widehat{T_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}] = -1 \qquad [\widehat{I_{\boldsymbol{X}_\mathrm{D},\sigma_\mathrm{D}}}] = 0$$

Conditional expectation value of time operator

Under the head-on condition

$$\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}} \propto \boldsymbol{P}_{\mathrm{S}},\ (\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}}) \cdot \boldsymbol{P}_{\mathrm{S}} > 0$$

$$\frac{\langle \widehat{T_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}} \rangle_{\hat{\rho}}}{\langle \widehat{I_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}} \rangle_{\hat{\rho}}} \simeq T_{\mathrm{S}} + m \frac{|\boldsymbol{X}_{\mathrm{D}} - \boldsymbol{X}_{\mathrm{S}}|}{|\boldsymbol{P}_{\mathrm{S}}|}$$

This suggests $\widehat{T_{\boldsymbol{X}_{\mathrm{D}},\sigma_{\mathrm{D}}}}$ is indeed operator corresponding to time

Commutator with free Hamiltonian

Analogous to CCR!

$$[\hat{T}, \widehat{H_{\mathrm{free}}}] = -i\hat{I}$$

$$[\hat{I}^{-1/2}\hat{T}\hat{I}^{-1/2}, \widehat{H_{\mathrm{free}}}] = -i\hat{1}$$

$$[\hat{I}, \widehat{H_{\mathrm{free}}}] = 0$$

Due to nonorthogonality,

$$[\hat{T}, \hat{I}] = O(\gamma^2) \neq 0$$

Nonexistence of partial inverse of and divergence of Lee error of $\widehat{H_{\mathrm{free}}}$

$$\mathrm{Tr}[\widehat{H_{\mathrm{free}}}\,\widehat{\rho_{\mathrm{S}}}] = \int \mathrm{d}T_{\mathrm{D}}\,[\mathrm{M}\widehat{\rho_{\mathrm{S}}}](T_{\mathrm{D}})\mathrm{M}^{\star}|^{-1}\widehat{H_{\mathrm{free}}}(T_{\mathrm{D}})$$

Performing Fourier transformation,

$$\widetilde{f_H}(\omega) = \frac{(2\pi)^d}{\sqrt{2\pi\gamma}} E(\boldsymbol{p}) e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot \boldsymbol{X}_{\mathrm{D}} + \frac{\sigma_{\mathrm{D}}}{4}(\boldsymbol{p}'-\boldsymbol{p})^2} \quad \omega = E(\boldsymbol{p}) - E(\boldsymbol{p}')$$

In reality, there is no solution, so

$$\widehat{H_{\mathrm{free}}} \notin \mathrm{ran}\,\mathrm{M}^{\star}_{\boldsymbol{X}_{\mathbf{D}},\sigma_{\mathrm{D}}}$$

Lee error of $\widehat{H_{\mathrm{free}}}$ for considered measurement blows up by definition:

$$\boxed{\tilde{\varepsilon}_{\widehat{\rho_{\mathrm{S}}}}[\widehat{H_{\mathrm{free}}}\,; \mathrm{M}_{\boldsymbol{X}_{\mathbf{D}},\sigma_{\mathrm{D}}}] = \infty}$$

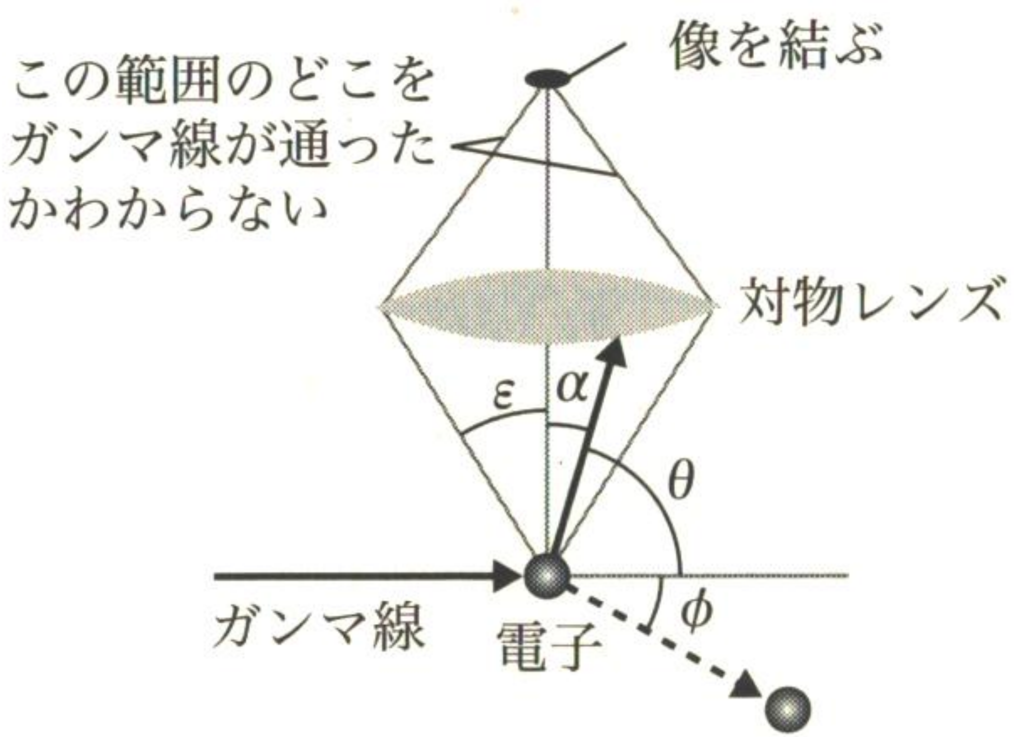
Summary

- Time is usually parameter, not operator in QM / **symmetric operator** / **POVM**
- **LT formalism:** measurement-induced **pullback** maps classical observables to quantum ones
- **Pullback time operator** and its **conditional expectation**
- **Commutator analogous to CCR**
- Nonexistence of partial inverse and **divergence of Lee error of free Hamiltonian**

Prospects

- **Bell-CHSH inequality**
- **Angle** and **orbital angular momentum**, **phase operator**, and **QCD axion**
- Application of similar approach to **Lorentz invariant** / **covariant Gaussian wave-packet basis**, achieving **relativistic extension**

当初のハイゼンベルクの理念：位置の測定誤差と運動量が受ける撹乱



「アインシュタイン vs 量子力学」（森田邦久著，化学同人，2015年）より引用

ハイゼンベルクの写真
by unknown author
File:Bundesarchiv Bild183-R57262, Werner Heisenberg.jpg - Wikimedia Commons

