

Tripartite Entanglement in Particle Scattering and Decays

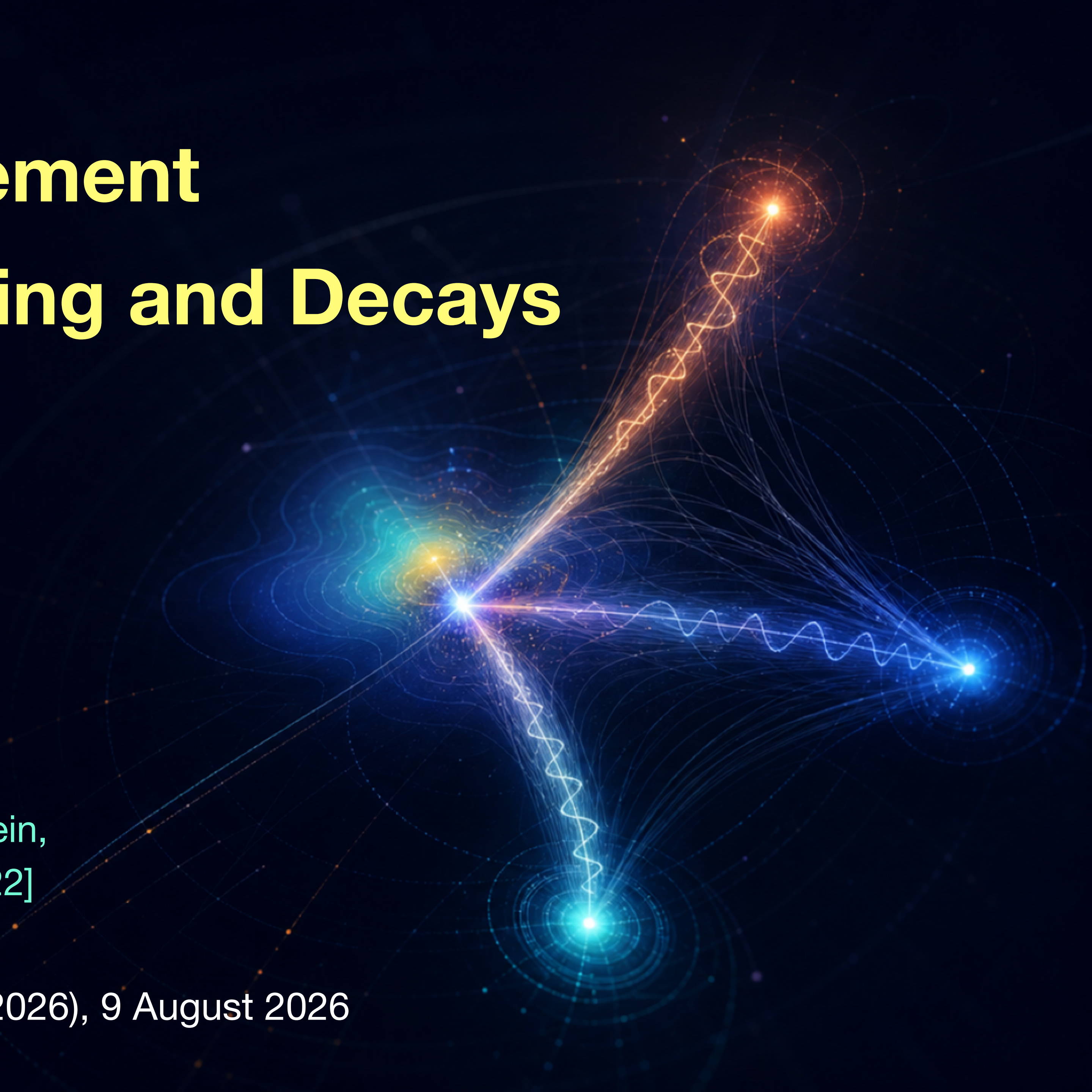
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Goncalves, Navarro, KS [2606.11296]

Banacki, Ohta, Shaleena, Caban, Eckstein,
KS, Spannowsky, Horodecki [2607.26122]

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Plan

⌘ A review on entanglement

⌘ Tripartite Entanglement

- $h \rightarrow \tau^- \tau^+ Z$
- $e^+ e^- \rightarrow t \bar{t} Z$

⌘ Conclusion

- General quantum states are faithfully described by **density operators**:

Ensemble: $\{ q_1; |\psi_1\rangle, q_2; |\psi_2\rangle, \dots \}$ $q_i \geq 0, \sum_i q_i = 1$

$$\hat{\rho} = \sum_i q_i |\psi_i\rangle \langle \psi_i|$$

Time Evolution: $\hat{\rho}(t) = U(t)\hat{\rho}(0)U(t)^\dagger$ $\hat{U}(t) = e^{-i\hat{H}t}$

Measurement: $\langle \hat{O} \rangle_\rho = \text{Tr}(\hat{O}\hat{\rho})$

- For a given density operator, the **pure state decomposition** is **not unique**!

$$\hat{\rho} = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| = \frac{1}{2}|\psi_+\rangle\langle\psi_+| + \frac{1}{2}|\psi_-\rangle\langle\psi_-|$$

$\begin{cases} 0\rangle & \cdots & p \\ 1\rangle & \cdots & 1-p \end{cases}$	physically equivalent $\longleftrightarrow$	$\begin{cases} \psi_+\rangle & \cdots & \frac{1}{2} \\ \psi_-\rangle & \cdots & \frac{1}{2} \end{cases}$
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$$|\psi_{\pm}\rangle = \sqrt{p}|0\rangle \pm \sqrt{1-p}|1\rangle$$

“Entanglement is not just one, but *the* characteristic trait of QM”

Erwin Schrödinger



⌘ Definition:

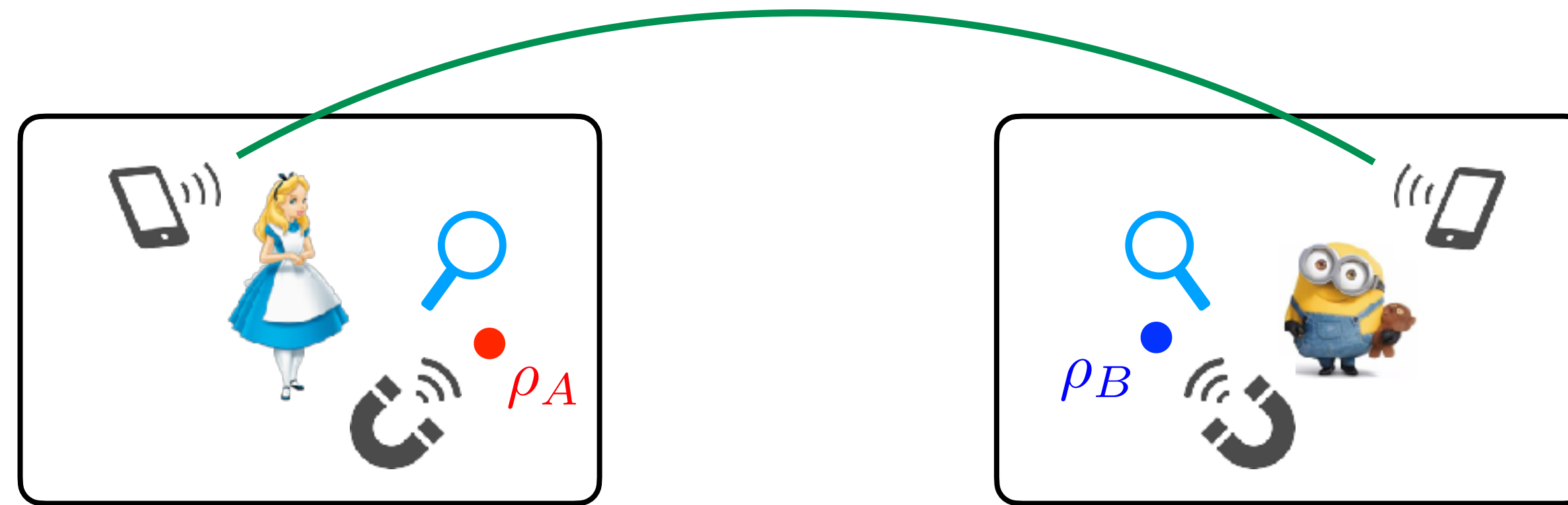
The state is called **separable** if it admits the expression

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

$$p_i \geq 0, \sum_i p_i = 1$$

otherwise it is called **entangled**.

Local Operation & Classical Communication

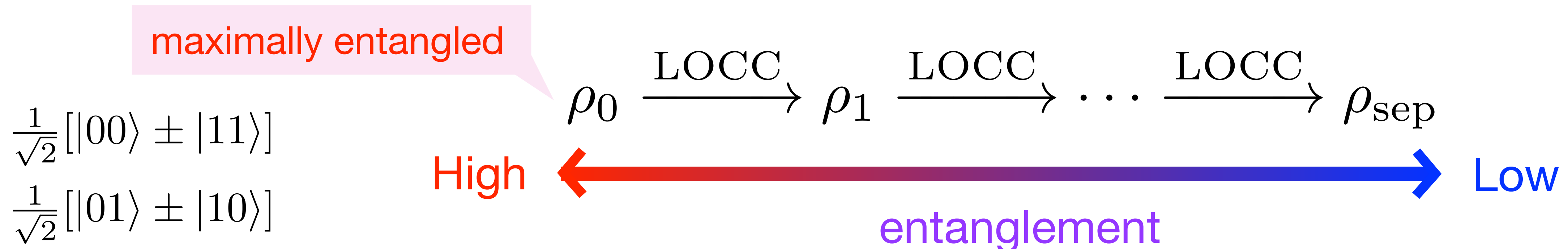


entangled

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right] \xrightarrow{\text{measurement}} |0\rangle_A \otimes |1\rangle_B$$

not entangled

⌘ LOCC introduces a partial **ordering** on quantum states.



- **Concurrence** *quantifies* entanglement:

$$(\text{separable}) \quad 0 \leq C(\rho) \leq 1 \quad (\text{maximally entangled})$$

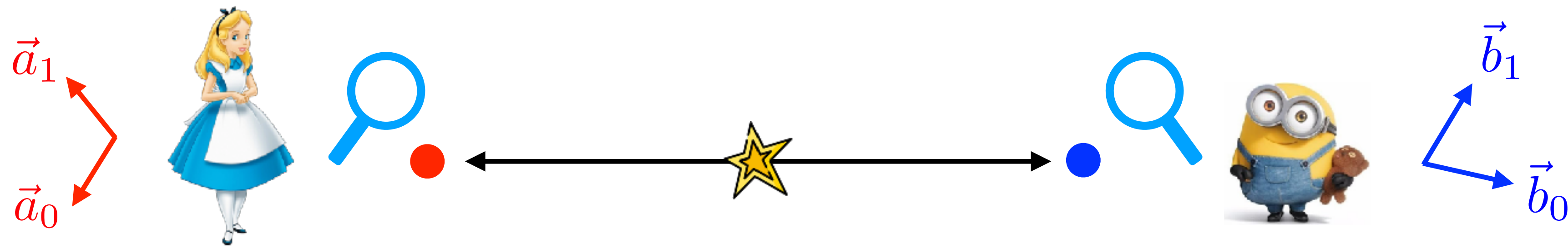
⌘ Pure states:

$$C(|\psi\rangle) \equiv \sqrt{2(1 - \text{Tr} \rho_A^2)} \quad \rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$$

⌘ Mixed states: **For 2-qubit**

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \quad \text{Wootters' (1998)}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of $\sqrt{\rho \tilde{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$.

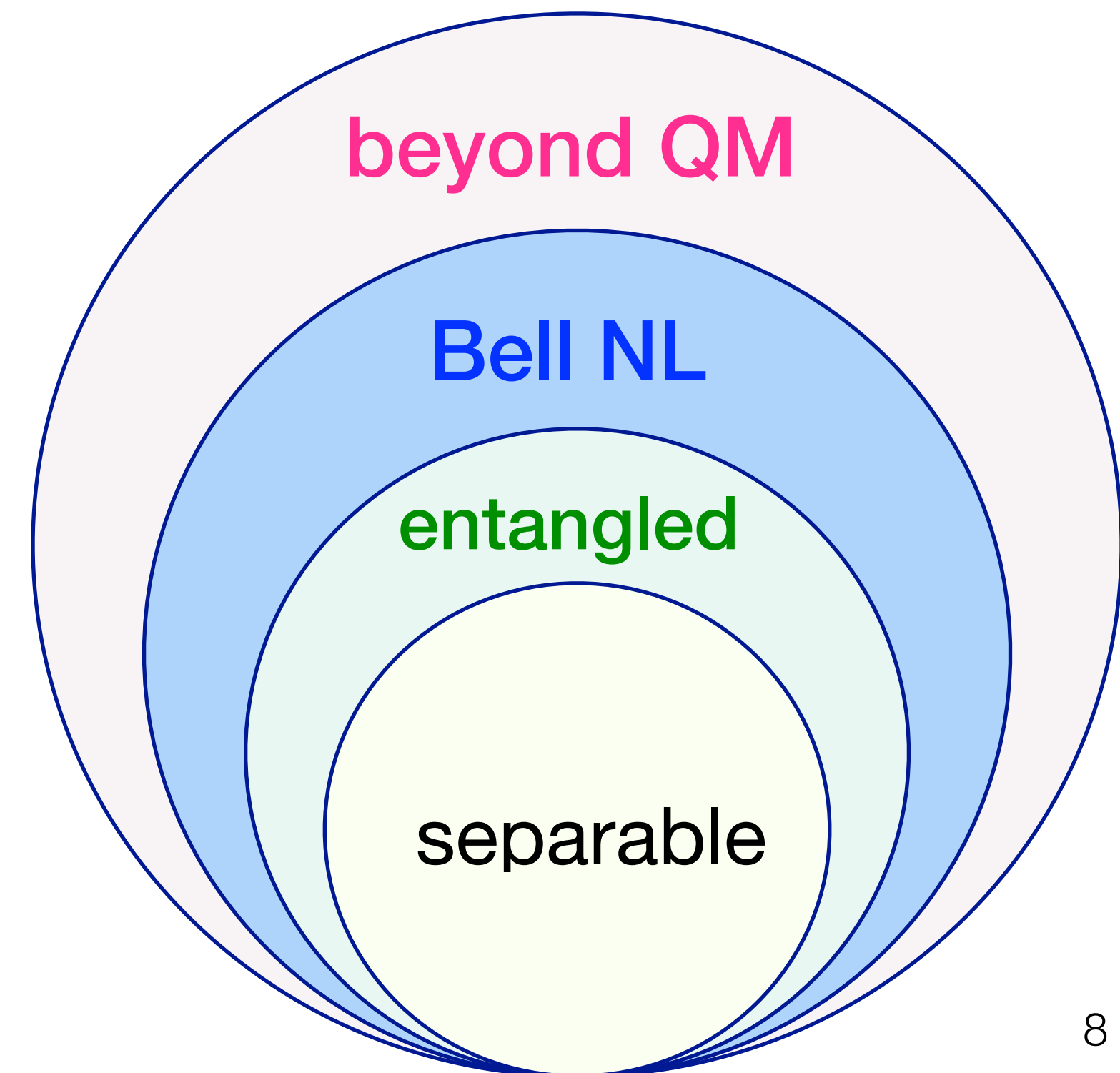


Bell test can certify **Bell nonlocality** and probe **beyond QM**

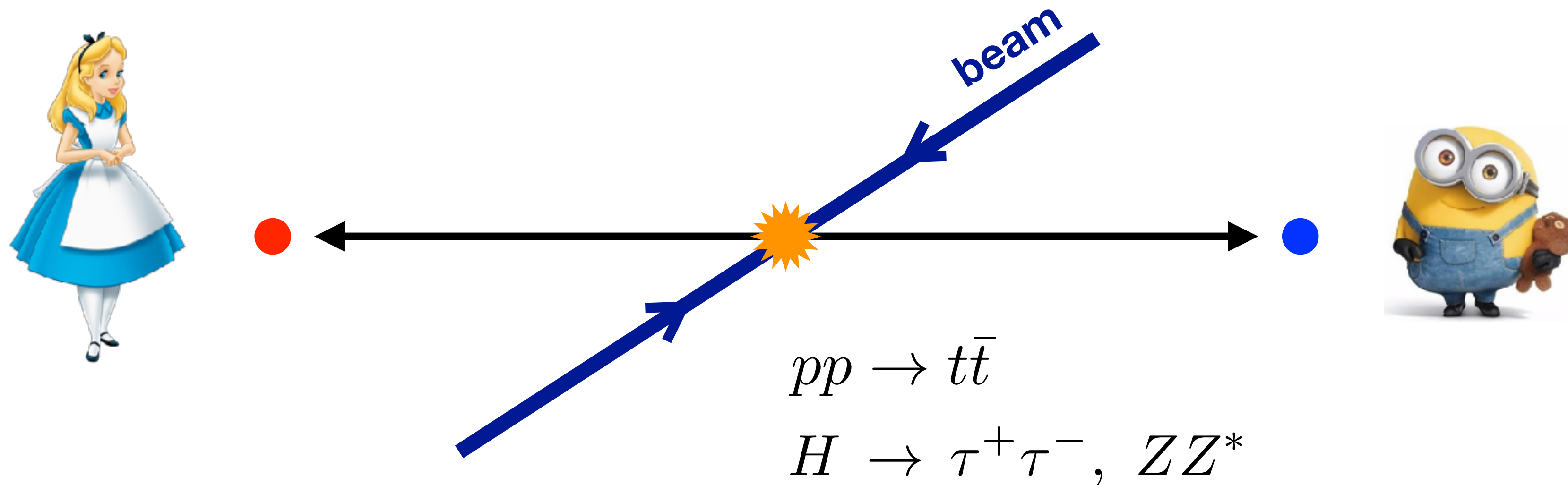
$$I(\{\vec{a}, \vec{b}\}) \equiv |\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle|$$

$$\mathcal{B}_{\text{CHSH}} \equiv \max_{\vec{a}, \vec{b}} I(\{\vec{a}, \vec{b}\}) \leq 2$$

$$\mathcal{B}_{\text{CHSH}} \leq \begin{cases} 2 & \text{LHV, } \rho_{\text{sep}} \\ 2\sqrt{2} & \text{QM} \\ 4 & \text{beyond QM} \end{cases}$$



Can we measure various quantum properties at collider?