ハイゼンベルクの不等式

$$\boxed{\varepsilon[\hat{x}]\eta[\widehat{p_x}] \geq \frac{\hbar}{2}}$$

他の主な不確定性関係

ケナード・ワイル・ロバートソン

$$\sigma[\hat{x}]\sigma[\widehat{p_x}] \geq \frac{\hbar}{2}$$

内在的な揺らぎのみ

小澤

$$\varepsilon^{O}[\hat{A}]\varepsilon^{O}[\hat{B}] + \varepsilon^{O}[\hat{A}]\sigma[\hat{B}] + \sigma[\hat{A}]\varepsilon^{O}[\hat{B}] \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$$

内在的な揺らぎだけでなく測定誤差（と擾乱）

運動量表示

$$\boxed{\langle \mathrm{p}|\mathrm{X},\mathrm{P};\sigma\rangle \coloneqq \left(\frac{\sigma}{\pi}\right)^{\frac{1}{4}} \exp\left(-\mathrm{ipX} - \frac{\sigma}{2}(\mathrm{p}-\mathrm{P})^2\right)}$$

非直交

$$\langle \mathrm{X},\mathrm{P};\sigma|\mathrm{X}',\mathrm{P}';\sigma\rangle$$

$$= \exp\left(i\,\frac{\mathrm{P}+\mathrm{P}'}{2}(\mathrm{X}-\mathrm{X}') - \frac{(\mathrm{X}-\mathrm{X}')^2}{4\sigma} - \frac{\sigma}{4}(\mathrm{P}-\mathrm{P}')^2 \right)$$

ガウス波束 POVM

$\mathbb{R}^2 := \{(X,P)\}$

- $\{|X,P;\sigma\rangle\langle X,P;\sigma|\}_{\mathbb{R}^2}$ は POVM：

$$|X,P;\sigma\rangle\langle X,P;\sigma| \geq 0\ ,$$ （正値性）

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}X\mathrm{d}P}{2\pi} |X,P;\sigma\rangle\langle X,P;\sigma| = \hat{1}$$ (完全性)

- ガウス波束基底による POVM 測定 M：
  - 状態 $\hat{\rho}$ の下で (X,P) を得る測定確率は

$$\mathrm{Tr}[|X,P;\sigma\rangle\langle X,P;\sigma|\hat{\rho}]$$

  - 位置と運動量の同時測定の有効な手段

Oda Ogawa(2025) からの引用

「It would be fascinating to explore the time-energy uncertainty relation on this ground.」

その “fascinating” を実際にやった
ただし同時測定ではなく **到着時刻測定**で

点測定器 $\mathrm{M}_{\boldsymbol{x}}$　　　測定結果の空間： $\Omega = \{\mathbb{R}, \mathrm{ND}\}$

γ: 測定器の詳細に依存する結合定数　　$[\gamma] = -(d-1)$

時刻 t に粒子が検出される事象に対応する **POVM 密度**：

$$\widehat{\Pi_{\boldsymbol{x}}}(t) = \gamma |\vec{x}\rangle\langle\vec{x}| \qquad [\widehat{\Pi_{\boldsymbol{x}}}(t)] = 1$$

区間 $\mathcal{J} = [T_{\mathrm{i}}, T_{\mathrm{f}}]$ 内の時刻に粒子が検出される事象に対応する **POVM 要素**

$$\hat{E}(\mathcal{J}) := \int_{\mathcal{J}} \mathrm{d}t\, \widehat{\Pi_{\boldsymbol{x}}}(t) \qquad [\hat{E}(\mathcal{J})] = 0$$

検出されない事象=ND に対応する **POVM 要素**：

$$\widehat{\Pi_{\boldsymbol{x}}}(\mathrm{ND}) = \hat{1} - \int_{\mathcal{J}} \mathrm{d}t\, \widehat{\Pi_{\boldsymbol{x}}}(t) \qquad [\widehat{\Pi_{\boldsymbol{x}}}(\mathrm{ND})] = 0$$

状態 $\hat{\rho}$ についての全検出確率（規格化因子）：

$$P_D[\hat{\rho}; M_{\boldsymbol{x}}] := \mathrm{Tr}\left[\hat{\rho}\int_{\mathcal{J}} \mathrm{d}t\, \widehat{\Pi_{\boldsymbol{x}}}(t)\right] = \int_{\mathcal{J}} \mathrm{d}t\, [M_{\boldsymbol{x}}\hat{\rho}](t)$$

検出時間の条件付き期待値：

$$\frac{\mathrm{Tr}\left[\hat{\rho}\int_{\mathcal{J}} \mathrm{d}t\, t\,\widehat{\Pi_{\boldsymbol{x}}}(t)\right]}{P_D[\hat{\rho}; M_{\boldsymbol{x}}]} = \frac{\int_{\mathcal{J}} \mathrm{d}t\, t\,[M_{\boldsymbol{x}}\hat{\rho}](t)}{\int_{\mathcal{J}} \mathrm{d}t\, [M_{\boldsymbol{x}}\hat{\rho}](t)} = \frac{\langle \mathrm{id}_{\mathcal{J}} \rangle_{M_{\boldsymbol{x}}\hat{\rho}}}{\langle \chi_{\mathcal{J}} \rangle_{M_{\boldsymbol{x}}\hat{\rho}}}$$

$$\chi_{\mathcal{J}}(t) = 1,\ \chi_{\mathcal{J}}(\mathrm{ND}) = 0$$
$$\mathrm{id}_{\mathcal{J}}(t) = t,\ \mathrm{id}_{\mathcal{J}}(\mathrm{ND}) = 0$$

点測定器についての引き戻し時間作用素

$$\widehat{T_x} = \widehat{\mathrm{M}_x^\star \mathrm{id}_{\mathbb{R}}} \qquad \widehat{I_x} = \widehat{\mathrm{M}_x^\star \chi_{\mathbb{R}}}$$

$$\langle \boldsymbol{p} | \widehat{T_x} | \boldsymbol{p}' \rangle = \lambda \int_{\mathcal{J}} \mathrm{d}t\, t \exp\left[-i \left(\frac{\boldsymbol{p}'^2}{2m} - \frac{\boldsymbol{p}^2}{2m} \right) + i(\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{x} \right]$$

$$\langle \boldsymbol{p} | \widehat{I_x} | \boldsymbol{p}' \rangle = \lambda \int_{\mathcal{J}} \mathrm{d}t\, \exp\left[-i \left(\frac{\boldsymbol{p}'^2}{2m} - \frac{\boldsymbol{p}^2}{2m} \right) + i(\boldsymbol{p}' - \boldsymbol{p}) \cdot \boldsymbol{x} \right]$$

古典的観測量 $\mathrm{id}_{\mathcal{J}}$ を、測定 M_x を介して量子側へ引き戻すことで演算子として定義した

時間作用素の条件付き期待値

粒子がまっすぐ検出器に向かってくるという条件

$$\boldsymbol{x}-\boldsymbol{X}_{\mathrm{S}} \propto \boldsymbol{P}_{\mathrm{S}},\ (\boldsymbol{x}-\boldsymbol{X}_{\mathrm{S}}) \cdot \boldsymbol{P}_{\mathrm{S}} > 0$$

の下で

$$\frac{\langle \widehat{T_{\boldsymbol{x}}} \rangle_{\hat{\rho}}}{\langle \widehat{I_{\boldsymbol{x}}} \rangle_{\hat{\rho}}} \simeq T_S + m \frac{|\boldsymbol{x}-\boldsymbol{X}_{\mathrm{S}}|}{|\boldsymbol{P}_{\mathrm{S}}|}$$

$\widehat{T_{\boldsymbol{x}}}$ が確かに時間に対応する演算子であることを示唆

ケナードの不等式（1927）

位置と運動量の測定値の内在的なゆらぎに関する不確定性関係

$$\boxed{\sigma[\hat{x}]\sigma[\widehat{p_x}] \geq \frac{\hbar}{2}}$$

ガウス波束：

$$\boxed{\sigma[\hat{x}]\sigma[\widehat{p_x}] = \frac{\hbar}{2}}$$

等号を達成

ロバートソンが一般の自己共役作用素について拡張

小澤の不等式（2003）

$$\boxed{\varepsilon^{O}[\hat{A}]\varepsilon^{O}[\hat{B}] + \varepsilon^{O}[\hat{A}]\sigma[\hat{B}] + \sigma[\hat{A}]\varepsilon^{O}[\hat{B}] \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|}$$

小澤の誤差 $\varepsilon^{O}[\hat{A}] := \sqrt{\langle(\hat{\mathbb{A}} - \hat{A}\otimes\hat{1})^2\rangle}$

$\hat{\mathbb{A}}$：測定器のメーターに相当する量

- 重力波検出で標準量子限界を超える可能性を指摘
- 測定誤差と擾乱についても同様に成り立つ
- $\hat{\mathbb{A}} - \hat{A}\otimes\hat{1}$ という演算子の和の操作的意味が不明確

小澤の定式化では位置の測定誤差と運動量が受ける擾乱の積は 0 になりうる

Ozawa, 1988;
Ozawa, 2002

$$\boxed{\varepsilon^{O}[\hat{x}]\eta^{O}[\widehat{p_x}] \geq 0}$$

von Neumann のモデル: $\boxed{\varepsilon^{O}[\hat{x}]\eta^{O}[\widehat{p_x}] \geq \frac{\hbar}{2}}$

収縮状態測定モデル： $\boxed{\varepsilon^{O}[\hat{x}]\eta^{O}[\widehat{p_x}] = 0}$

時間とエネルギーの不確定性関係

$$\Delta E \Delta \mathrm{t} \gtrsim h$$

時間は自己共役演算子ではない（ cf. **Pauli の定理**）ので、さまざまな議論の対象となってきた（ロバートソン型の不等式ではない？）

詳しく知りたい方は谷村省吾「時間とエネルギーの不確定性関係―腑に落ちない関係」素粒子論研究電子版Vol. 16 (2014) No. 3 を参照

Aharonov-Bohm の時間作用素（非相対論的）

時間に対応する演算子を構成する試みは過去にあった

$$\widehat{T_{\mathrm{AB}}} = -\frac{m}{2}\left(\widehat{p_x}^{-1}\hat{x} + \hat{x}\widehat{p_x}^{-1}\right)$$

$$[\widehat{T_{\mathrm{AB}}}, \widehat{H_{\mathrm{free}}}] = -i\hat{1}$$

※ $\widehat{T_{\mathrm{AB}}}$ は対称作用素であって、自己共役作用素でないとされている（参考：パウリの定理）

※ $\widehat{T_{\mathrm{AB}}}$ の物理的意味は「到着時刻」であると解釈されている

李・筒井の誤差

$$\varepsilon_{\hat{\rho}}[\hat{A};\mathrm{M}] := \sqrt{\underbrace{\| \hat{A} \|_{\hat{\rho}}^{2}}_{\text{測定前}} - \underbrace{\| \mathrm{M}_{\star}\hat{A} \|_{\mathrm{M}\hat{\rho}}^{2}}_{\text{測定後}}}$$

李の誤差

$$\tilde{\varepsilon}_{\hat{\rho}}[\hat{A};\mathrm{M}] := \sqrt{\underbrace{\| \mathrm{M}_{\hat{\rho}}^{\star}|^{-1}\hat{A} \|_{\mathrm{M}\hat{\rho}}^{2}}_{\text{測定後}} - \underbrace{\| \hat{A} \|_{\hat{\rho}}^{2}}_{\text{測定前}}}$$

$$\boxed{\tilde{\varepsilon}_{\hat{\rho}}[\hat{A};\mathrm{M}] \geq \varepsilon_{\hat{\rho}}[\hat{A};\mathrm{M}]}$$

操作的意味が明確

李・筒井/李の不等式の導出

Lee and Tsutsui, 2020;
Lee, 2022

$\hat{A},\hat{B}\in S(\mathcal{H})$ $\mathrm{f},\mathrm{g}\in R(\Omega)$

$S(\mathcal{H})\times R(\Omega)$ 上の半内積：

$$\langle(\hat{A},\mathrm{f}),(\hat{B},\mathrm{g})\rangle \coloneqq \langle\hat{A}\hat{B}\rangle_{\hat{\rho}}+\langle\mathrm{fg}\rangle_{M\hat{\rho}}-\langle\widehat{\mathrm{M}^{\star}\mathrm{f}}\,\widehat{\mathrm{M}^{\star}\mathrm{g}}\rangle_{\hat{\rho}}$$

$S(\mathcal{H})\times R(\Omega)$ 上の半ノルム：

$$\|(\hat{A},\mathrm{f})\| \coloneqq \sqrt{\langle(\hat{A},\mathrm{f}),(\hat{A},\mathrm{f})\rangle}$$

対応する Cauchy-Schwarz の不等式:

$$\|(\hat{A},\mathrm{f})\|\|(\hat{B},\mathrm{g})\|\geq|\langle(\hat{A},\mathrm{f}),(\hat{B},\mathrm{g})\rangle|$$

李・筒井：$\hat{A}\to\hat{A}-\mathrm{M}^{\star}\mathrm{M}_{\star}\hat{A},\ \mathrm{f}\to\mathrm{M}_{\star}\hat{A}$

李：$\hat{A}\to\hat{A}-\mathrm{M}^{\star}\mathrm{M}^{\star -1}\hat{A},\ \mathrm{f}\to\mathrm{M}^{\star}|^{-1}\hat{A}$