In the past few years, entanglement and other quantum properties in bipartite systems have been extensively studied.

● $t \bar{t}$

Afik, de Nova [2003.02280], Fabbrichesi, Floreanini, Panizzo [2102.11883], Severi, Boschi, Maltoni, Sioli [2110.10112], Aoude, Madge, Maltoni, Mantani [2203.05619], Afik, de Nova [2203.05582], Aguilar-Saavedra, Casas [2205.00542], Fabbrichesi, Floreanini, Gabrielli [2208.11723], Afik, de Nova [2209.03969], Severi, Vryonidou [2210.09330], Dong, Gonçalves, Kong, Navarro [2305.07075], Aguilar-Saavedra [2307.06991], ...

● $\tau^+ \tau^-$

Altakach, Lamba, Maltoni, Mawatari, KS [2211.10513], Ma, Li [2309.08103], Ehatäht, Fabbrichesi, Marzola, Veelken [2311.17555], Fabbrichesi, Marzola [2405.09201], Fabbrichesi, Low, Marzola [2501.03311], Han, Low, Su [2501.04801], Zhang, Zhou, Liu, Wu, Li, Han et al. [2504.01496], Jeans [2603.12609], Lee, Lawless, Riggall [2603.28868], ...

● $W W, Z Z$

Barr [2106.01377], Aguilar-Saavedra, Bernal, Casas, Moreno [2209.13441], Aguilar-Saavedra [2209.14033], Ashby-Pickering, Barr, Wierzchucka [2209.13990], Fabbrichesi, Floreanini, Gabrielli, Marzola [2302.00683], Fabbrichesi, Floreanini, Gabrielli, Marzola [2304.02403], Morales [2306.17247], Aoude, Madge, Maltoni, Mantani [2307.09675], Fabbri, Howarth, Maurin [2307.13783], Bernal, Caban, Rembieliński [2307.13496], ...

ATLAS and CMS observed entanglement in $t\bar{t}$

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Observation of quantum entanglement with top quarks at the ATLAS detector

The ATLAS Collaboration

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Reports on Progress in Physics

<https://doi.org/10.1088/1361-6633/ab9881>

Observation of quantum entanglement in top quark pair production in proton–proton collisions at $\sqrt{s} = 13$ TeV

The CMS Collaboration

Abstract

Entanglement is an intrinsic property of quantum mechanics and is predicted to be exhibited in the particles produced at the Large Hadron Collider. A measurement of the extent of entanglement in top quark-antiquark ($t\bar{t}$) events produced in proton–proton collisions at a center-of-mass energy of 13 TeV is performed with the data recorded by the CMS experiment at the CERN LHC in 2016, and corresponding to an integrated luminosity of 36.3 fb^{-1} . The events are selected based on the presence of two leptons with opposite charges and high transverse momentum. An entanglement-sensitive observable D is derived from the top quark spin-dependent parts of the $t\bar{t}$ production density matrix and measured in the region of the $t\bar{t}$ production threshold. Values of $D < -1/3$ are evidence of entanglement and D is observed (expected) to be $-0.480^{+0.026}_{-0.029}$ ($-0.467^{+0.026}_{-0.029}$) at the parton level. With an observed significance of 5.1 standard deviations with respect to the non-entangled hypothesis, this provides observation of quantum mechanical entanglement within $t\bar{t}$ pairs in this phase space. This measurement provides a new probe of quantum mechanics at the highest energies ever produced.

Entanglement is a key feature of quantum mechanics [1–3], with applications in fields such as metrology, cryptography, quantum information, and quantum computation [4–8]. It has been observed in a wide variety of systems and length scales, ranging from the microscopic [9–13] to the macroscopic [14–16]. However, entanglement remains largely unexplored at the highest accessible energy scales. Here we report the highest-energy observation of entanglement, in top–antitop quark events produced at the Large Hadron Collider, using a proton–proton collision dataset with a center-of-mass energy of $\sqrt{s} = 13$ TeV and an integrated luminosity of 140 fb^{-1} recorded with the ATLAS experiment. Spin entanglement is detected from the

Also, very recently, in $H \rightarrow ZZ^*$

ATLAS [2603.26463]

Measurements of Z-boson pair entanglement in decays of Higgs bosons at the ATLAS experiment

The ATLAS Collaboration

Entanglement is a key property of quantum systems. In this Letter the first measurements of quantum entanglement between spins in pairs of Z bosons are reported, using proton–proton collision data from the Large Hadron Collider (LHC) at center-of-mass energies of 13 TeV and 13.6 TeV, recorded with the ATLAS detector. Measurements of angular observables sensitive to ZZ^* spin-density-matrix elements in the $H \rightarrow ZZ^* \rightarrow \ell^+ \ell^- \ell^+ \ell^-$ process yield coefficients $C_{2,1,2,-1} = -0.71 \pm 0.45$ and $C_{2,2,2,-2} = 0.08 \pm 0.44$, consistent with their Standard Model predictions. A complementary hypothesis test using the full angular distribution, and relying on several Standard Model assumptions in the decays, provides substantially higher sensitivity to quantum correlations and disfavors the separable-state hypothesis at a significance of 4.7 standard deviations (expected 4.9σ) relative to the entangled Standard Model hypothesis. These results provide strong evidence of quantum entanglement between massive bosons (spin qutrits) at the electroweak scale.

[CMS PAS HIG-25-011]

Study of spin correlations in Higgs boson decays to four leptons at CMS

The CMS Collaboration

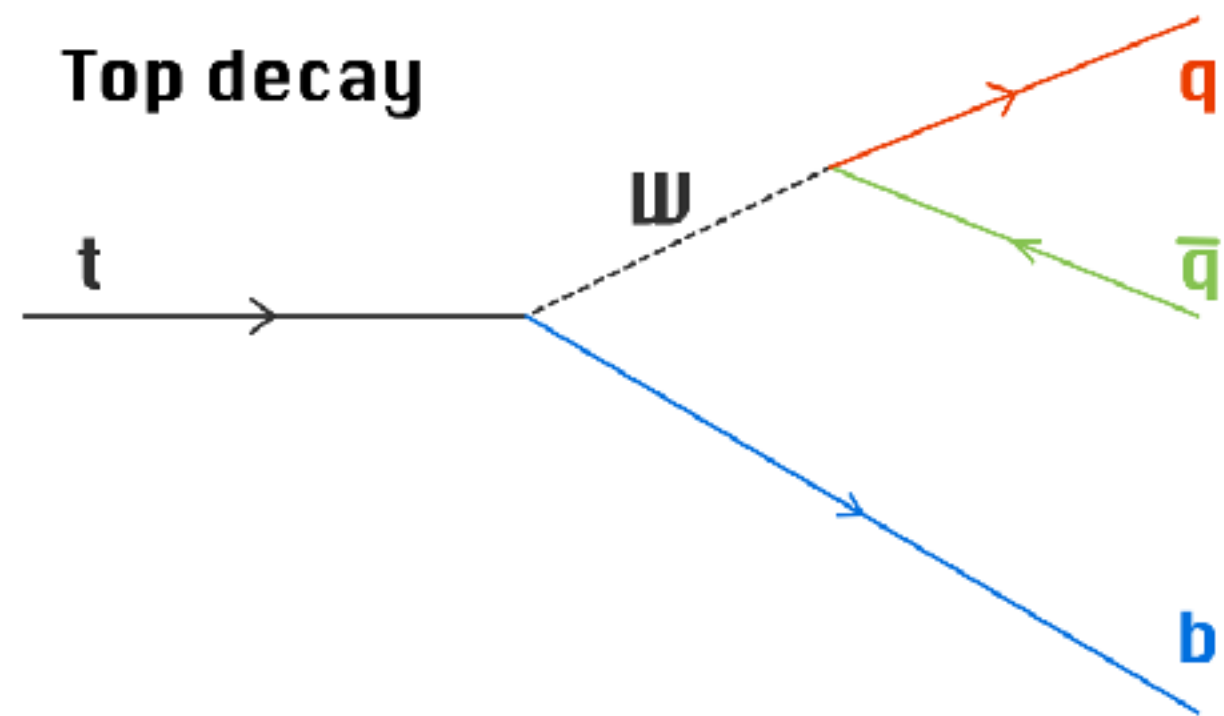
Abstract

Kinematic distributions in Higgs boson decay to four leptons $H \rightarrow 4\ell$ are studied using matrix element techniques, optimizing sensitivity to the tensor structure of Higgs boson interactions and spin correlations in the decay process. The data were collected by the CMS experiment at the LHC, corresponding to integrated luminosities of 138 and 62 fb^{-1} at proton-proton center of mass energies of 13 and 13.6 TeV, respectively, covering the 2016–2018 and 2022–2023 data-taking periods. A simultaneous measurement of eight Higgs boson couplings to electroweak vector bosons is carried out within the frameworks of anomalous couplings and effective field theory. Under the assumption of CP symmetry conservation, the polarization density matrix in the $H \rightarrow ZZ$ decay is measured, and a search for CP violation in the polarization of the Z bosons is conducted. Quantum mechanical interference is tested through the permutation of identical leptons. Implications regarding quantum entanglement and the violation of a Bell-type inequality are discussed.

QI properties in *tripartite* systems are *much less explored*.

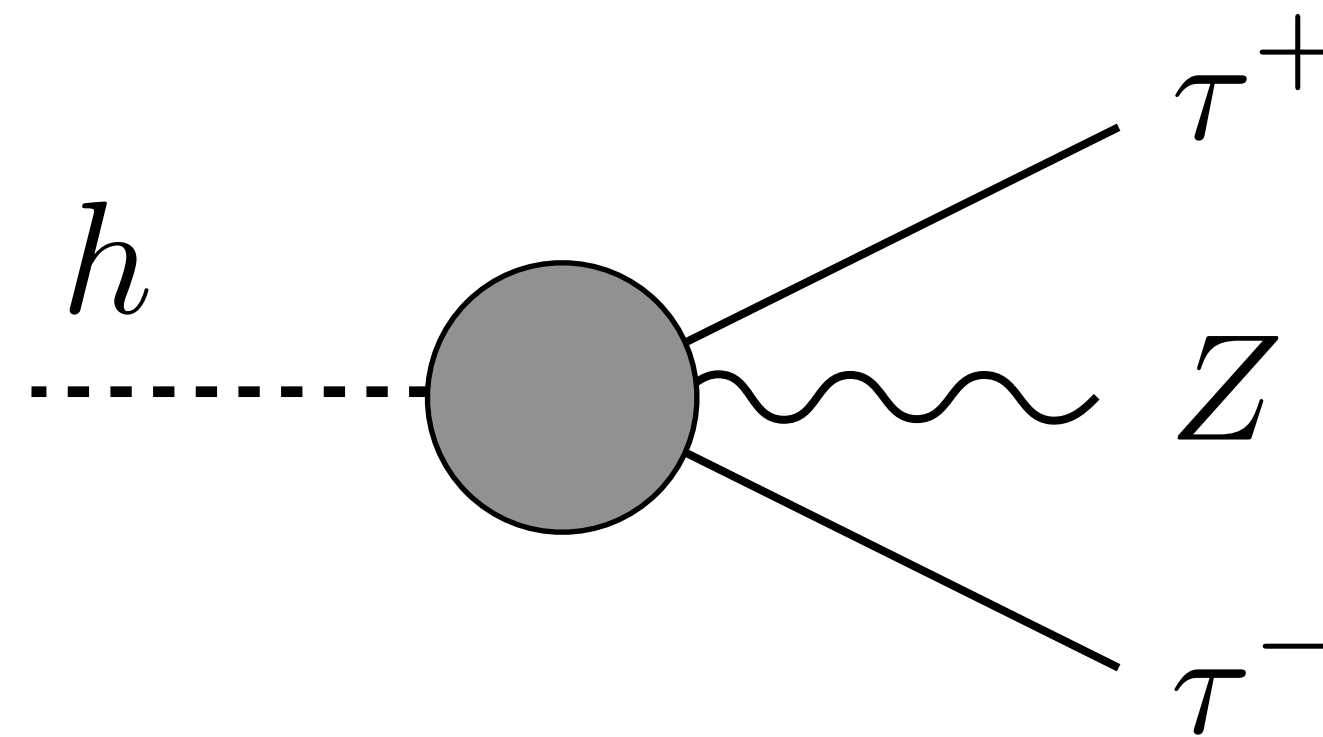
- There are several opportunities for colliders:

$$2 \rightarrow 222$$



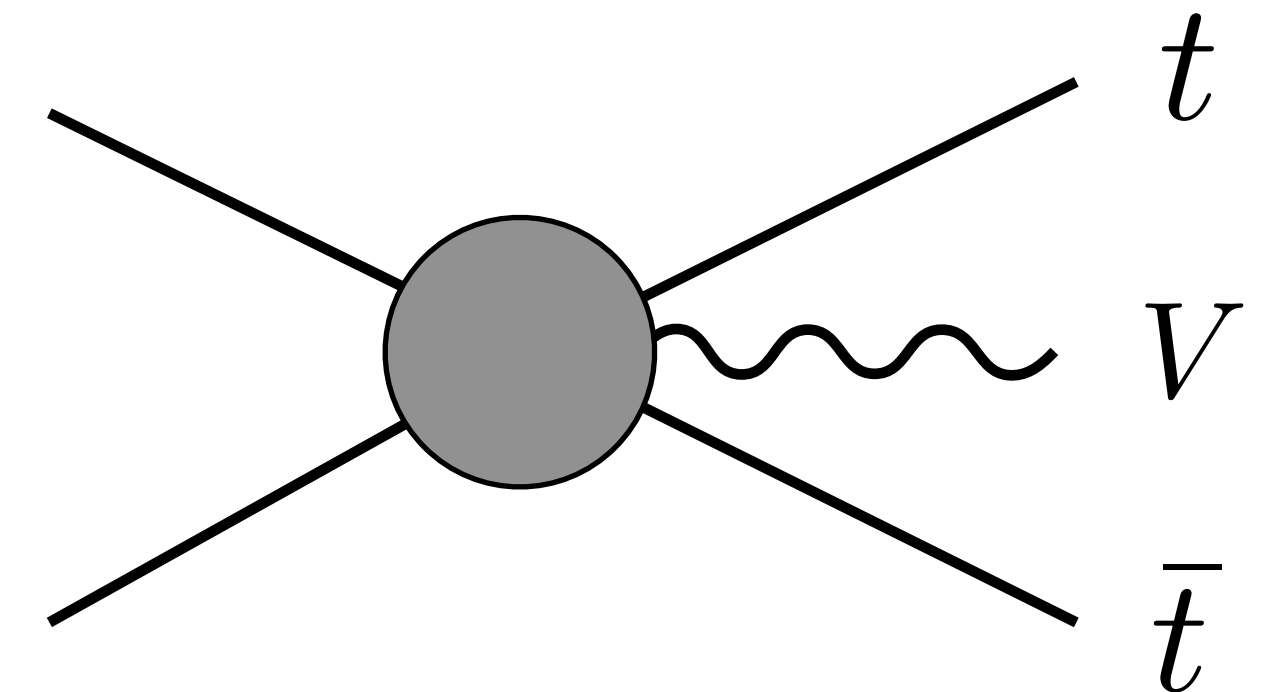
Horodecki, KS, Shaleena
Spannowsky [2502.19470],
KS, Spannowsky [2310.01477]