李・筒井の不等式は**小澤の不等式**を含んでいる

$$\epsilon^{\mathrm{O}}_{\widehat{\rho}}\left[\widehat{A}\right]\epsilon^{\mathrm{O}}_{\widehat{\rho}}\left[\widehat{B}\right] \geq \varepsilon_{\widehat{\rho}}\left[\widehat{A}\right]\varepsilon_{\widehat{\rho}}\left[\widehat{B}\right] \overset{\mathrm{LT}}{\geq} \sqrt{\mathcal{I}^2_{\widehat{\rho}}\left[\widehat{A},\widehat{B}\right]+\mathcal{R}^2_{\widehat{\rho}}\left[\widehat{A},\widehat{B}\right]}$$

$$\geq \left|\mathcal{I}_{\widehat{\rho}}\left[\widehat{A},\widehat{B}\right]\right| \geq \frac{1}{2}\left|\left\langle\left[\widehat{A},\widehat{B}\right]\right\rangle_{\widehat{\rho}}\right| - \epsilon^{\mathrm{O}}_{\widehat{\rho}}\left[\widehat{A}\right]\sigma_{\widehat{\rho}}\left[\widehat{B}\right] - \sigma_{\widehat{\rho}}\left[\widehat{A}\right]\epsilon^{\mathrm{O}}_{\widehat{\rho}}\left[\widehat{B}\right]$$

他にも
ケナード・ワイル・ロバートソンの不等式も含む

不確定性関係の大雑把な歴史

- **ハイゼンベルクの不等式**： ガンマ線顕微鏡の思考実験；電子の**位置の測定誤差**と**運動量が受ける擾乱**
- **ケナード・ワイル・ロバートソンの不等式**： 測定と無関係な内在的揺らぎについて（標準的）
- **小澤の不等式**： 内在的揺らぎに加えて測定誤差（と擾乱）を合わせた不等式；測定誤差の操作的意味が不明確
- **李・筒井形式**： 近年、李と筒井が提案；以上のような不確定性関係を含む＝「**普遍的**」；「**誤差の操作的意味が明確**」（後で解説）

Ozawa, 1988;
Ozawa, 2002

von Neumann のモデル:

$$\boxed{\hat{U} = \exp(-i\hbar\hat{x}\widehat{p_y})}$$

$$\boxed{\varepsilon^O[\hat{x}]\eta^O[\widehat{p_x}] \geq \frac{\hbar}{2}}$$

収縮状態測定モデル:

$$\boxed{\hat{U} = \exp\left(-i\frac{K\pi}{3\sqrt{3}\hbar}\{2(\hat{x}\widehat{p_y} - \widehat{p_x}\hat{y}) + (\hat{x}\widehat{p_x} - \hat{y}\widehat{p_y})\}\right)}$$

相互作用時間とパラメータ K を適切に選ぶと

$$\boxed{\varepsilon^O[\hat{x}]\eta^O[\widehat{p_x}] = 0}$$

物理量と状態

	物理量	**状態**
古典	実関数 $\mathrm{f}(\omega) \in R(\Omega)$	確率密度関数 $\mathrm{p}(\omega) \in W(\Omega)$
量子	自己共役演算子 $\hat{A} \in S(\mathcal{H})$	密度演算子 $\hat{\rho} \in Z(\mathcal{H})$

物理量空間: すべての物理量から成る集合 $(R(\Omega), S(\mathcal{H}))$

状態空間: すべての状態から成る集合 $(W(\Omega), Z(\mathcal{H}))$

半ノルムと半内積

	期待値	**半ノルム**	**半内積**
古典	$\langle f \rangle_p := \int_\Omega d\omega\, f(\omega) p(\omega)$	$\lVert f \rVert_p^2 := \langle f^2 \rangle_p$	$\langle f, g \rangle_p := \langle fg \rangle_p$
量子	$\langle \hat{A} \rangle_{\hat{\rho}} := \mathrm{Tr}[\hat{A}\hat{\rho}]$	$\lVert \hat{A} \rVert_{\hat{\rho}}^2 := \langle \hat{A}^2 \rangle_{\hat{\rho}}$	$\langle \hat{A}, \hat{B} \rangle_{\hat{\rho}} := \langle \frac{\{\hat{A}, \hat{B}\}}{2} \rangle_{\hat{\rho}}$

押し出し

測定 M, 量子物理量 $\hat{A}$, 与えられた量子状態 $\hat{\rho}$ についての**押し出し** $\mathrm{M}_\star\hat{A}$ を定める $\widehat{\mathrm{M}^\star \mathrm{f}}$ を用いた双対関係 :

$$\boxed{\langle \hat{A}, \widehat{\mathrm{M}^\star \mathrm{f}} \rangle_{\hat{\rho}} = \langle \mathrm{M}_\star \hat{A}, \mathrm{f} \rangle_{\mathrm{M}\hat{\rho}}}$$

$$\boxed{\langle \hat{A}, \hat{B} \rangle_{\hat{\rho}} \coloneqq \langle \frac{\{\hat{A}, \hat{B}\}}{2} \rangle_{\hat{\rho}}}$$

$$\boxed{\langle \mathrm{f}, \mathrm{g} \rangle_{\mathrm{p}} \coloneqq \langle \mathrm{fg} \rangle_{\mathrm{p}}}$$

Lee-Tsutsui(LT) error

$$\varepsilon_{\hat{\rho}}[\hat{A}; \mathrm{M}] := \sqrt{\| \hat{A} \|_{\hat{\rho}}^2 - \| \mathrm{M}_{\star}\hat{A} \|_{\mathrm{M}\hat{\rho}}^2}$$

Pre-measurement

Post-measurement

$\| \hat{A} \|_{\hat{\rho}}^2 := \langle \hat{A}^2 \rangle_{\hat{\rho}}$

$\| \mathrm{f} \|_{\mathrm{p}}^2 := \langle \mathrm{f}^2 \rangle_{\mathrm{p}}$

Clear operational meaning

李・筒井の不等式

状態 $\hat{\rho}$ と測定 M について、

$$\boxed{\varepsilon[\hat{A}]\varepsilon[\hat{B}] \geq \sqrt{\mathcal{J}_{\hat{\rho}}^2[\hat{A},\hat{B}] + \mathcal{R}_{\hat{\rho}}^2[\hat{A},\hat{B}]}}$$

$$\mathcal{J}_{\hat{\rho}}[\hat{A},\hat{B}] := \left\langle \frac{[\hat{A},\hat{B}]}{2i} \right\rangle_{\hat{\rho}} - \left\langle \frac{[\widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}\hat{A}},\hat{B}]}{2i} \right\rangle_{\hat{\rho}} - \left\langle \frac{[\hat{A},\widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}\hat{B}}]}{2i} \right\rangle_{\hat{\rho}}$$ 純量子項

$$\mathcal{R}_{\hat{\rho}}[\hat{A},\hat{B}] := \left\langle \frac{\{\hat{A},\hat{B}\}}{2} \right\rangle_{\hat{\rho}} - \langle \mathrm{M}_{\star}\hat{A}, \mathrm{M}_{\star}\hat{B} \rangle_{\mathrm{M}\hat{\rho}}$$ 半古典項