$$0 \rightarrow 223$$



Banacki, Ohta, Shaleena,
Caban, Eckstein, KS,
Spannowsky, Horodecki
[2607.26122]

$$22 \rightarrow 223$$

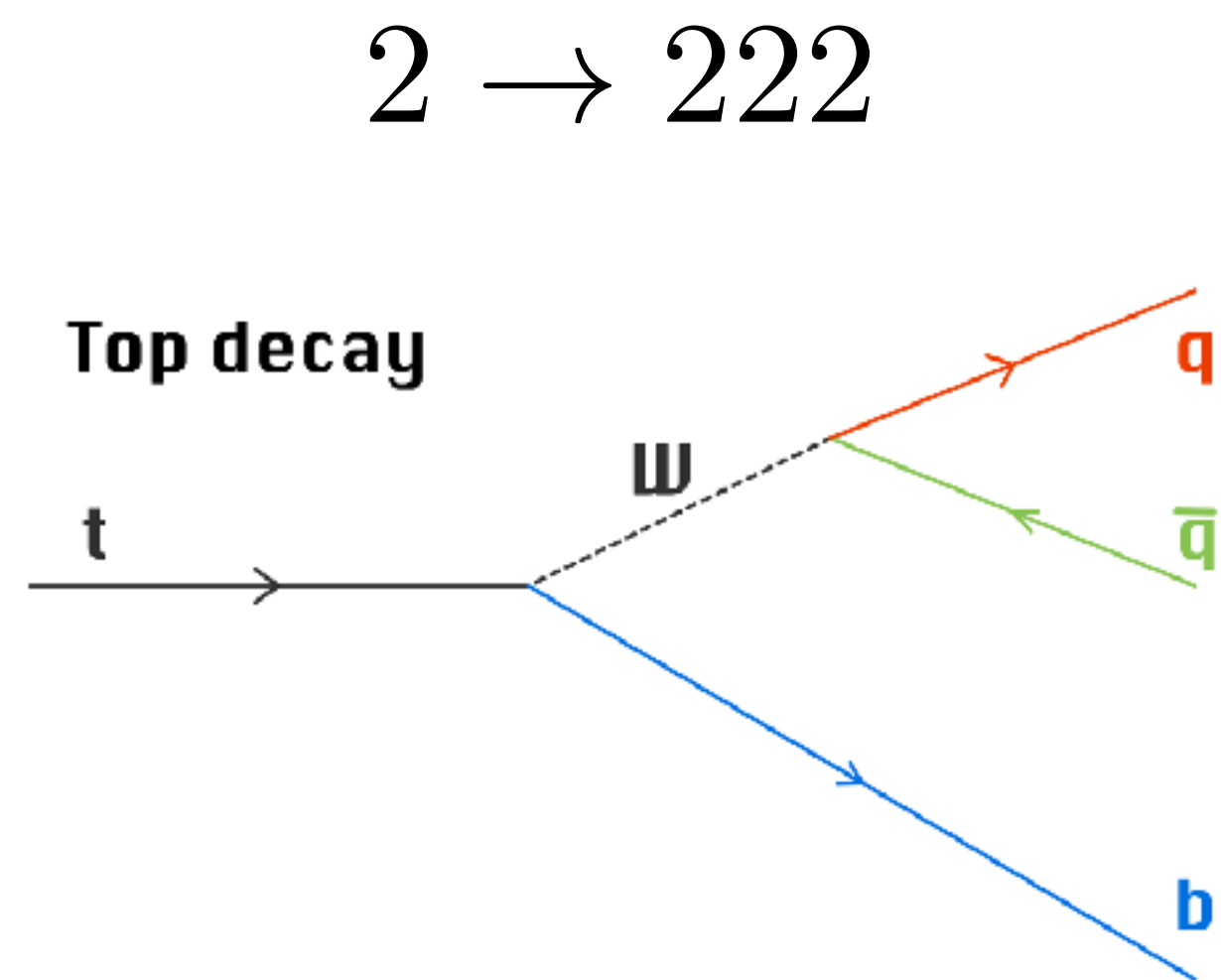


Goncalves, Navarro, KS
[2606.11296]

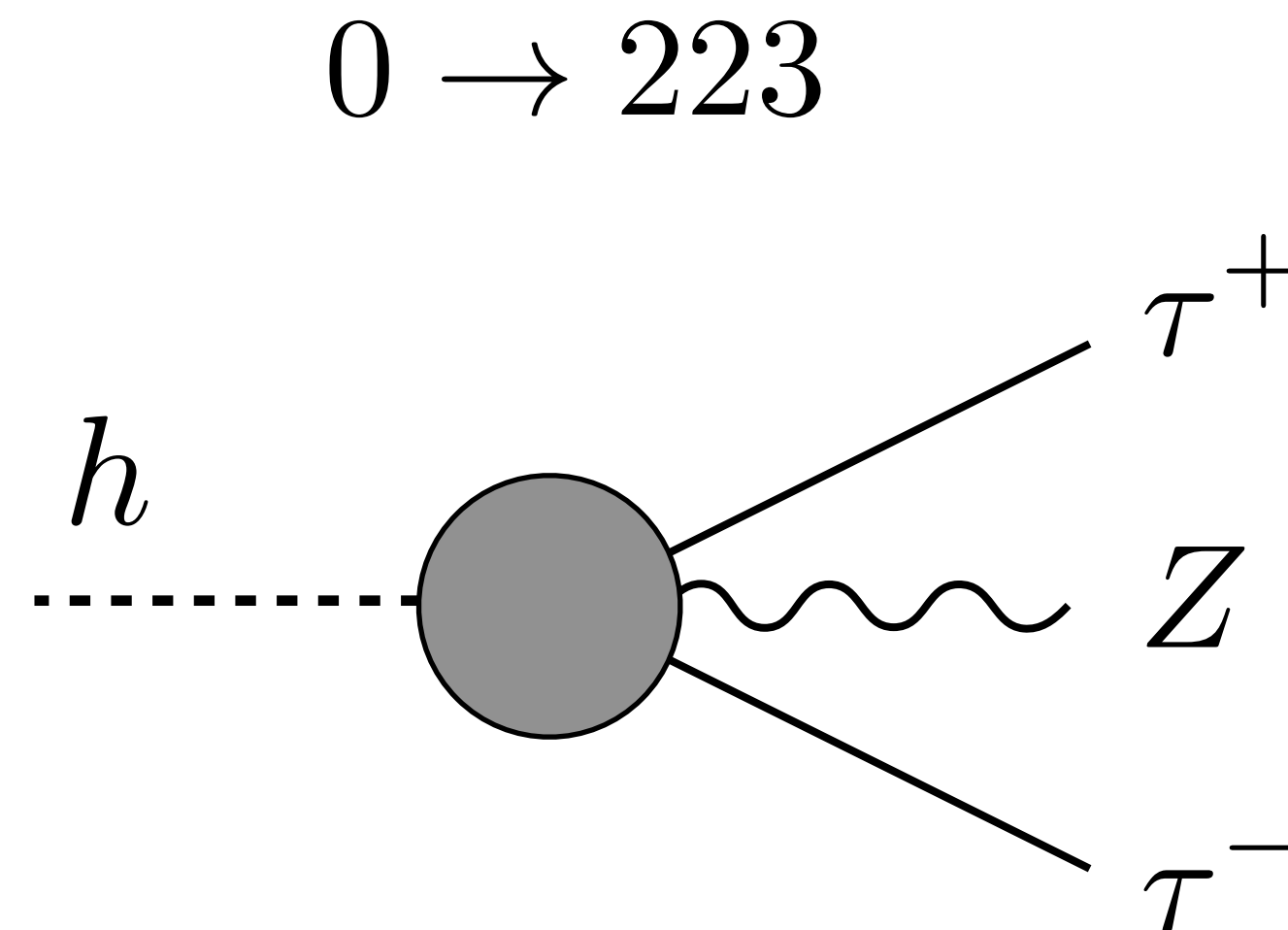
Spin reconstruction is possible for weakly decaying particles; $t, \tau, W^\pm, Z$

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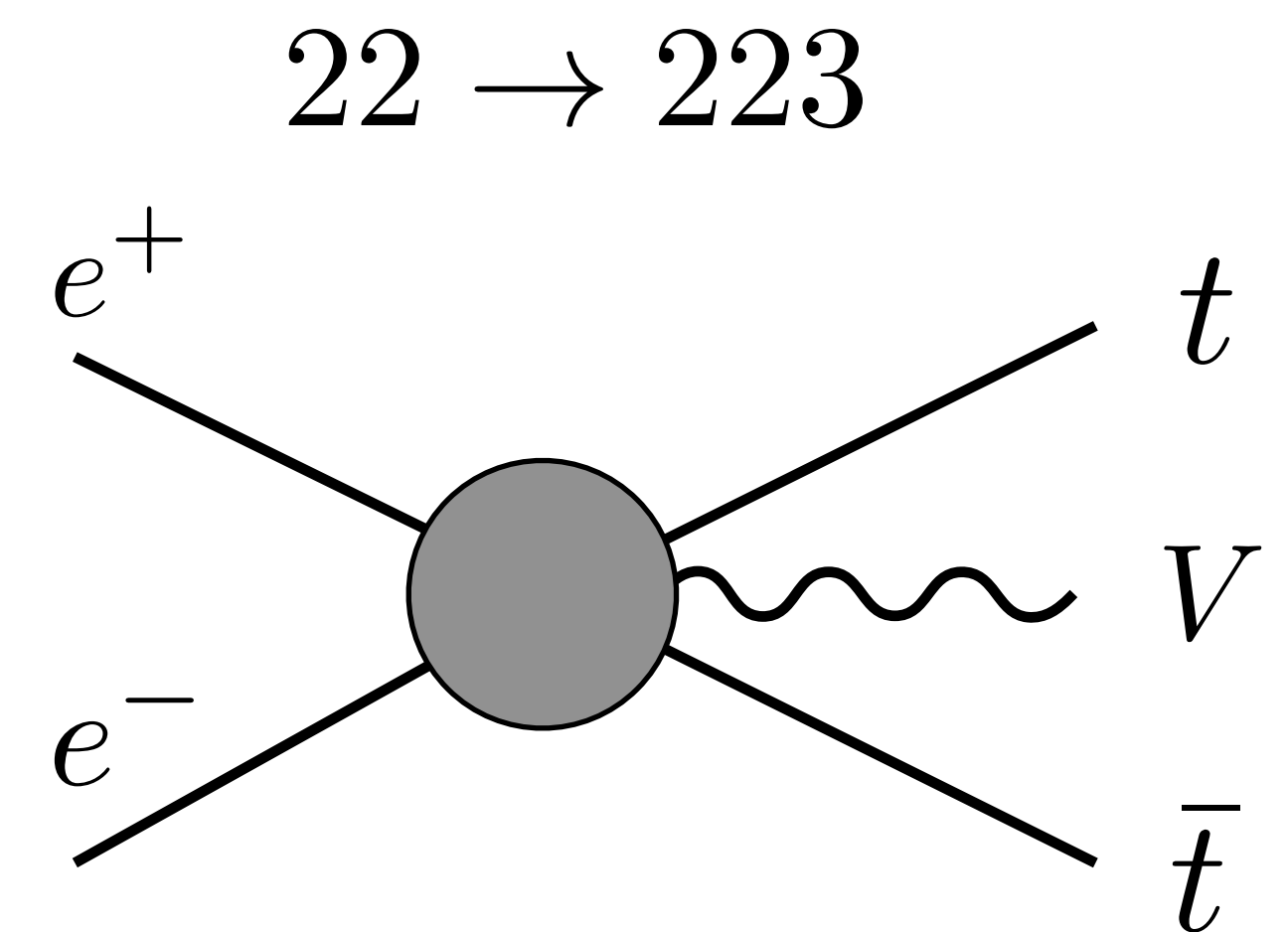
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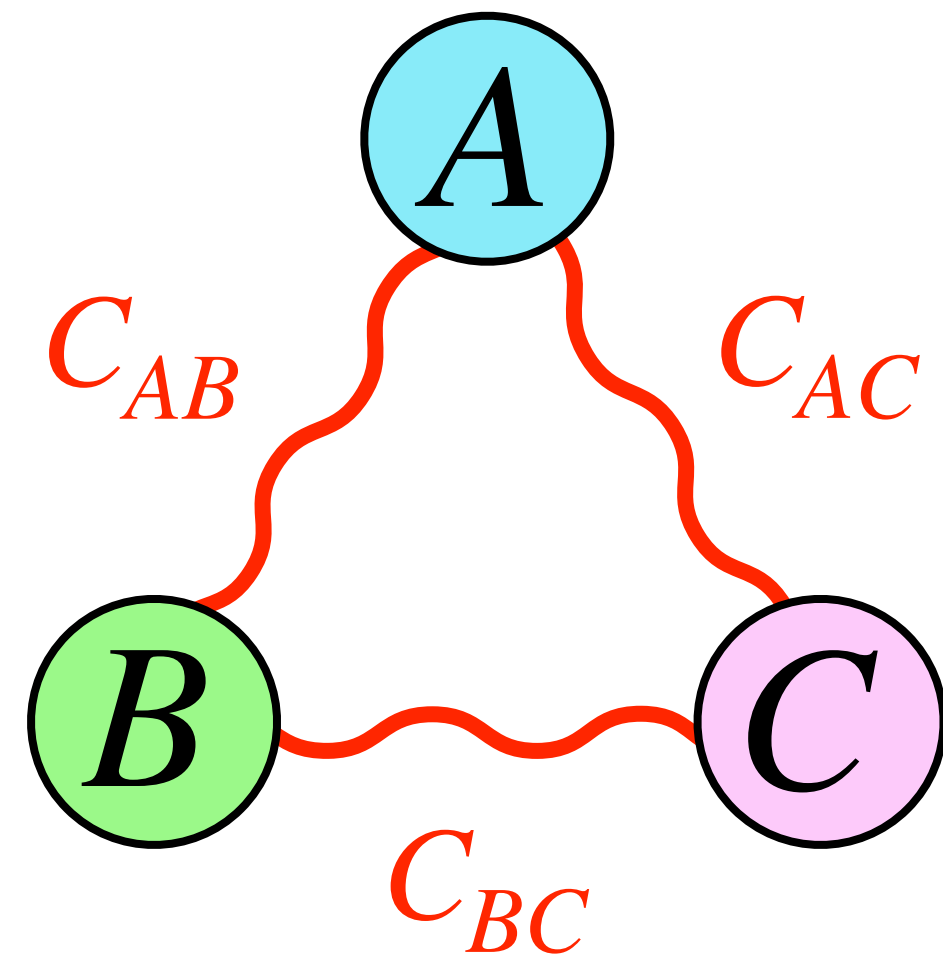
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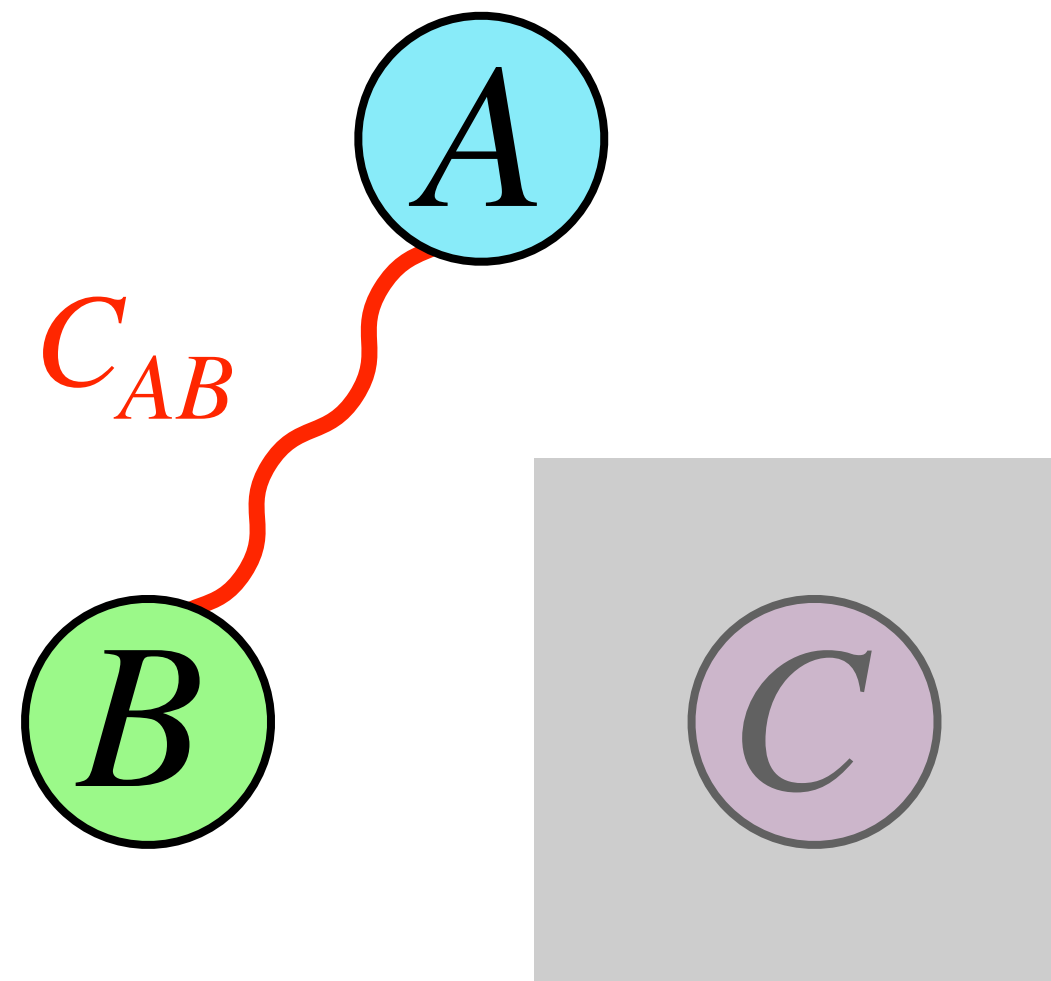
3-partite entanglement has a much *richer structure* than 2-partite one !



- Entanglement btw individual particles (one-to-one)

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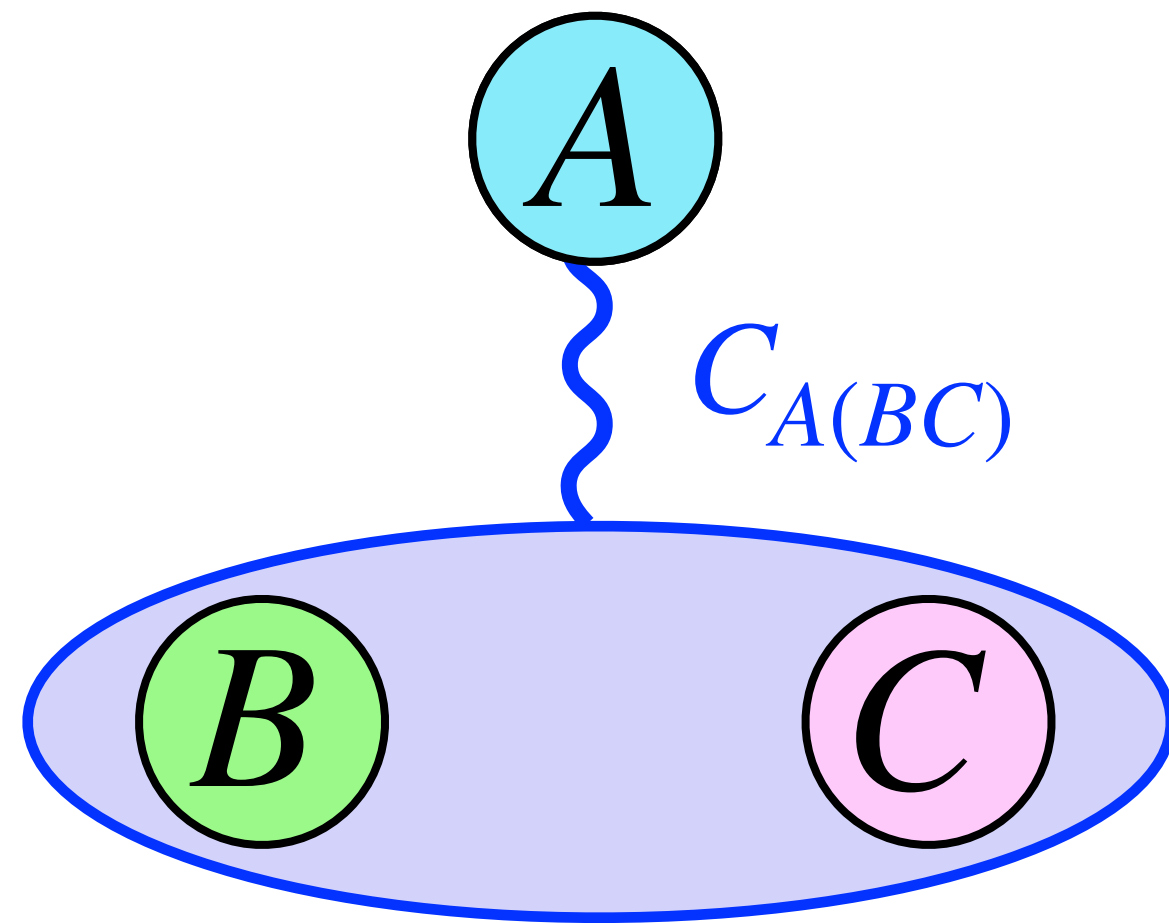
$$C_{AB} = C(\rho_{AB}) \quad \rho_{AB} = \text{Tr}_C |\psi\rangle\langle\psi|$$

Wootters 2-qubit formula:

$$C(\rho) = \max (0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

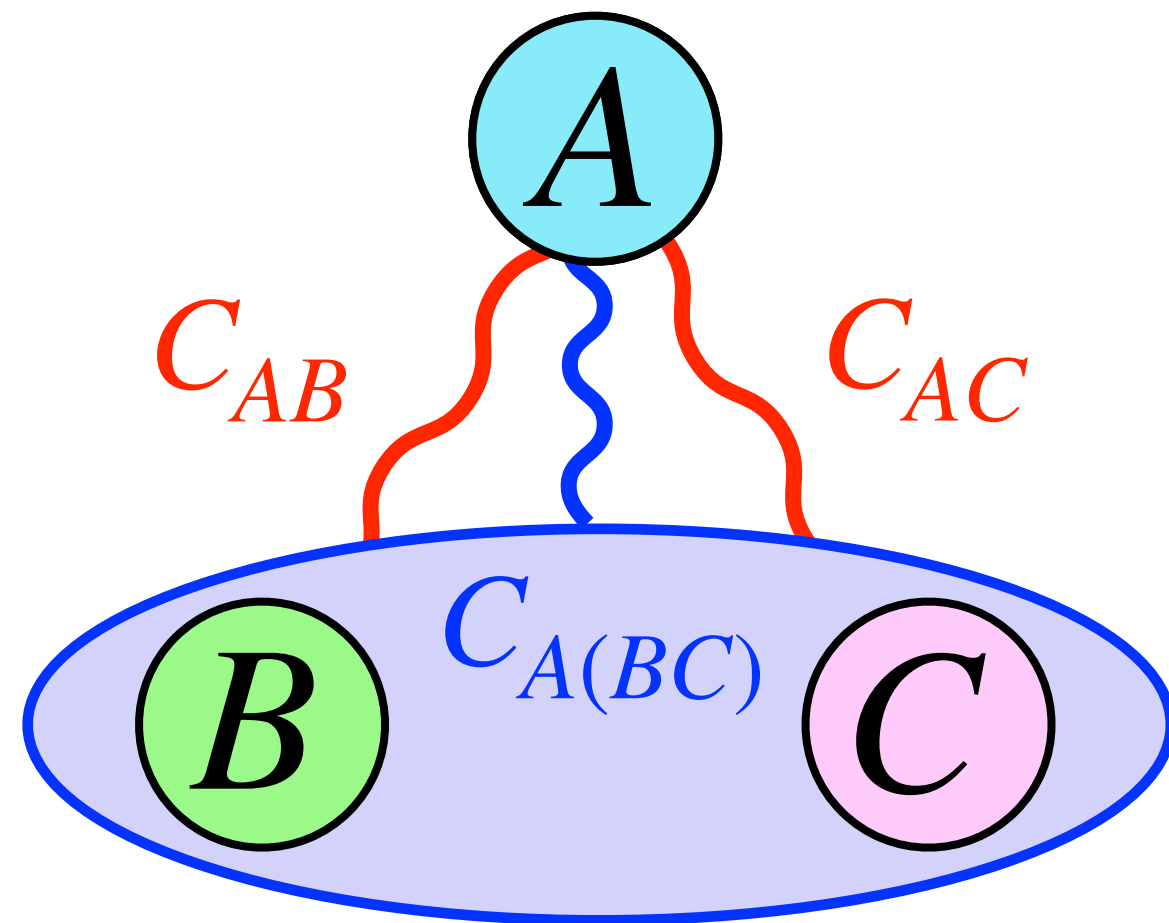
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- Entanglement btw individual particles (one-to-one)
- Entanglement btw one particle and the other subsystem

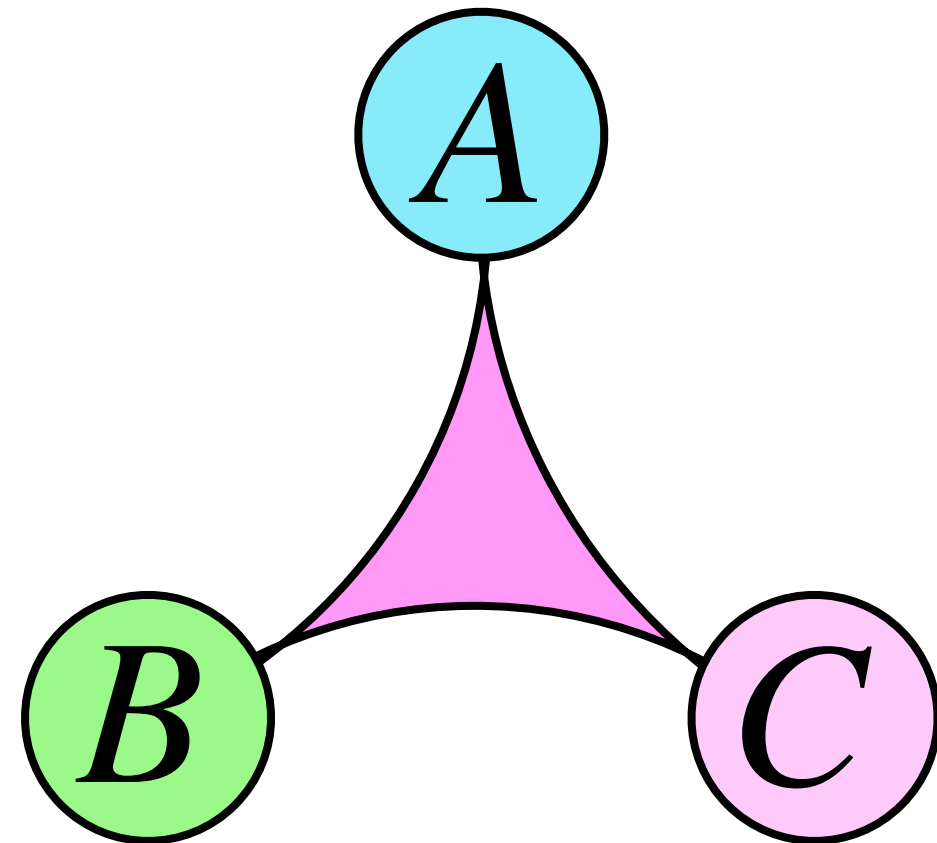
$$C_{A(BC)} = \sqrt{2(1 - \text{Tr} \rho_A^2)} \quad \rho_A = \text{Tr}_{BC} |\psi\rangle\langle\psi|$$

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- Entanglement btw one particle and the other subsystem
- Monogamy of entanglement: $C_{A(BC)}^2 \geq C_{AB}^2 + C_{AC}^2$

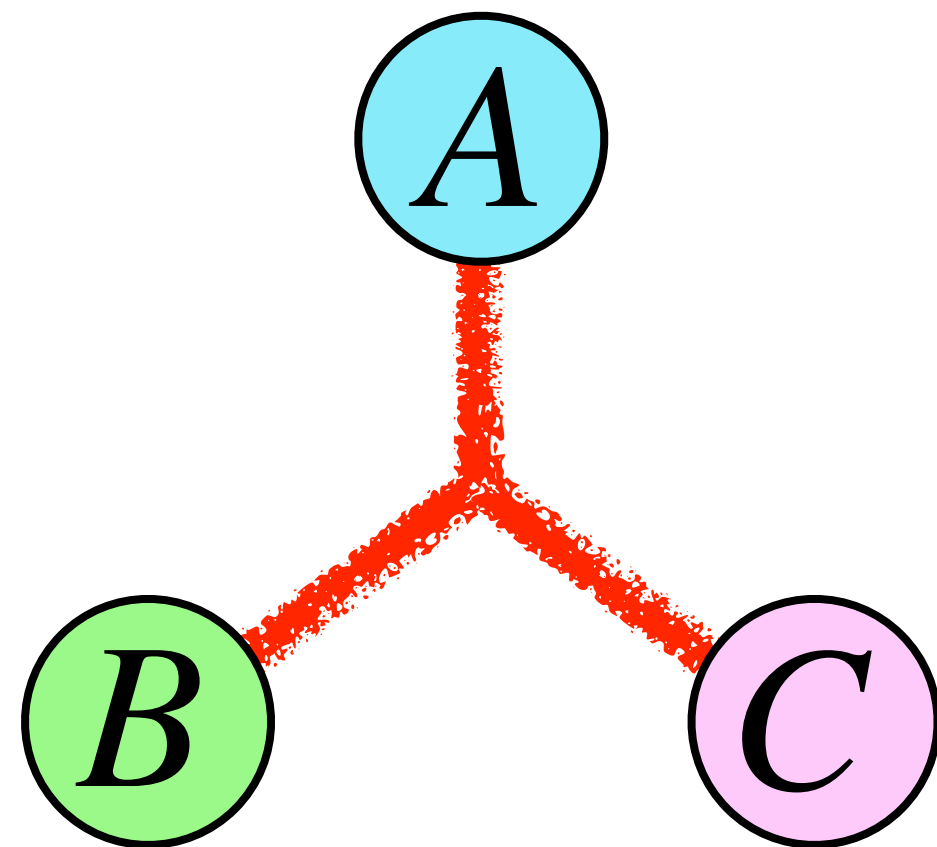
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- Genuinely Multipartite Entanglement (GME)

~~$$|\psi\rangle_A \otimes (|00\rangle_{BC} + |11\rangle_{BC})$$~~

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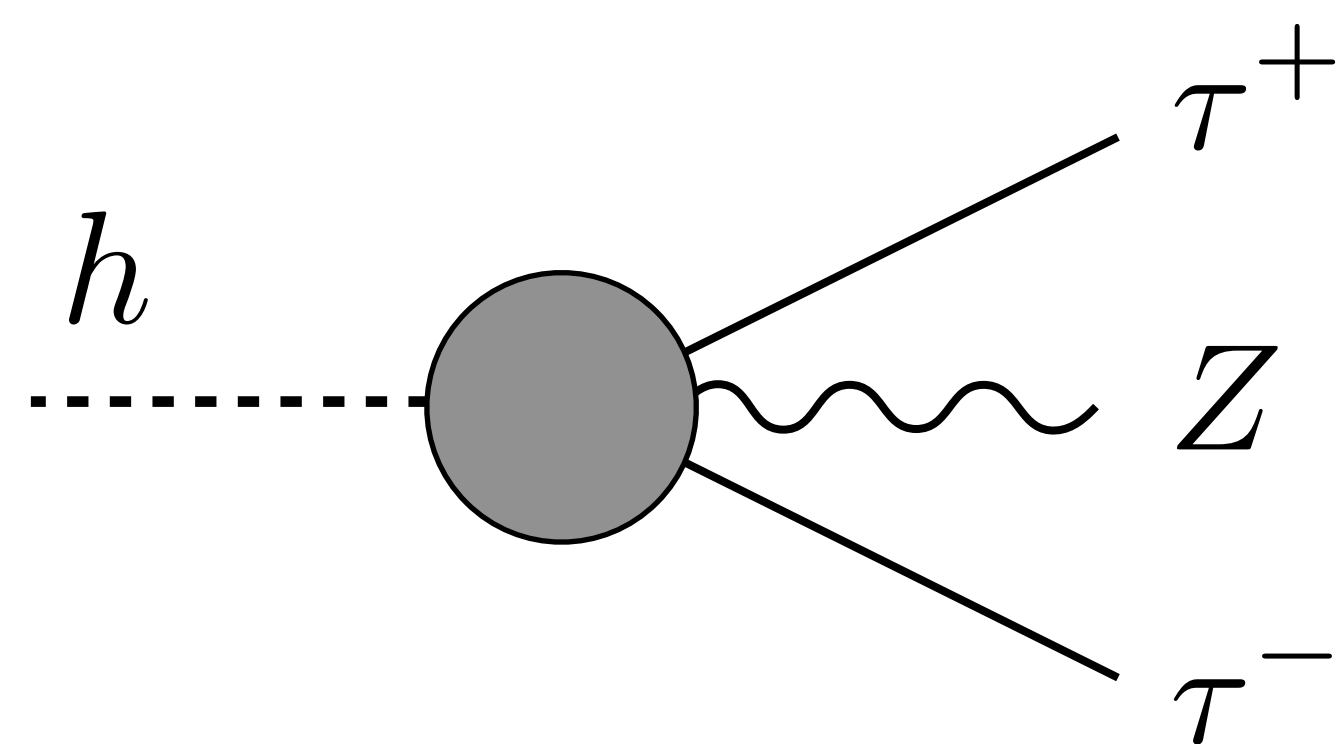