李・筒井の不等式は Cauchy-Schwarz 不等式

李・筒井と小澤の誤差の関係

- 小澤の誤差は李・筒井の誤差より小さくない： $\varepsilon^{O}_{\hat{\rho}}[\hat{x}] \geq \varepsilon_{\hat{\rho}}[\hat{x}]$
  - $\because$ $\hat{\mathbb{X}}$ の半ノルムは $\mathrm{M}^{\star}_{\hat{\rho}}|^{-1}$ の下で不変： $\|\hat{\mathbb{X}}\|_{\hat{\rho}} = \|\mathrm{M}^{\star}|^{-1}\hat{\mathbb{X}}\|_{\mathrm{M}\hat{\rho}}$

- $\left|\mathcal{J}_{\hat{\rho}}[\hat{x},\hat{p}]\right| \geq |\langle\frac{[\hat{A},\hat{B}]}{2i}\rangle_{\hat{\rho}}|$- $\sigma_{\hat{\rho}}[\hat{x} - \widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}}\hat{x}]\ \sigma_{\hat{\rho}}[\hat{p}]$- $\sigma_{\hat{\rho}}[\hat{x}]\sigma_{\hat{\rho}}[\hat{p} - \widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}}\hat{p}]$

- $\sigma_{\hat{\rho}}[\hat{A} - \widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}}\hat{A}] \leq \varepsilon_{\hat{\rho}}[\hat{A}]$

李の不等式

状態 $\hat{\rho}$ と測定 M について、

$$\tilde{\mathcal{E}}[\hat{A}]\tilde{\mathcal{E}}[\hat{B}] \geq \sqrt{\mathcal{J}_{0\hat{\rho}}^2[\hat{A},\hat{B}] + \tilde{\mathcal{R}}_{\hat{\rho}}^2[\hat{A},\hat{B}]}$$

$$\mathcal{J}_{0\hat{\rho}}[\hat{A},\hat{B}] := \left\langle \frac{[\hat{A},\hat{B}]}{2i} \right\rangle_{\hat{\rho}}$$ 純量子項

$$\tilde{\mathcal{R}}_{\hat{\rho}}[\hat{A},\hat{B}] := \left\langle \frac{\{\hat{A},\hat{B}\}}{2} \right\rangle_{\hat{\rho}} - \langle \mathrm{M}^{\star}|^{-1}\hat{A}, \mathrm{M}^{\star}|^{-1}\hat{B} \rangle_{\mathrm{M}\hat{\rho}}$$ 半古典項

李の不等式は Cauchy-Schwarz 不等式

Arthurs-Kelly-Goodman の不等式を含む

李・筒井/李の不等式は誤差・擾乱関係も全く同様

$$\boxed{\varepsilon[\hat{A}]\eta[\hat{B}] \geq \sqrt{\mathcal{J}_{\hat{\rho}}^2[\hat{A},\hat{B}] + \mathcal{R}_{\hat{\rho}}^2[\hat{A},\hat{B}]}}$$

$$\boxed{\tilde{\varepsilon}[\hat{A}]\tilde{\eta}[\hat{B}] \geq \sqrt{\mathcal{J}_{0\hat{\rho}}^2[\hat{A},\hat{B}] + \tilde{\mathcal{R}}_{\hat{\rho}}^2[\hat{A},\hat{B}]}}$$

位置でも運動量でもガウス分布

遷移確率　$\boxed{|\langle X, P; \sigma | X', P'; \sigma\rangle|^2}$

運動量について先に積分：　幅 $\sim \sqrt{\sigma}$ で X' 周りに局在

$$\frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(X - X')^2}{2\sigma}\right)$$

位置について先に積分：　幅 $\sim \sqrt{1/\sigma}$ で P' 周りに局在

$$\sqrt{\frac{\sigma}{2\pi}} \exp\left(-\frac{\sigma}{2}(P - P')^2\right)$$

ガウス波束基底 $\{|X,P;\sigma\rangle\}$ の運動量表示：

$$\boxed{\langle \mathrm{p}|\mathrm{X},\mathrm{P};\sigma\rangle := \left(\frac{\sigma}{\pi}\right)^{\frac{1}{4}} \exp\left(-\mathrm{ipX} - \frac{\sigma}{2}(\mathrm{p}-\mathrm{P})^2\right)}$$

完全性関係：

$$\boxed{\int_{\mathbb{R}^2} \frac{\mathrm{d}X\mathrm{d}P}{2\pi} |X,P;\sigma\rangle\langle X,P;\sigma| = \hat{1}}$$

異なるガウス波束基底同士は非直交

$$\boxed{\begin{aligned}&\langle \mathrm{X,P};\sigma|\mathrm{X',P'};\sigma\rangle\\&=\exp\left(i\frac{\mathrm{P+P'}}{2}(\mathrm{X}-\mathrm{X'})-\frac{(\mathrm{X-X'})^2}{4\sigma}-\frac{\sigma}{4}(\mathrm{P}-\mathrm{P'})^2\right)\end{aligned}}$$

平面波と異なり規格化可能なので Hilbert 空間の元になっている

ガウス波束基底による測定は射影測定にはならないが、次で見るように **Positive Operator Valued Measure (POVM) 測定**になっている

小澤の不等式のレビュー

対象系

$\mathcal{H}, \hat{\rho}, \hat{x}, \hat{p},$
$\hat{\Pi}(X, P)$

相互作用

$\hat{U}$

測定器系

$\mathcal{K}, \hat{\xi}, \hat{X}, \hat{P},$
$\hat{E}^{\hat{X}}(X), \hat{E}^{\hat{P}}(P)$

メーター物理量

$$[\hat{X}, \hat{P}] = 0, \ \hat{X}|X, P\rangle = X|X, P\rangle, \hat{P}|X, P\rangle = P|X, P\rangle$$

射影作用素： $\hat{E}^{\hat{X}}(X), \ \hat{E}^{\hat{P}}(P)$

小澤の主張：任意の POVM $\hat{\Pi}(X,P)$について、次を実現する5つ組 $(\mathcal{K}, \hat{\xi}, \hat{U}, \hat{X}, \hat{P})$ が存在する:

$$\hat{\Pi}(X,P) = \mathrm{Tr}_{\mathcal{K}}\left\{\hat{U}^{\dagger}\left(\hat{1}\otimes|X,P\rangle\langle X,P|\right)\hat{U}\left(\hat{1}\otimes\hat{\xi}\right)\right\}$$

メーターに対応する量：

$$\hat{\mathbb{X}} := \hat{U}^{\dagger}\left(\hat{1}\otimes\hat{X}\right)\hat{U} \quad \hat{\mathbb{P}} := \hat{U}^{\dagger}\left(\hat{1}\otimes\hat{P}\right)\hat{U}$$

誤差演算子の定義：

$$\widehat{N}_{\hat{q}} := \widehat{\mathbb{Q}} - \hat{q} \otimes \hat{1} \qquad \widehat{\mathbb{Q}} = \widehat{\mathbb{X}} \text{ or } \widehat{\mathbb{P}},\ \hat{q} = \hat{x} \text{ or } \hat{p}$$

小澤の誤差の定義：

$$\epsilon^{\mathrm{O}}_{\widehat{\rho}}[\widehat{q}] := \left\| \widehat{N}_{\widehat{q}} \right\|_{\widehat{\rho} \otimes \widehat{\xi}} = \sqrt{\mathrm{Tr}\left[\widehat{N}^2_{\widehat{q}} \left(\widehat{\rho} \otimes \widehat{\xi} \right) \right]}$$

$$[\widehat{\mathbb{X}}, \widehat{\mathbb{P}}] = 0 \;\rightarrow\; 0 = \left[\widehat{N}_{\widehat{x}}, \widehat{N}_{\widehat{p}}\right] + \left[\widehat{N}_{\widehat{x}}, \widehat{p} \otimes \widehat{1}\right] + \left[\widehat{x} \otimes \widehat{1}, \widehat{N}_{\widehat{p}}\right] + [\widehat{x}, \widehat{p}] \otimes \widehat{1}.$$

上の事実から、小澤は次の小澤の不等式を得た：

$$\epsilon^{\mathrm{O}}_{\widehat{\rho}}[\widehat{x}]\, \epsilon^{\mathrm{O}}_{\widehat{\rho}}[\widehat{p}] + \epsilon^{\mathrm{O}}_{\widehat{\rho}}[\widehat{x}]\, \sigma_{\widehat{\rho}}[\widehat{p}] + \sigma_{\widehat{\rho}}[\widehat{x}]\, \epsilon^{\mathrm{O}}_{\widehat{\rho}}[\widehat{p}] \geq \frac{1}{2} \left| \langle [\widehat{x}, \widehat{p}] \rangle_{\widehat{\rho}} \right|$$

李・筒井の誤差と不等式

不等式の計算のために必要な量を計算していく

※見せるだけ追わないでよい

$$\left\langle \frac{[\hat{x},\hat{p}]}{2i} \right\rangle_{\hat{\rho}_{\mathrm{in}}} = \frac{1}{2}$$

$$\left\langle \frac{\{\hat{x},\hat{p}\}}{2} \right\rangle_{\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}} P_{\mathrm{in}}$$

$$\|\hat{x}\|^2_{\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}}{2}$$

$$\|\hat{p}\|^2_{\hat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{1}{2\sigma_{\mathrm{in}}}$$

押し出し

※見せるだけ追わないでよい

$$\sigma_{\mathrm{red}} := \frac{\sigma\sigma_{\mathrm{in}}}{\sigma+\sigma_{\mathrm{in}}}$$

$$[\mathrm{M}_\star \hat{x}](X,P) = \bar{X} := \sigma_{\mathrm{red}}\left(\frac{X}{\sigma} + \frac{X_{\mathrm{in}}}{\sigma_{\mathrm{in}}}\right)$$

X と X_{in} を σ_{in} と σ でそれぞれ重みづけして平均化したもの

$$[\mathrm{M}_\star \hat{p}](X,P) = \bar{P} := \frac{\sigma P + \sigma_{\mathrm{in}} P_{\mathrm{in}}}{\sigma+\sigma_{\mathrm{in}}}$$

P と P_{in} を σ と σ_{in} でそれぞれ重みづけして平均化したもの

$$\sigma_{\mathrm{sum}} := \sigma + \sigma_{\mathrm{in}}$$

$$\|\mathrm{M}_\star \hat{x}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}^2}{2\sigma_{\mathrm{sum}}}$$

$$\|\mathrm{M}_\star \hat{p}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{\sigma}{2\sigma_{\mathrm{in}}\sigma_{\mathrm{sum}}}$$

$$\langle \mathrm{M}_\star \hat{x}, \mathrm{M}_\star \hat{p}\rangle_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}} P_{\mathrm{in}}$$

李・筒井の誤差

※見せるだけ追わないでよい

$$\varepsilon_{\hat{\rho}_{\mathrm{in}}}[\hat{x};\mathrm{M}] \coloneqq \sqrt{\|\hat{x}\|^2_{\hat{\rho}_{\mathrm{in}}} - \|\mathrm{M}_\star\hat{x}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}}}$$

$$\|\hat{x}\|^2_{\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}}{2}$$

$$\|\mathrm{M}_\star\hat{x}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}^2}{2\sigma_{\mathrm{sum}}}$$

$$\|\hat{p}\|^2_{\hat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{1}{2\sigma_{\mathrm{in}}}$$

$$\|\mathrm{M}_\star\hat{p}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{\sigma}{2\sigma_{\mathrm{in}}\sigma_{\mathrm{sum}}}$$

$$\varepsilon^2[\hat{x}] = \frac{\sigma_{\mathrm{red}}}{2},$$

$$\varepsilon^2[\hat{p}] = \frac{1}{2\sigma_{\mathrm{sum}}}$$

$$\therefore\ \varepsilon[\hat{x}]\varepsilon[\hat{p}] = \frac{1}{2}\sqrt{\frac{\sigma_{\mathrm{red}}}{\sigma_{\mathrm{sum}}}}$$

押し出しの引き戻し　※見せるだけ追わないでよい

$$\widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}\hat{x}} = \int_{\mathbb{R}^2} \frac{\mathrm{d}X\mathrm{d}P}{2\pi} \bar{X} |X, P; \sigma\rangle\langle X, P; \sigma|$$

$$\boxed{\bar{X} := \sigma_{\mathrm{red}} \left(\frac{X}{\sigma} + \frac{X_{\mathrm{in}}}{\sigma_{\mathrm{in}}} \right)}$$

$$\widehat{\mathrm{M}^{\star}\mathrm{M}_{\star}\hat{p}} = \int_{\mathbb{R}^2} \frac{\mathrm{d}X\mathrm{d}P}{2\pi} \bar{P} |X, P; \sigma\rangle\langle X, P; \sigma|$$

$$\boxed{\bar{P} := \frac{\sigma P + \sigma_{\mathrm{in}} P_{\mathrm{in}}}{\sigma + \sigma_{\mathrm{in}}}}$$