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$$\text{---} |\psi\rangle_A \otimes (|00\rangle_{BC} + |11\rangle_{BC}) \text{---}$$

- Bell nonlocality is extended to 3-partite system

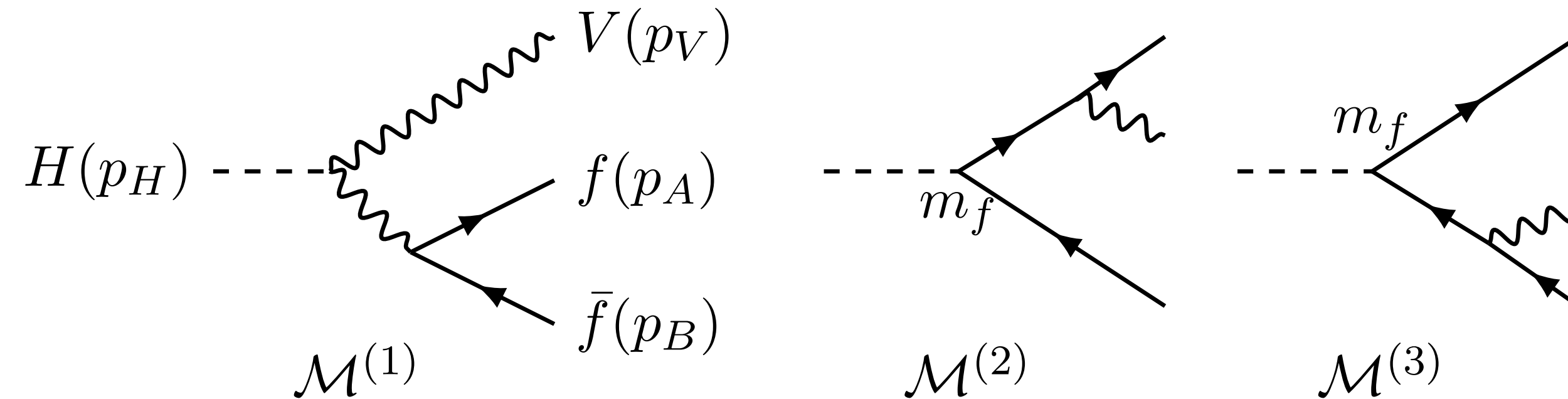
$$h \rightarrow \tau^+ \tau^- Z$$



$$0 \rightarrow 223$$

$$H \rightarrow f \bar{f} V$$

Diagrams:



Lagrangian:

$$\mathcal{L} \supset g_{Hff} H \bar{\psi}_f \psi_f + g_{HVV} H V_\mu V^\mu + g_{Vff} \bar{\psi}_f \gamma^\mu (c_L P_L + c_R P_R) \psi_f V_\mu$$

$$g_{Hff} = -\frac{m_f}{v}$$

$$g_{HVV} = \frac{2m_V^2}{v}$$

For $f = \tau$:

$$c_L = -\frac{1}{2} + \sin^2 \theta_W \simeq -0.269 \quad c_R = \sin^2 \theta_W \simeq 0.231$$

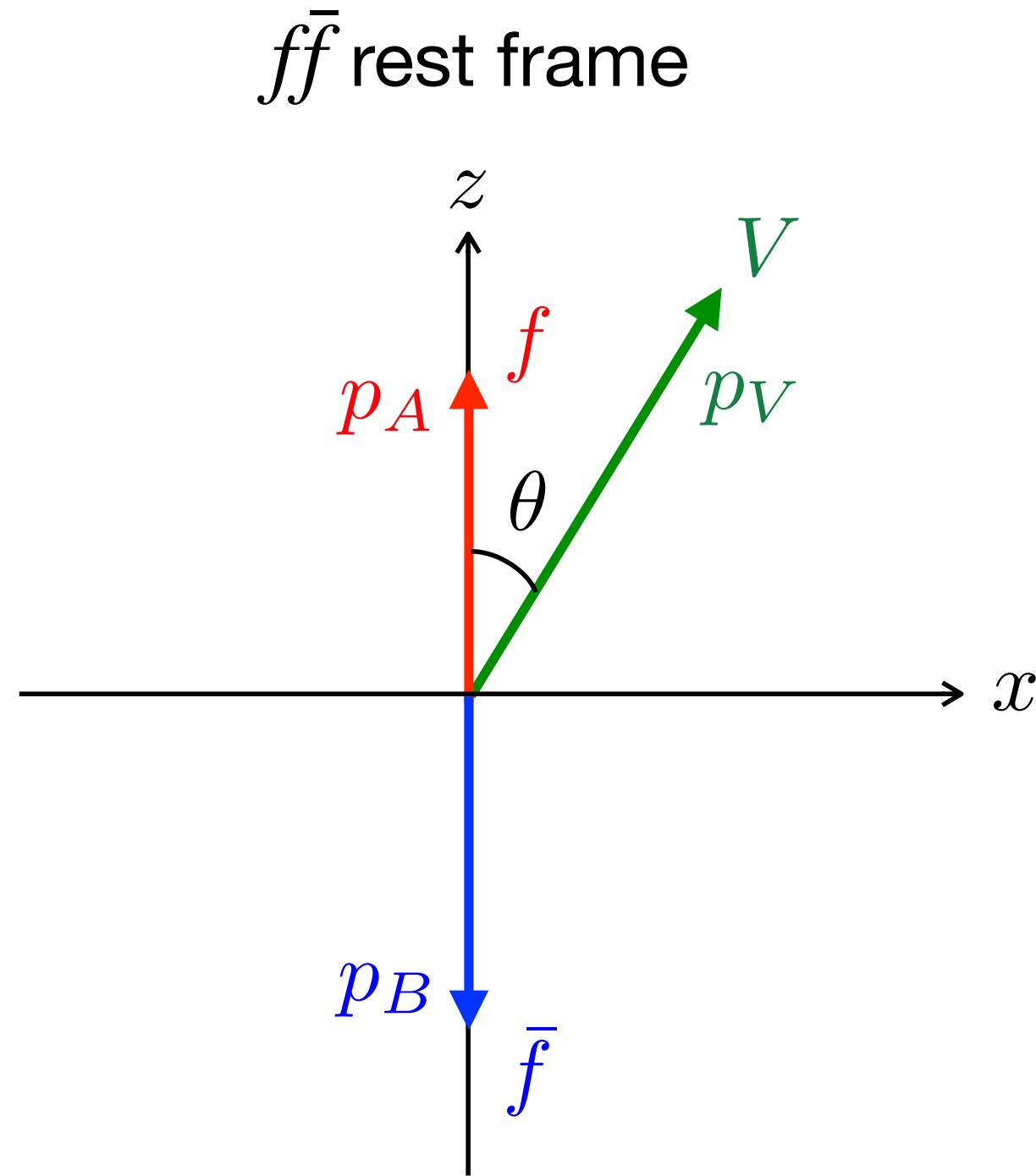
$$c_V \equiv \frac{1}{2}(c_R + c_L) \simeq -0.019 \quad c_A \equiv \frac{1}{2}(c_R - c_L) \simeq 0.250$$

Spin state:

$$|\psi\rangle \propto \sum_{h_A, h_B, h_V} \mathcal{M}_{h_A h_B h_V} |h_A, h_B, h_V\rangle$$

$$\underbrace{\mathcal{M}_{h_A h_B h_V}}_{\text{helicity}} = \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)}$$

Kinematics: we work at the **rest frame of the fermion pair**.



- Fermion momenta are fixed up to their invariant mass m_{AB}

$$p_A^\mu = \left(\frac{m_{AB}}{2}, 0, 0, k_A \right), \quad p_B^\mu = \left(\frac{m_{AB}}{2}, 0, 0, -k_A \right), \quad k_A = \frac{1}{2} \sqrt{m_{AB}^2 - 4m_f^2},$$

- Gauge boson momentum is fixed up to the angle θ

$$p_V^\mu = (E_V, k_V \sin \theta, 0, k_V \cos \theta),$$

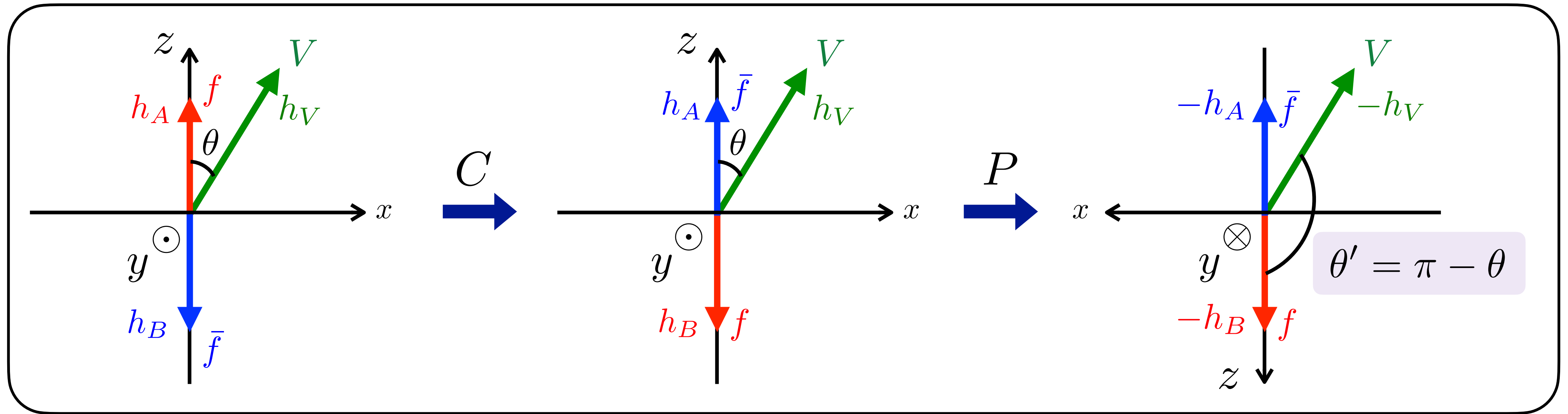
$$E_V = \frac{m_H^2 - m_{AB}^2 - m_V^2}{2m_{AB}}, \quad k_V = \sqrt{E_V^2 - m_V^2} = \frac{\lambda^{1/2}(m_H^2, m_{AB}^2, m_V^2)}{2m_{AB}},$$

- The kinematics is completely fixed by the two parameters:

$$m_{AB} \in [2m_f, m_H - m_V] \quad m_{AB}^{\min} = 2m_f : \quad E_V, k_V \sim \frac{m_H^2 - m_V^2}{4m_f}, \quad \text{: fermion threshold}$$

$$\theta \in [0, \pi] \quad m_{AB}^{\max} = m_H - m_V : \quad E_V = m_V, \quad k_V = 0. \quad \text{: gauge boson threshold}$$

CP properties:



$$|\psi\rangle = \sum_{h_A, h_B, h_V} \psi_{h_A, h_B, h_V}(\theta) |h_A, h_B, h_V\rangle$$

$$\psi_{h_A, h_B, h_V}(\theta) \xrightarrow{\text{CP}} \psi_{h'_A, h'_B, h'_V}^{\text{CP}}(\theta') = \psi_{-h_B, -h_A, -h_V}(\pi - \theta)$$

$$O(\textcolor{red}{A}, \textcolor{blue}{B}, \textcolor{green}{V}; \textcolor{violet}{\cos \theta}) \stackrel{\text{CP}}{=} O(\textcolor{blue}{B}, \textcolor{red}{A}, \textcolor{green}{V}; -\textcolor{violet}{\cos \theta})$$

Expansion in $\varepsilon \propto m_f$

$$|\psi\rangle = \frac{1}{\mathcal{N}} \left(|\psi^{(0)}\rangle + \varepsilon |\psi^{(1)}\rangle + \dots \right)$$

$$, \varepsilon \in [3.14, 3.21] \times 10^{-2} \text{ for } h \rightarrow \tau^+ \tau^- Z$$

$$\varepsilon \equiv \frac{\sqrt{2} m_f (m_V^2 - m_{AB}^2)}{m_V (m_H^2 - m_{AB}^2 - m_V^2)}$$

Expansion in $\varepsilon \propto m_f$

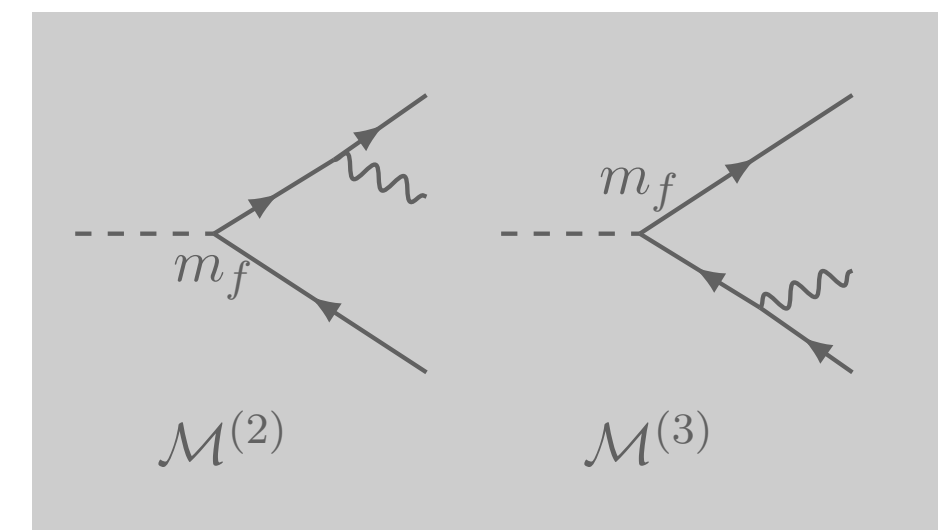
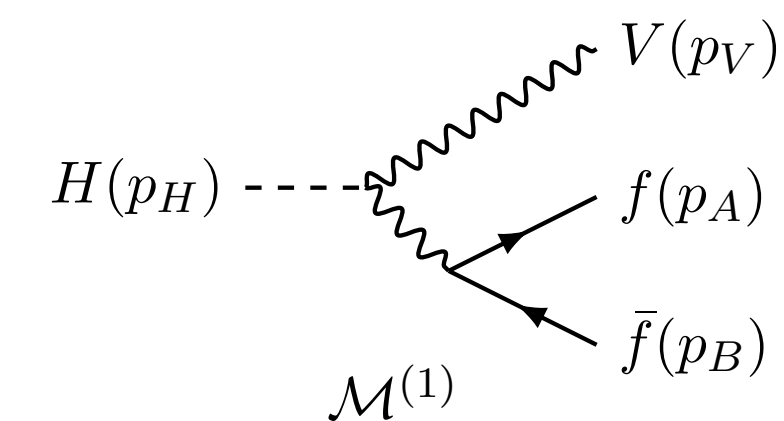
, $\varepsilon \in [3.14, 3.21] \times 10^{-2}$ for $h \rightarrow \tau^+ \tau^- Z$

$$|\psi\rangle = \frac{1}{\mathcal{N}} \left(|\psi^{(0)}\rangle + \varepsilon |\psi^{(1)}\rangle + \dots \right)$$

$$\varepsilon \equiv \frac{\sqrt{2} m_f (m_V^2 - m_{AB}^2)}{m_V (m_H^2 - m_{AB}^2 - m_V^2)}$$

- In $m_f \rightarrow 0$, only $|+-\rangle$ and $|-+\rangle$ in the fermion sector are populated:

$$|\psi^{(0)}\rangle = c_L |+-\rangle |\Psi_+\rangle + c_R |+-\rangle |\Psi_-\rangle$$



- Gauge boson states (pointer states) are

$$|\Psi_{\pm}\rangle = \pm \frac{m_V}{E_V} (1 \pm \cos \theta) |+\rangle - \sqrt{2} \sin \theta |0\rangle \pm \frac{m_V}{E_V} (1 \mp \cos \theta) |-\rangle$$

effectively $2 \otimes 2 \otimes 2$

$$\langle \Psi_+ | \Psi_+ \rangle = \langle \Psi_- | \Psi_- \rangle = 2 \left[1 + \frac{m_V^2}{E_V^2} - \frac{k_V^2 \cos^2 \theta}{E_V^2} \right], \quad \langle \Psi_+ | \Psi_- \rangle = \frac{2k_V^2}{E_V^2} \sin^2 \theta.$$

Greenberger, Horne, Zeilinger (1989)

$$\simeq |\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

maximally entangled state
in $2 \otimes 2 \otimes 2$



$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

$$c_L = -\frac{1}{2} + \sin^2 \theta_W \simeq -0.269$$

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At the *gauge boson threshold*, $m_{AB} \rightarrow m_{AB}^{\max}$ ($k_V \rightarrow 0$)

$$\langle \Psi_+ | \Psi_- \rangle = \frac{2k_V^2}{E_V^2} \sin^2 \theta = 0$$

- In the leading correction, only $|\pm\pm\rangle$ are populated: $|\psi\rangle = \frac{1}{\mathcal{N}} \left(|\psi^{(0)}\rangle + \varepsilon |\psi^{(1)}\rangle + \dots \right)$

$$|\psi^{(1)}\rangle = |++\rangle |\Phi_+\rangle + |--\rangle |\Phi_-\rangle$$

- The corresponding gauge boson states are

$$|\Phi_{\pm}\rangle = a_{\pm}^{(+)} |+\rangle + a_{\pm}^{(0)} |0\rangle + a_{\pm}^{(-)} |-\rangle$$

$$a_{\pm}^{(h_V)} = \mp h_V \sqrt{2} \sin \theta \left[\frac{2c_V m_V^2}{m_V^2 - m_{AB}^2} + \frac{m_{AB} (E_V + m_{AB} \pm h_V k_V) (c_V D \pm c_A m_{AB} k_V \cos \theta)}{D_+ D_-} \right] \quad D \equiv E_V m_{AB} + m_V^2$$

$$a_{\pm}^{(0)} = 2 \left[c_A \frac{k_V}{m_V} \left(2 + \frac{m_{AB}^2 D \sin^2 \theta}{D_+ D_-} \right) \mp c_V m_V \cos \theta \left(\frac{2E_V}{m_V^2 - m_{AB}^2} + \frac{m_{AB} m_H^2}{D_+ D_-} \right) \right] \quad D_{\pm} \equiv D \pm m_{AB} k_V \cos \theta$$

- At the **fermion threshold** region, $k_V \rightarrow \frac{m_H^2 - m_V^2}{4m_f}$.

The **growth** of longitudinal amplitude $a_{\pm}^{(0)}$ **compensates the ε suppression** !

- At the **fermion threshold** region $m_{AB} \rightarrow 2m_f$, the state approaches

$$|\psi_{\text{thr}}\rangle = \frac{\sqrt{2}}{\mathcal{N}} \left[- (c_V - \beta c_A) \sin \theta | - + \rangle - (c_V + \beta c_A) \sin \theta | + - \rangle \right. \\ \left. + (c_A - c_V \cos \theta) | ++ \rangle + (c_A + c_V \cos \theta) | -- \rangle \right] |0\rangle$$

separable, $AB \mid V$

effectively $2 \otimes 2$

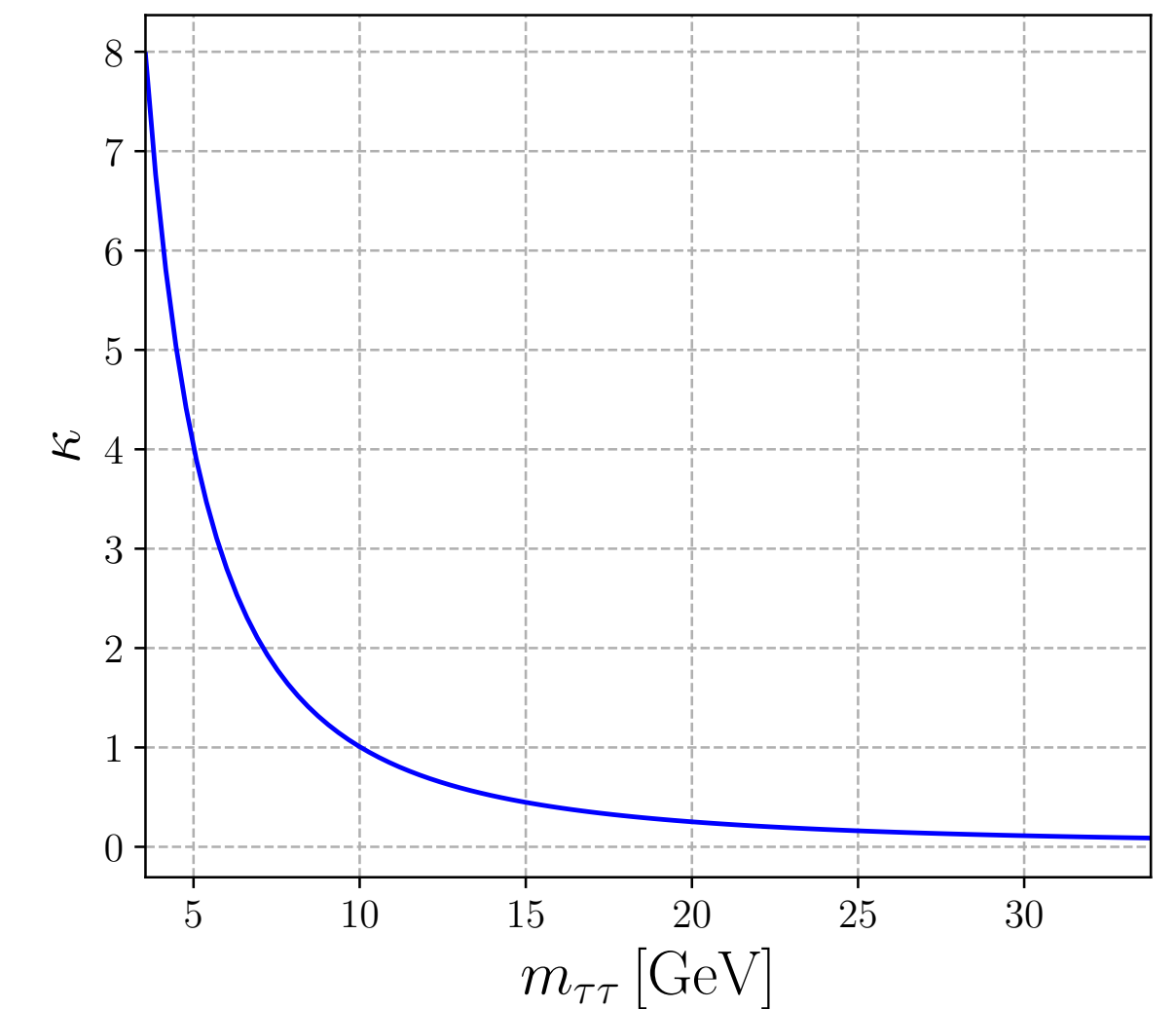
$\beta \equiv 2k_A/m_{AB} = \sqrt{1 - 4m_f^2/m_{AB}^2}$ is the fermion velocity

- Consider the *collinear regions*, $\sin \theta \rightarrow 0$.

Not only the subleading state grows, but the *leading state decreases* at lower m_{AB} .

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

$$|\Psi_{\pm}\rangle = \pm \frac{m_V}{E_V} (1 \pm \cos \theta) | + \rangle - \sqrt{2} \sin \theta | 0 \rangle \pm \frac{m_V}{E_V} (1 \mp \cos \theta) | - \rangle$$

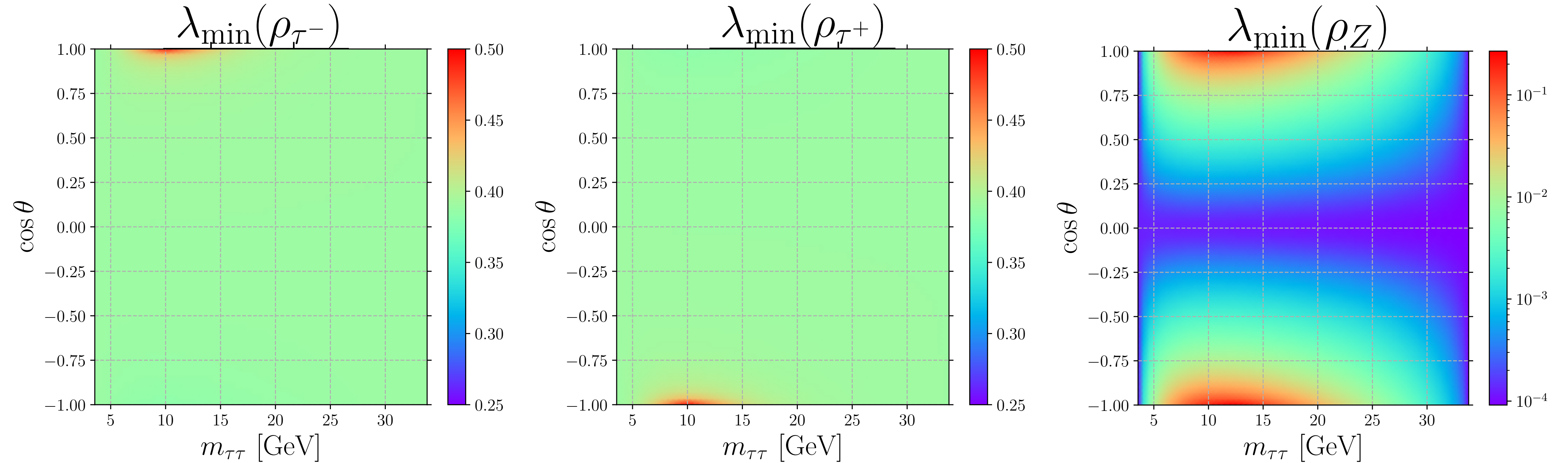


- At the collinaer regions, the state approaches

$$\begin{aligned}
 |\psi\rangle \Big|_{\cos \theta = +1} &\propto c_L | - + \rangle | + \rangle - c_R | + - \rangle | - \rangle + \kappa (c_R | - - \rangle - c_L | + + \rangle) | 0 \rangle \\
 |\psi\rangle \Big|_{\cos \theta = -1} &\propto c_L | - + \rangle | - \rangle - c_R | + - \rangle | + \rangle + \kappa (c_R | + + \rangle - c_L | - - \rangle) | 0 \rangle
 \end{aligned}
 \quad \kappa \equiv \frac{m_f (m_H^2 - m_V^2)}{\sqrt{2} m_{AB}^2 m_V}$$

Is this state ganuinely $2 \otimes 2 \otimes 3$?

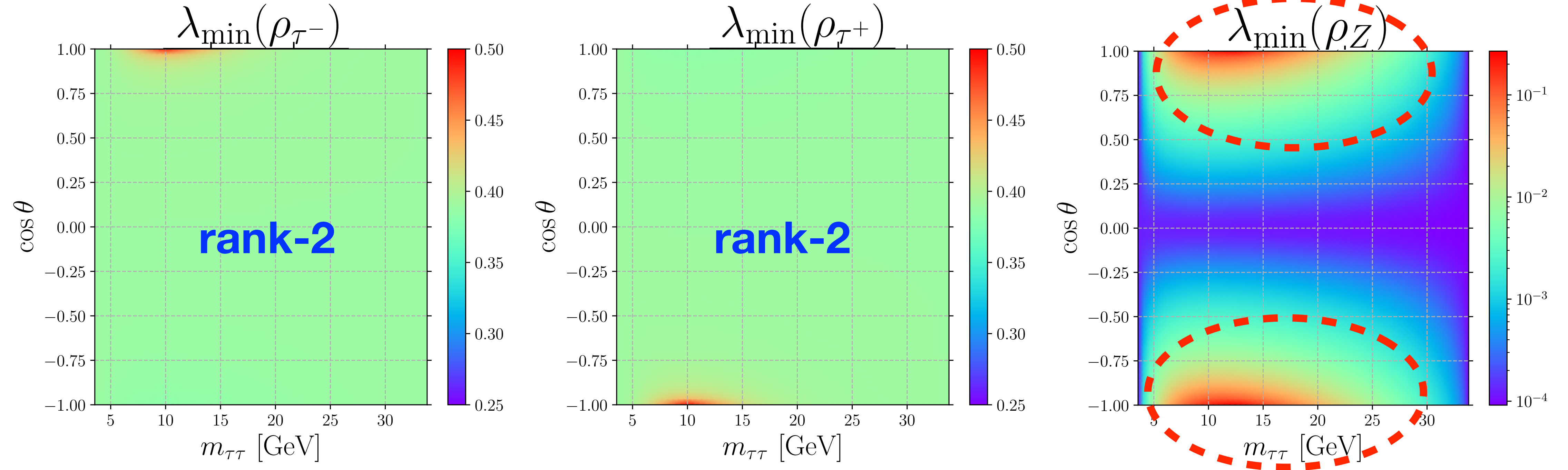
- The **smallest eigenvalue** of the one-particle reduced states:



- The state is almost **maximally** $2 \otimes 2 \otimes 3$ at the **collinear regions** at

$$\kappa_* = \sqrt{\frac{1 - |\hat{c}|}{2}} \simeq 0.65 \quad \hat{c} \equiv \frac{2 c_A c_V}{c_A^2 + c_V^2} \simeq -0.151 \quad (m_{\tau\tau} \simeq 12 \text{ GeV})$$

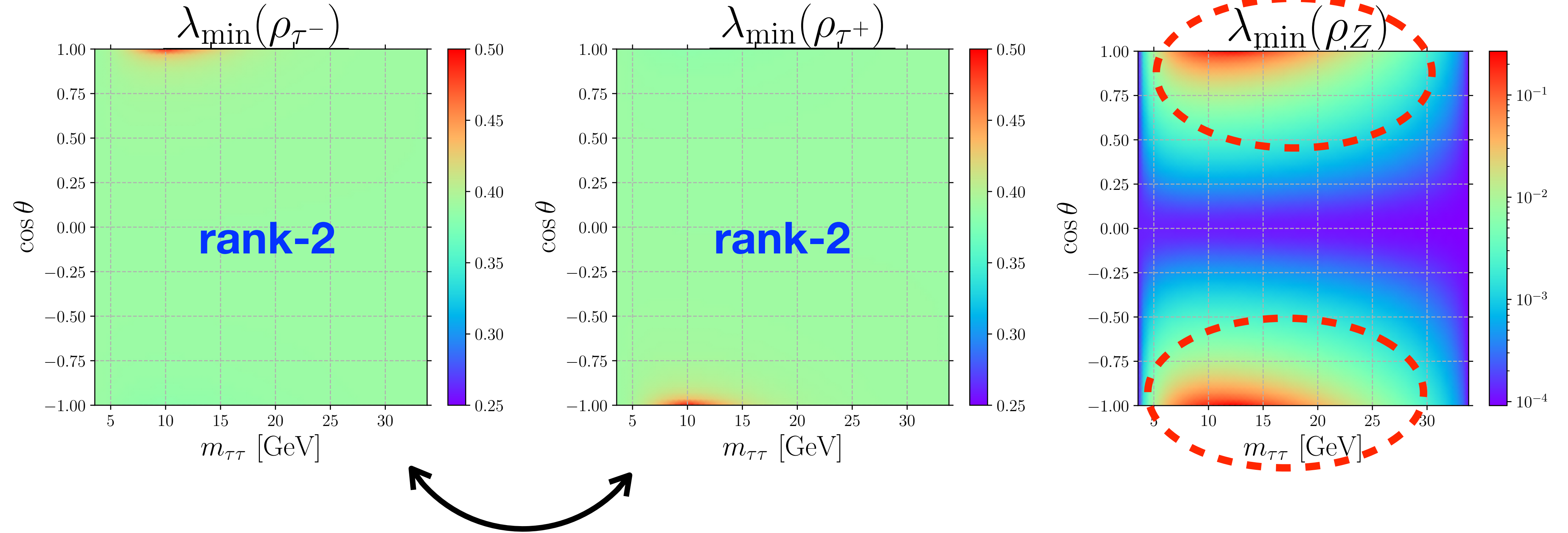
- The **smallest eigenvalue** of the one-particle reduced states:



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- The **smallest eigenvalue** of the one-particle reduced states:

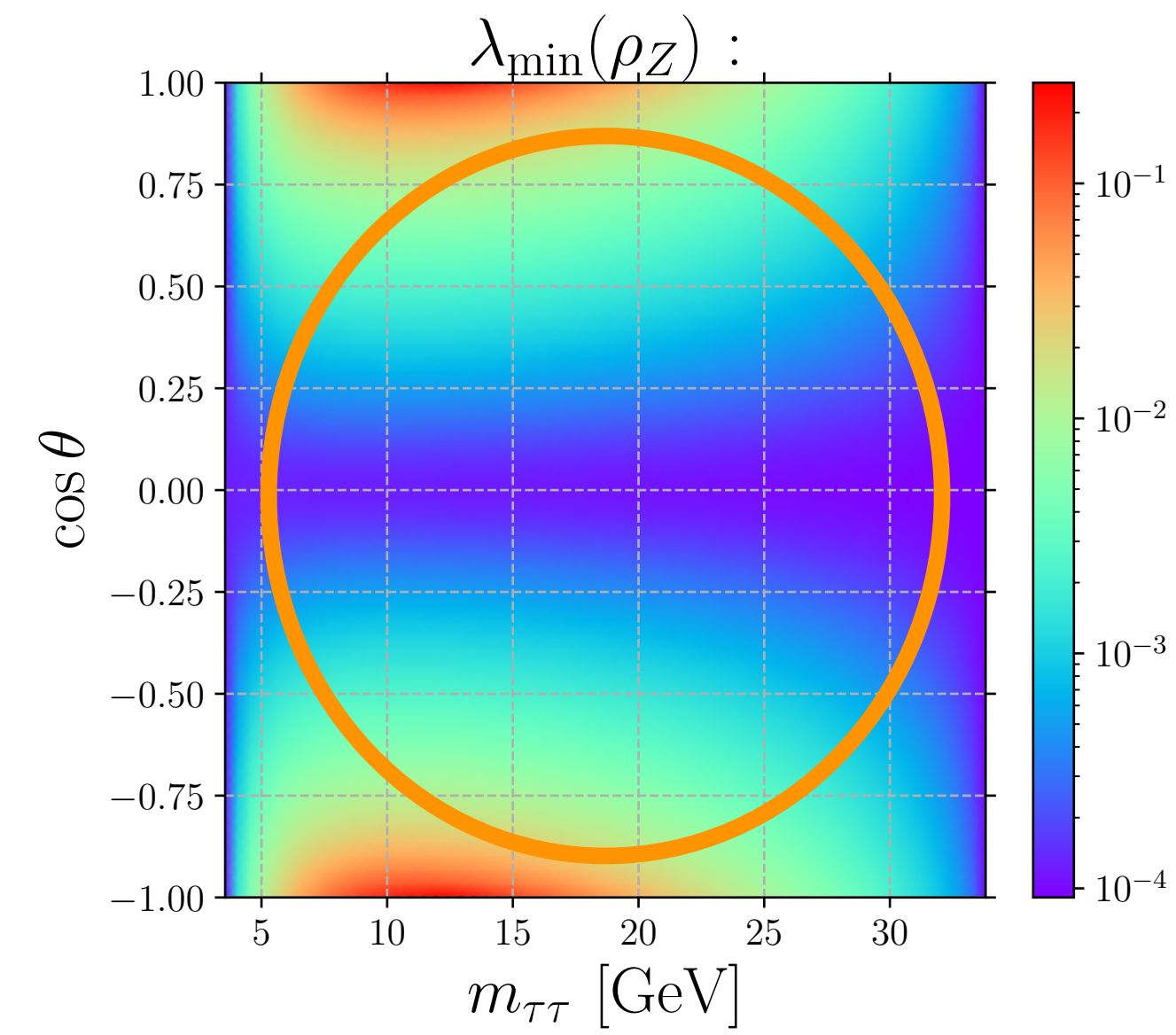


$$O(\textcolor{red}{A}, \textcolor{blue}{B}, \textcolor{green}{V}; \textcolor{violet}{\cos\theta}) \stackrel{\text{CP}}{=} O(\textcolor{blue}{B}, \textcolor{red}{A}, \textcolor{green}{V}; -\textcolor{violet}{\cos\theta})$$

Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle \quad \langle \Psi_+ | \Psi_- \rangle \neq 0$$

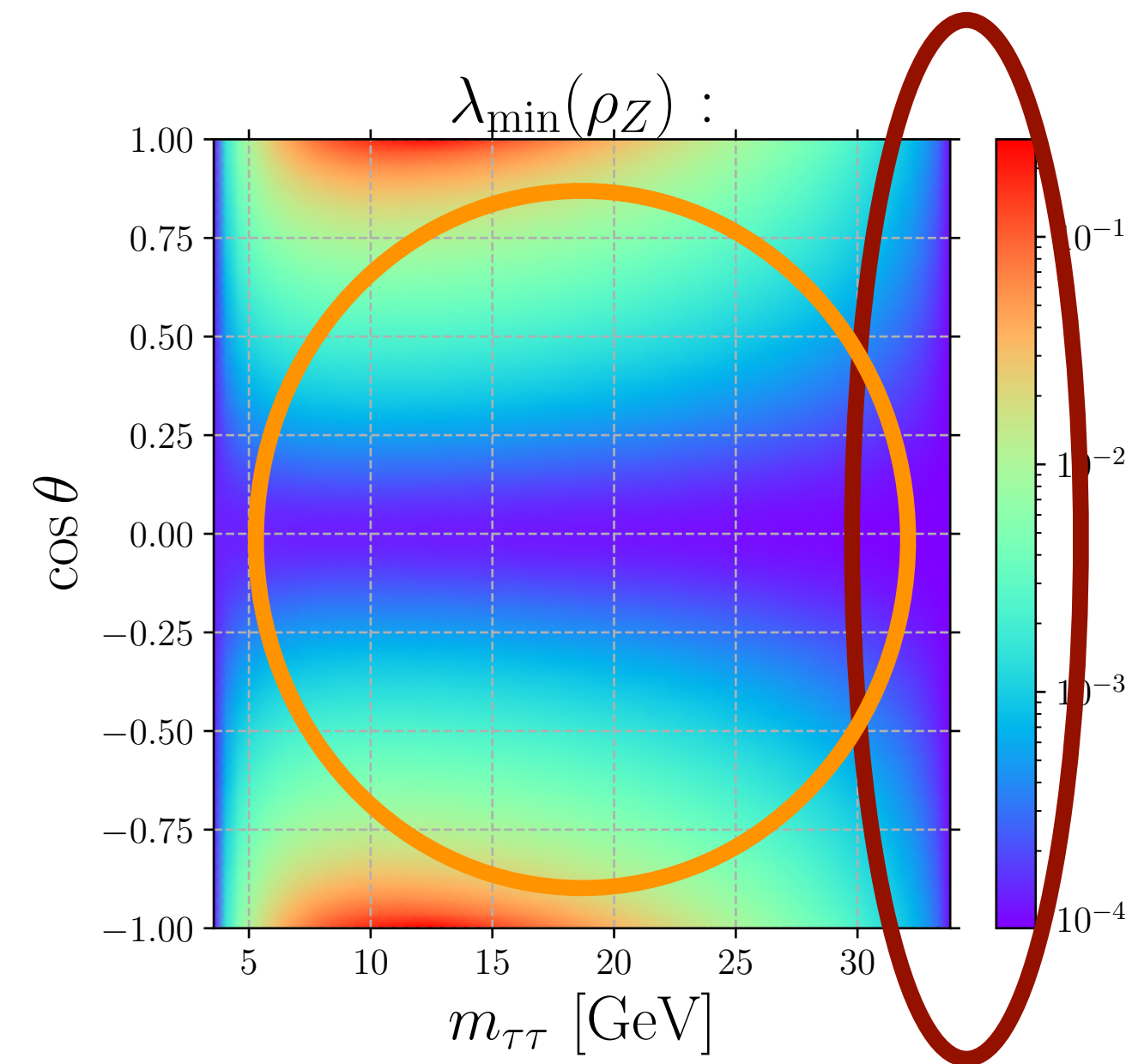
effectively
 $2 \otimes 2 \otimes 2$



Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle \quad \langle \Psi_+ | \Psi_- \rangle \neq 0$$

effectively
 $2 \otimes 2 \otimes 2$



Gauge boson threshold

$$\langle \Psi_+ | \Psi_- \rangle = 0$$

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

$$\simeq |\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

max ent. GHZ in $2 \otimes 2 \otimes 2$

Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle \quad \langle \Psi_+ | \Psi_- \rangle \neq 0$$

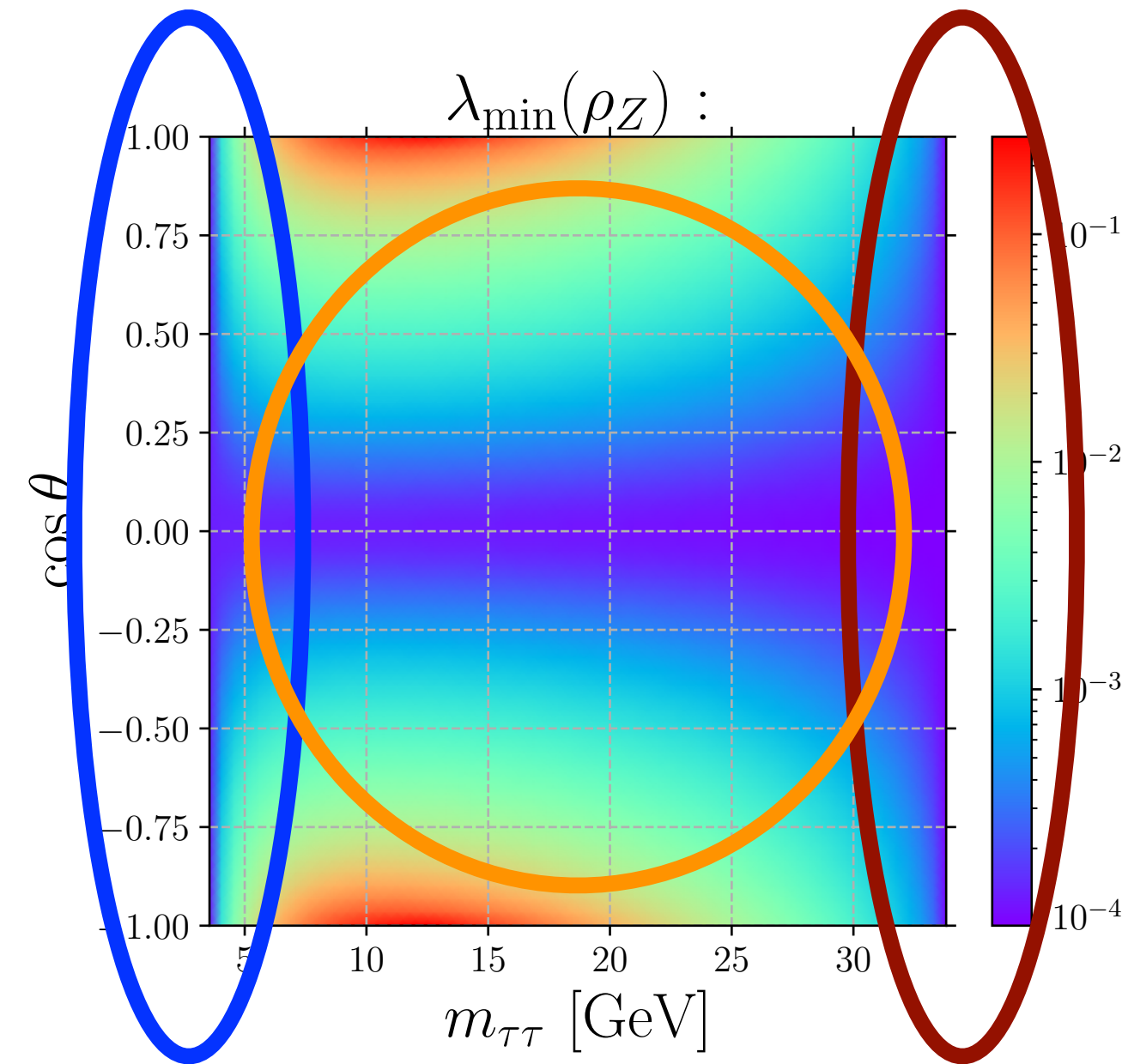
effectively
 $2 \otimes 2 \otimes 2$

Fermion threshold

$$|\psi_{\text{thr}}\rangle = \frac{\sqrt{2}}{\mathcal{N}} \left[\begin{aligned} & - (c_V - \beta c_A) \sin \theta | - + \rangle \\ & - (c_V + \beta c_A) \sin \theta | + - \rangle \\ & + (c_A - c_V \cos \theta) | ++ \rangle \\ & + (c_A + c_V \cos \theta) | -- \rangle \end{aligned} \right] |0\rangle$$

separable AB | V

effectively $2 \otimes 2$



Gauge boson threshold

$$\begin{aligned} \langle \Psi_+ | \Psi_- \rangle &= 0 \\ |\psi^{(0)}\rangle &= c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle \\ &\simeq |\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \end{aligned}$$

max ent. GHZ in $2 \otimes 2 \otimes 2$

Bulk

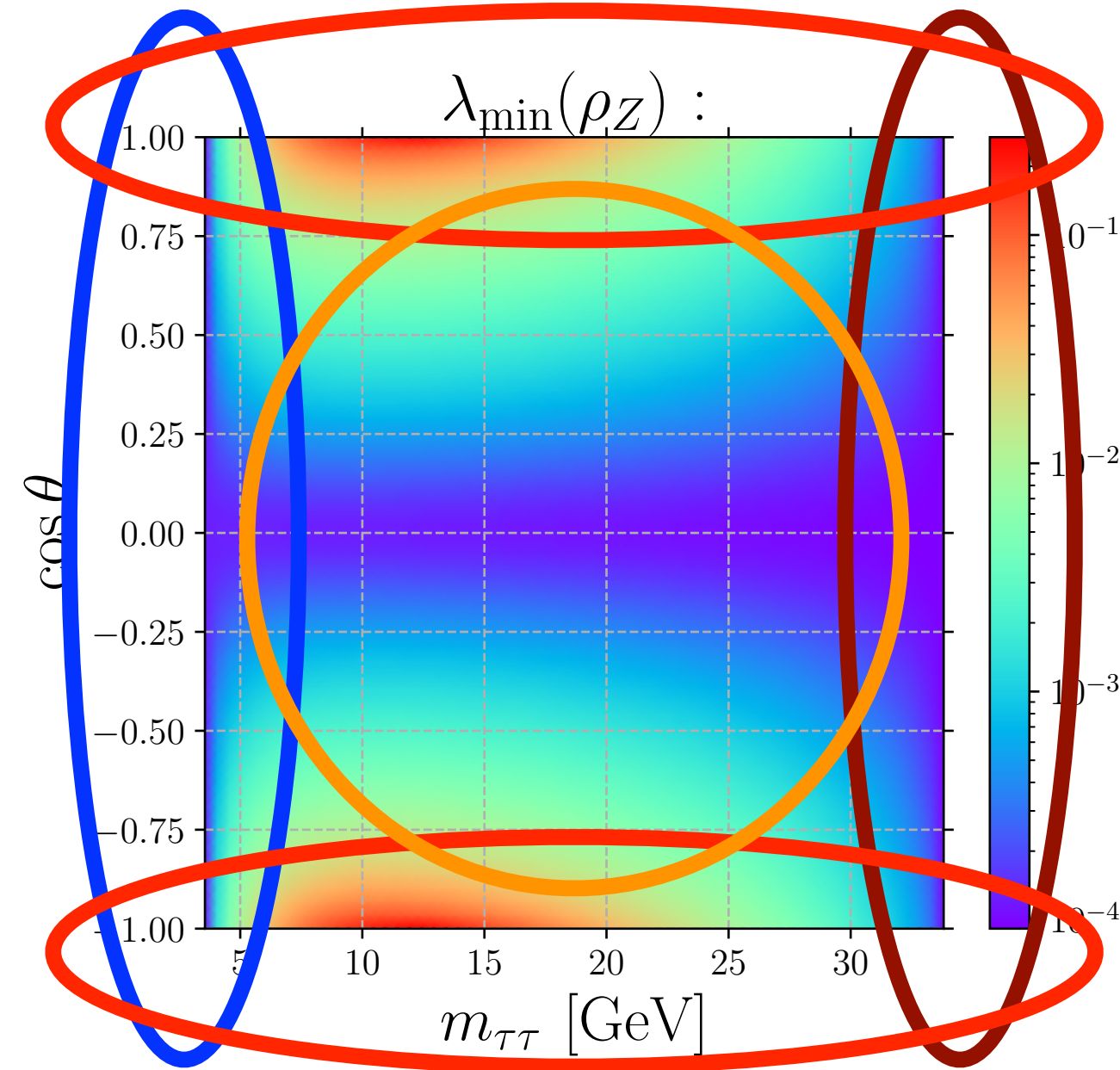
$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle \quad \langle \Psi_+ | \Psi_- \rangle \neq 0$$

effectively
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Fermion threshold