不等式の右辺の量

※見せるだけ追わないでよい

$$\left\langle\frac{[\widehat{\mathrm{M}^\star\mathrm{M}_\star\hat{x}},\hat{p}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}=\frac{\sigma_{\mathrm{in}}}{2\sigma_{\mathrm{sum}}}\quad\left\langle\frac{[\hat{x},\widehat{\mathrm{M}^\star\mathrm{M}_\star\hat{p}}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}=\frac{\sigma}{2\sigma_{\mathrm{sum}}}$$

$$\boxed{\mathcal{J}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]\coloneqq\left\langle\frac{[\hat{x},\hat{p}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}-\left\langle\frac{[\widehat{\mathrm{M}^\star\mathrm{M}_\star\hat{x}},\hat{p}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}-\left\langle\frac{[\hat{x},\widehat{\mathrm{M}^\star\mathrm{M}_\star\hat{p}}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}}$$

$$\boxed{\mathcal{R}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]\coloneqq\left\langle\frac{\{\hat{x},\hat{p}\}}{2}\right\rangle_{\hat{\rho}_{\mathrm{in}}}-\langle\mathrm{M}_\star\hat{x},\mathrm{M}_\star\hat{p}\rangle_{\mathrm{M}\hat{\rho}_{\mathrm{in}}}}\quad\boxed{\left\langle\frac{\{\hat{x},\hat{p}\}}{2}\right\rangle_{\hat{\rho}_{\mathrm{in}}}=X_{\mathrm{in}}P_{\mathrm{in}}}\quad\boxed{\left\langle\frac{[\hat{x},\hat{p}]}{2i}\right\rangle_{\hat{\rho}_{\mathrm{in}}}=\frac{1}{2}}$$

$$\therefore\ \mathcal{J}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]=0,\ \mathcal{R}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]=0\qquad\boxed{\langle\mathrm{M}_\star\hat{x},\mathrm{M}_\star\hat{p}\rangle_{\mathrm{M}\hat{\rho}_{\mathrm{in}}}=X_{\mathrm{in}}P_{\mathrm{in}}}$$

李・筒井の不等式

$\sigma_{\mathrm{sum}} := \sigma + \sigma_{\mathrm{in}}$

$\sigma_{\mathrm{red}} := \frac{\sigma\sigma_{\mathrm{in}}}{\sigma+\sigma_{\mathrm{in}}}$

$$\left(\varepsilon[\hat{x}]\varepsilon[\widehat{p_x}] =\right) \frac{1}{2}\sqrt{\frac{\sigma_{\mathrm{red}}}{\sigma_{\mathrm{sum}}}} \geq 0$$

右辺が 0 なので下限が自明

小澤の不等式と整合的

$\varepsilon^{O}[\hat{x}]\eta^{O}[\widehat{p_x}] \geq 0$

下限は射影測定になる極限である $\sigma \to 0$ または $\sigma \to \infty$ のときに達成

李の誤差と不等式

引き戻しの逆写像

※見せるだけ追わないでよい

$$[\mathrm{M}^\star|^{-1}\hat{x}](X,P) = X$$

$$[\mathrm{M}^\star|^{-1}\hat{p}](X,P) = P$$

$$\|\mathrm{M}^\star|^{-1}\hat{x}\|^2_{\mathrm{M}_{\hat{\rho}_{\mathrm{in}}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{sum}}}{2}$$

$$\sigma_{\mathrm{sum}} := \sigma + \sigma_{\mathrm{in}}$$

$$\|\mathrm{M}^\star|^{-1}\hat{p}\|^2_{\mathrm{M}_{\hat{\rho}_{\mathrm{in}}}} = P_{\mathrm{in}}^2 + \frac{1}{2\sigma_{\mathrm{red}}}$$

$$\sigma_{\mathrm{red}} := \frac{\sigma\sigma_{\mathrm{in}}}{\sigma+\sigma_{\mathrm{in}}}$$

$$\langle \mathrm{M}^\star|^{-1}\hat{x}, \mathrm{M}^\star|^{-1}\hat{p}\rangle_{\mathrm{M}_{\hat{\rho}_{\mathrm{in}}}} = X_{\mathrm{in}}P_{\mathrm{in}}$$

李の誤差

※見せるだけ追わないでよい

$$\boxed{\tilde{\varepsilon}_{\hat{\rho}}[\hat{A}; \mathrm{M}] := \sqrt{\| \mathrm{M}^{\star}|^{-1}\hat{A} \|^2_{\mathrm{M}\hat{\rho}} - \| \hat{A} \|^2_{\hat{\rho}}}}$$

$$\|\hat{x}\|^2_{\hat{\rho}_{\mathrm{in}}} = X^2_{\mathrm{in}} + \frac{\sigma_{\mathrm{in}}}{2}$$

$$\|\mathrm{M}^{\star}|^{-1}\hat{x}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = X^2_{\mathrm{in}} + \frac{\sigma_{\mathrm{sum}}}{2}$$

$$\|\hat{p}\|^2_{\hat{\rho}_{\mathrm{in}}} = P^2_{\mathrm{in}} + \frac{1}{2\sigma_{\mathrm{in}}}$$

$$\|\mathrm{M}^{\star}|^{-1}\hat{p}\|^2_{\mathrm{M}\hat{\rho}_{\mathrm{in}}} = P^2_{\mathrm{in}} + \frac{1}{2\sigma_{\mathrm{red}}}$$

$$\tilde{\varepsilon}^2[\hat{x}] = \frac{\sigma}{2},\ \tilde{\varepsilon}^2[\hat{p}] = \frac{1}{2\sigma}$$

$$\therefore\ \tilde{\varepsilon}[\hat{x}]\tilde{\varepsilon}[\hat{p}] = \frac{1}{2}$$

不等式の右辺の量

※見せるだけ追わないでよい

$$\mathcal{J}_{0\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] := \langle\frac{[\hat{x},\hat{p}]}{2i}\rangle_{\hat{\rho}_{\mathrm{in}}}$$

$$\tilde{\mathcal{R}}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] := \langle\frac{\{\hat{x},\hat{p}\}}{2}\rangle_{\hat{\rho}_{\mathrm{in}}} - \langle \mathrm{M}^{\star}|^{-1}\hat{x}, \mathrm{M}^{\star}|^{-1}\hat{p}\rangle_{\mathrm{M}\ \hat{\rho}_{\mathrm{in}}}$$

$$\langle\frac{[\hat{x},\hat{p}]}{2i}\rangle_{\hat{\rho}_{\mathrm{in}}} = \frac{1}{2}$$

$$\langle\frac{\{\hat{x},\hat{p}\}}{2}\rangle_{\hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}P_{\mathrm{in}}$$

$$\langle \mathrm{M}^{\star}|^{-1}\hat{x}, \mathrm{M}^{\star}|^{-1}\hat{p}\rangle_{\mathrm{M}\ \hat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}P_{\mathrm{in}}$$

$$\therefore \mathcal{J}_{0\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] = \frac{1}{2},\ \tilde{\mathcal{R}}_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] = 0$$

李の不等式

$$\boxed{\left(\tilde{\varepsilon}[\hat{x}]\ \tilde{\varepsilon}[\widehat{p_x}] =\right)\frac{1}{2} \geq \frac{1}{2}}$$

※自然単位系

通常の不確定性原理の下限を与え，かつ等号を常に達成！

$$\boxed{\varepsilon[\hat{x}]\eta[\widehat{p_x}] \geq \frac{\hbar}{2}}$$

ハイゼンベルクの不等式

$$\boxed{\varepsilon^{O}[\hat{x}]\eta^{O}[\widehat{p_x}] \geq \frac{\hbar}{2}}$$

小澤の不等式（von Neumann の測定モデル）

$$\boxed{\sigma[\hat{x}]\sigma[\widehat{p_x}] = \frac{\hbar}{2}}$$

ガウス波束について

との関係は…？

時間を含む位置表示

$$\langle\vec{x}|\vec{X},\boldsymbol{P};\sigma\rangle \coloneqq \left(\frac{1}{\pi\sigma\left(1+i\frac{t-T}{\sigma m}\right)^2}\right)^{\frac{d}{4}} \exp\left(\frac{-\mathrm{i}\frac{\boldsymbol{p}^2}{2m}(t-T)-\mathrm{i}\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{X})-\frac{1}{2\sigma}(\boldsymbol{x}-\boldsymbol{X})^2}{1+i\frac{t-T}{\sigma m}}\right)$$

$$\left|\langle\vec{x}|\vec{X},\boldsymbol{P};\sigma\rangle\right|^2 \coloneqq \left(\frac{1}{\pi\sigma\left(1+\left(\frac{t-T}{\sigma m}\right)^2\right)}\right)^{\frac{d}{2}} \exp\left(-\frac{\left[\boldsymbol{x}-\boldsymbol{X}-\frac{\boldsymbol{P}}{m}(t-T)\right]^2}{\sigma+\frac{(t-T)^2}{\sigma m^2}}\right)$$

状態 $|\vec{X},\boldsymbol{P};\sigma\rangle$ は$\boldsymbol{X}(t)\coloneqq\boldsymbol{X}+\frac{\boldsymbol{P}}{m}(t-T)$ の周りに幅 $\sigma+\frac{(t-T)^2}{\sigma m^2}$ で局在している

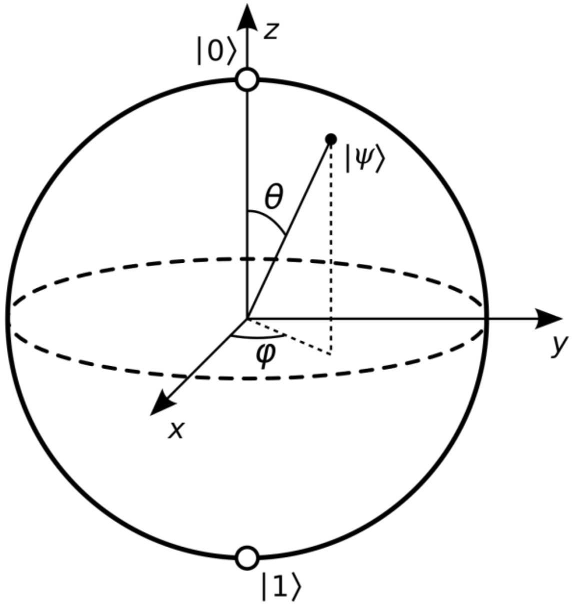
李・筒井による適用例: スピン2準位系

Lee, J.; Tsutsui, I. Uncertainty Relation for Errors Focusing on General POVM Measurements with an Example of Two-State Quantum Systems. *Entropy* **2020**, *22*, 1222. https://doi.org/10.3390/e22111222 に基づく

$$\vec{\sigma} = (\widehat{\sigma_1}, \widehat{\sigma_2}, \widehat{\sigma_3})\ (\widehat{\sigma_i}(i = 1,2,3)\text{: パウリ行列})$$

$\hat{\rho}$ の Bloch 表示:

$$\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{r} \cdot \vec{\sigma})$$



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where

$$\vec{r} := (r_1, r_2, r_3)\ (|\vec{r}| \leq 1)$$

スピン-1/2 粒子のスピン角運動量:

$$\widehat{S_i} = \frac{1}{2}\widehat{\sigma_i}\ (i = 1,2,3)$$

$\widehat{S_3}$ の測定（M）から $\widehat{S_1}$ と $\widehat{S_2}$ の押し出しを計算

$$\mathrm{M}_\star \widehat{S_1}(m) = \begin{cases} \dfrac{1}{2}\dfrac{r_1}{1+r_3} & \left(m = +\dfrac{1}{2}\right) \\ \dfrac{1}{2}\dfrac{r_1}{1-r_3} & \left(m = -\dfrac{1}{2}\right) \end{cases}$$

$$\mathrm{M}_\star \widehat{S_2}(m) = \begin{cases} \frac{1}{2}\frac{r_2}{1+r_3} & \left(m = +\frac{1}{2}\right) \\ \frac{1}{2}\frac{r_2}{1-r_3} & \left(m = -\frac{1}{2}\right) \end{cases}$$

李・筒井の誤差を計算

$$\varepsilon_{\hat{\rho}}[\widehat{S_1}]=\frac{1}{2}\sqrt{1-\frac{r_1^2}{1-r_3^2}}$$

$$\varepsilon_{\hat{\rho}}[\widehat{S_2}]=\frac{1}{2}\sqrt{1-\frac{r_2^2}{1-r_3^2}}$$

$$\varepsilon[\widehat{S_1}]\varepsilon[\widehat{S_2}]=\left(\frac{1}{2}\right)^2\sqrt{\left(\frac{r_1 r_2}{1-r_3^2}\right)^2+\left(1-\frac{r_1^2+r_2^2}{1-r_3^2}\right)}$$

李・筒井の不等式の右辺の量を計算

$$\mathcal{R}_{\hat{\rho}} = -\left(\frac{1}{2}\right)^2 \frac{r_1 r_2}{1 - r_3^2}$$

$$\mathcal{J}_{\hat{\rho}} = \left(\frac{1}{2}\right)^2 \left(1 - \frac{r_1^2 + r_2^2}{1 - r_3^2}\right) r_3$$

$$\sqrt{\mathcal{R}_{\hat{\rho}}^2 + \mathcal{J}_{\hat{\rho}}^2} = \left(\frac{1}{2}\right)^2 \sqrt{\left(\frac{r_1 r_2}{1 - r_3^2}\right)^2 + \left(1 - \frac{r_1^2 + r_2^2}{1 - r_3^2}\right)^2 r_3^2}$$