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Gauge boson threshold

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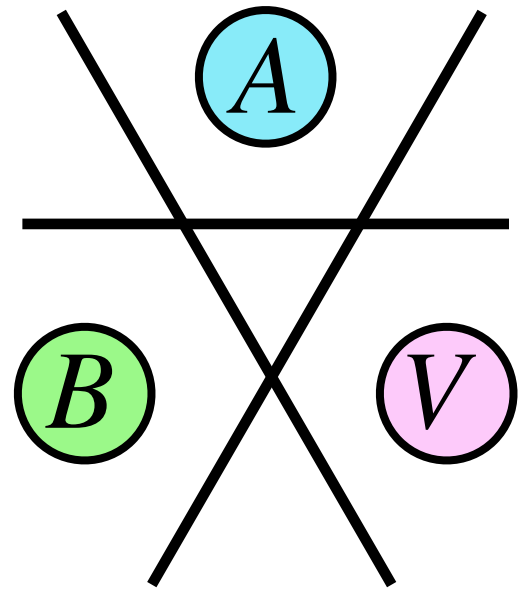
max ent. GHZ in $2 \otimes 2 \otimes 2$

Collinear

$$\begin{aligned} |\psi\rangle \Big|_{\cos \theta = +1} &\propto c_L | - + \rangle | + \rangle - c_R | + - \rangle | - \rangle + \kappa (c_R | -- \rangle - c_L | ++ \rangle) |0\rangle \\ |\psi\rangle \Big|_{\cos \theta = -1} &\propto c_L | - + \rangle | - \rangle - c_R | + - \rangle | + \rangle + \kappa (c_R | ++ \rangle - c_L | -- \rangle) |0\rangle \end{aligned}$$

maximally $2 \otimes 2 \otimes 3$ at $m_{\tau\tau} \sim 12 \text{ GeV}$

One-to-other entanglement

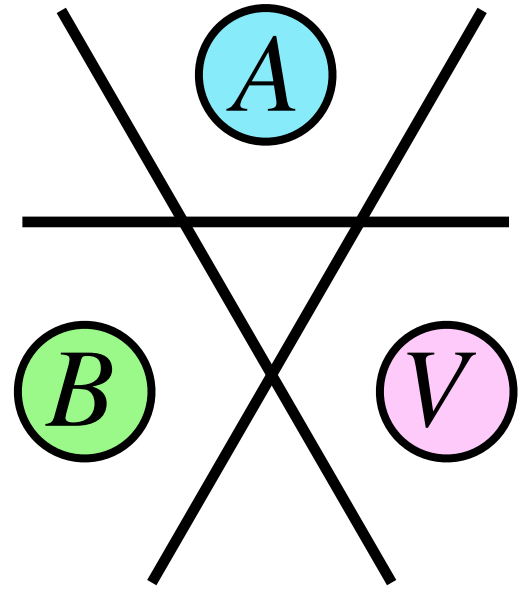


$$C_{A|BV} = \sqrt{2(1 - \text{Tr } \rho_A^2)},$$

$$C_{B|AV} = \sqrt{2(1 - \text{Tr } \rho_B^2)},$$

$$C_{V|AB} = \sqrt{\frac{3}{2}(1 - \text{Tr } \rho_V^2)},$$

One-to-other entanglement

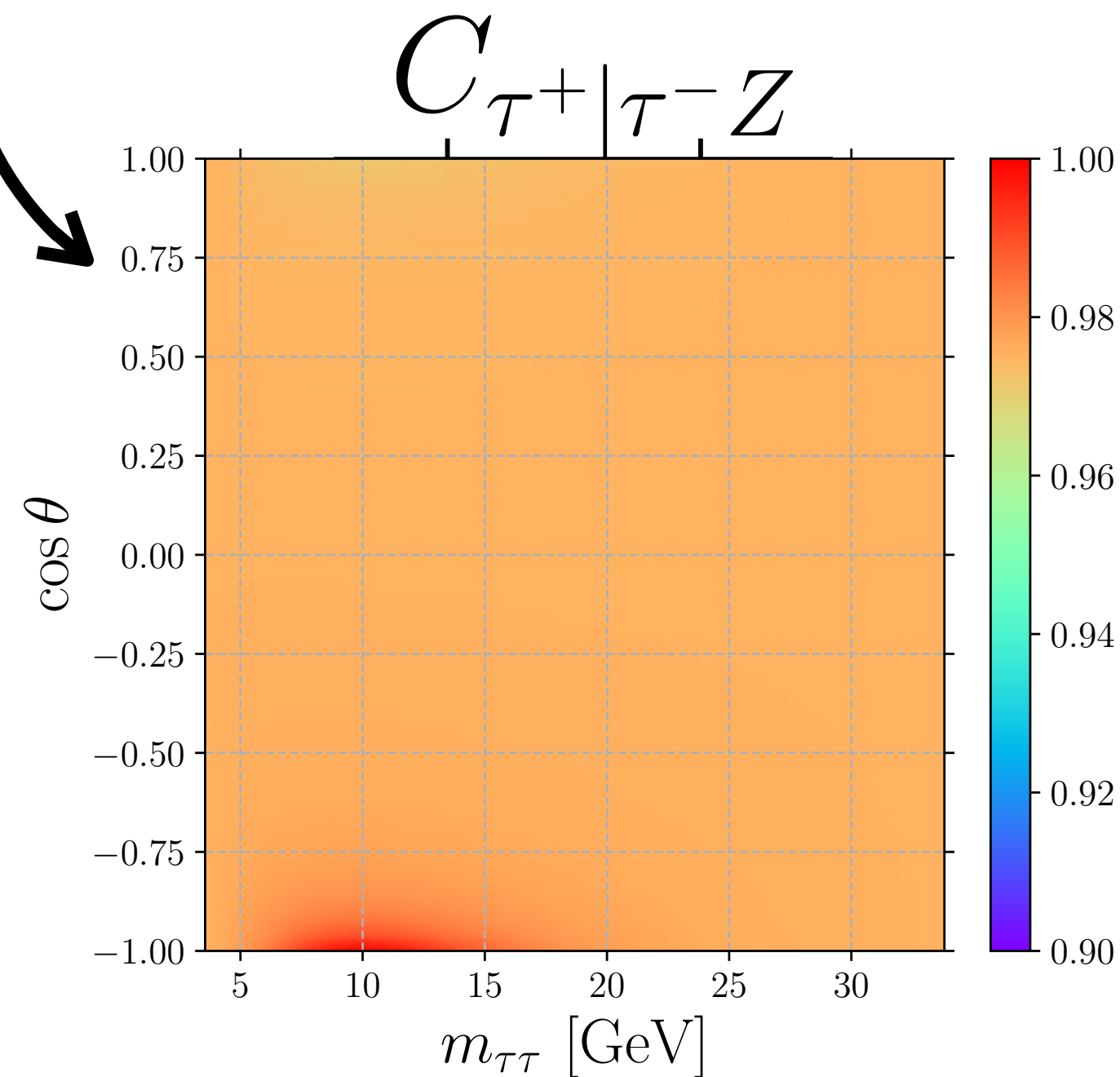
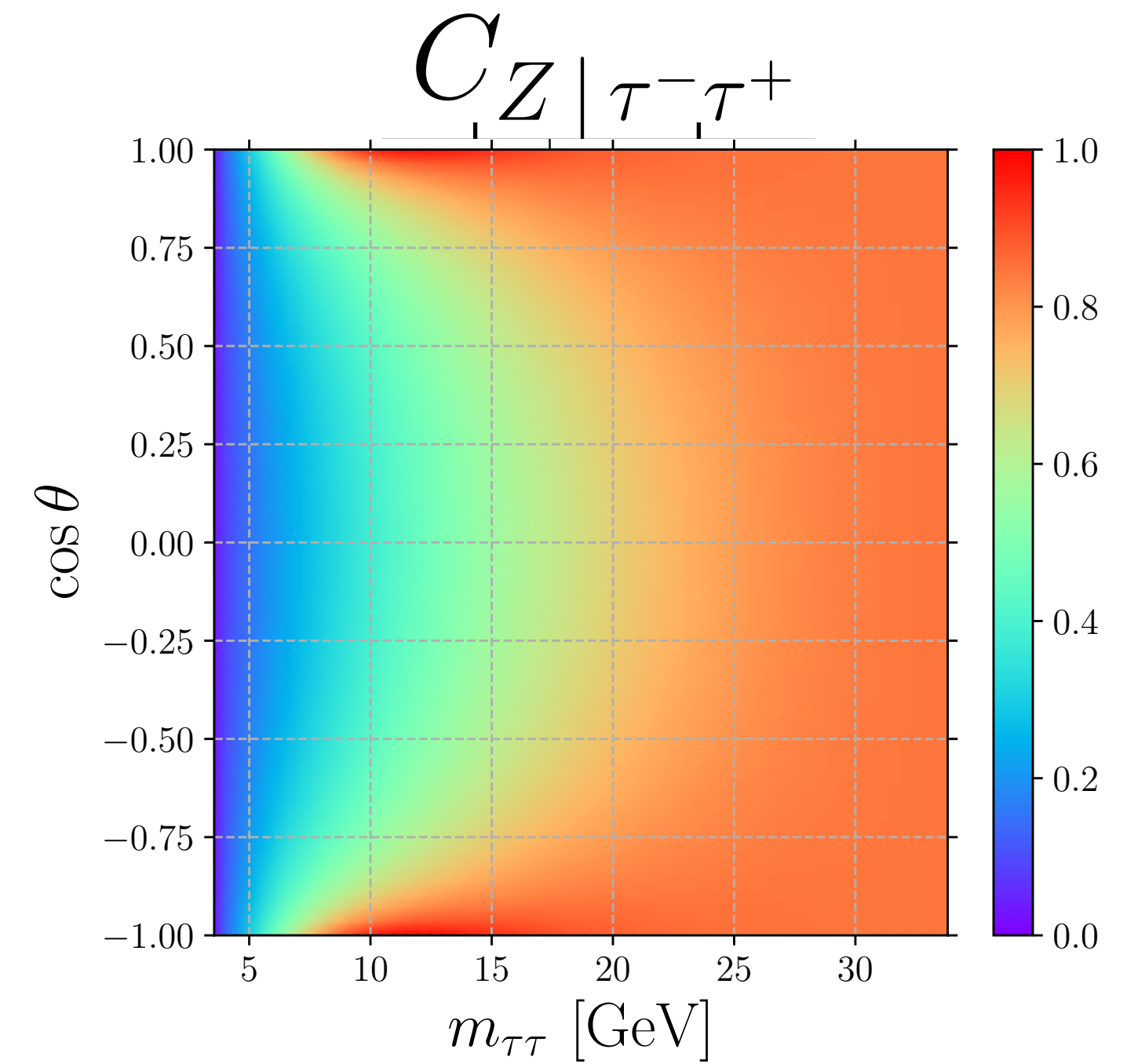
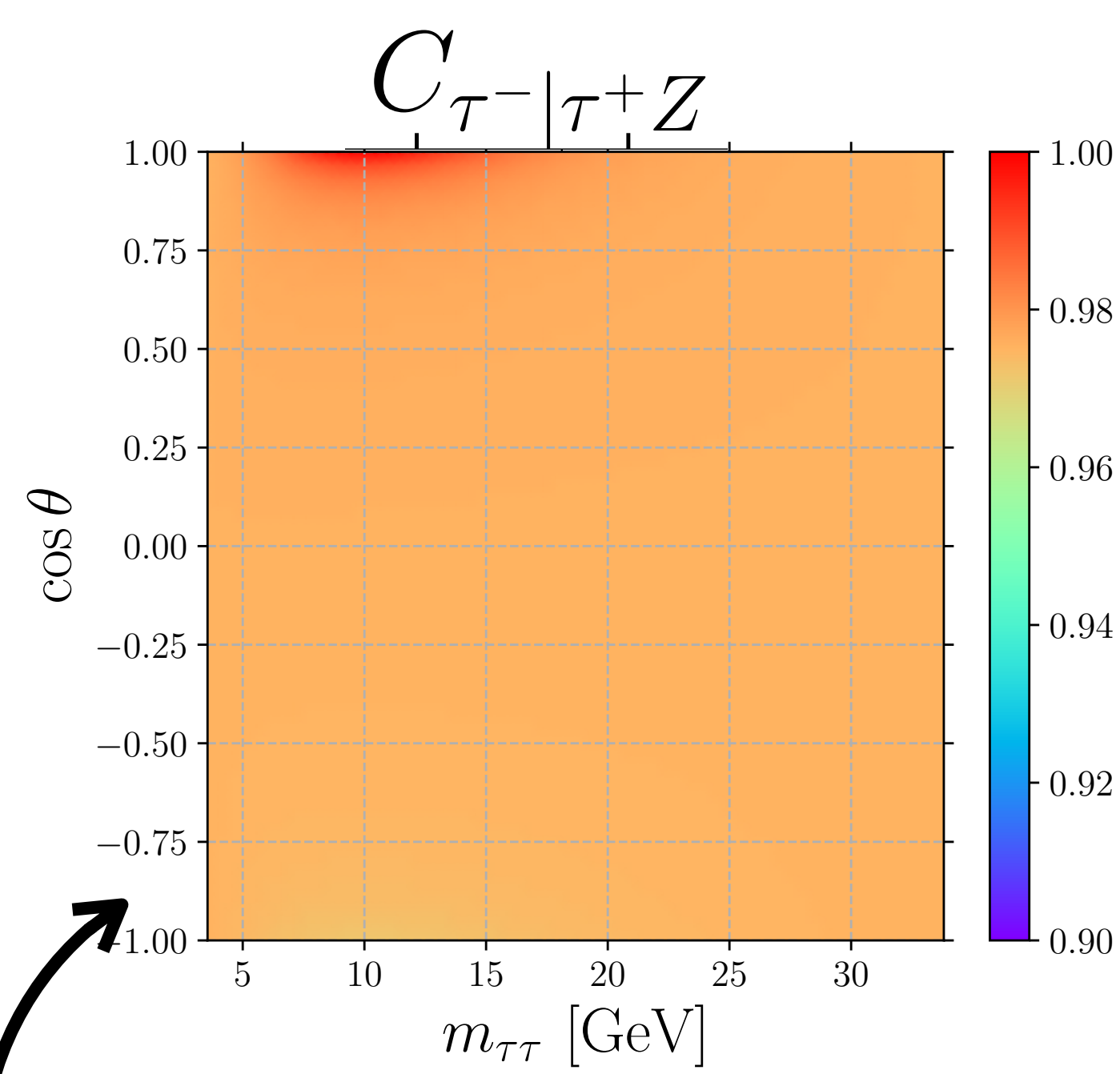


$$C_{A|BV} = \sqrt{2(1 - \text{Tr } \rho_A^2)},$$

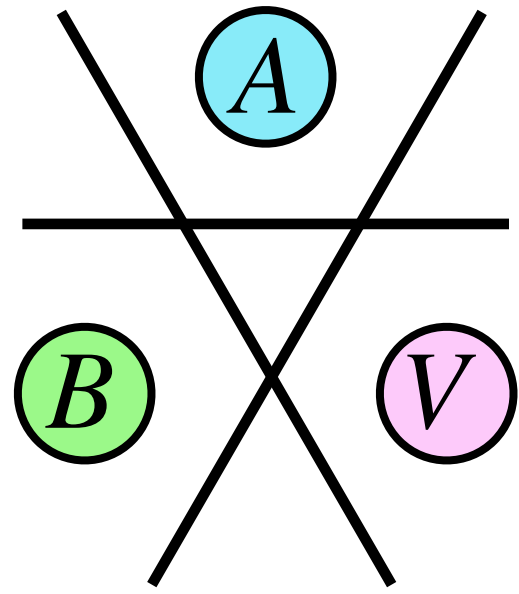
$$C_{B|AV} = \sqrt{2(1 - \text{Tr } \rho_B^2)},$$

$$C_{V|AB} = \sqrt{\frac{3}{2}(1 - \text{Tr } \rho_V^2)},$$

$$O(A, B, V; \cos \theta) \stackrel{\text{CP}}{=} O(B, A, V; -\cos \theta)$$



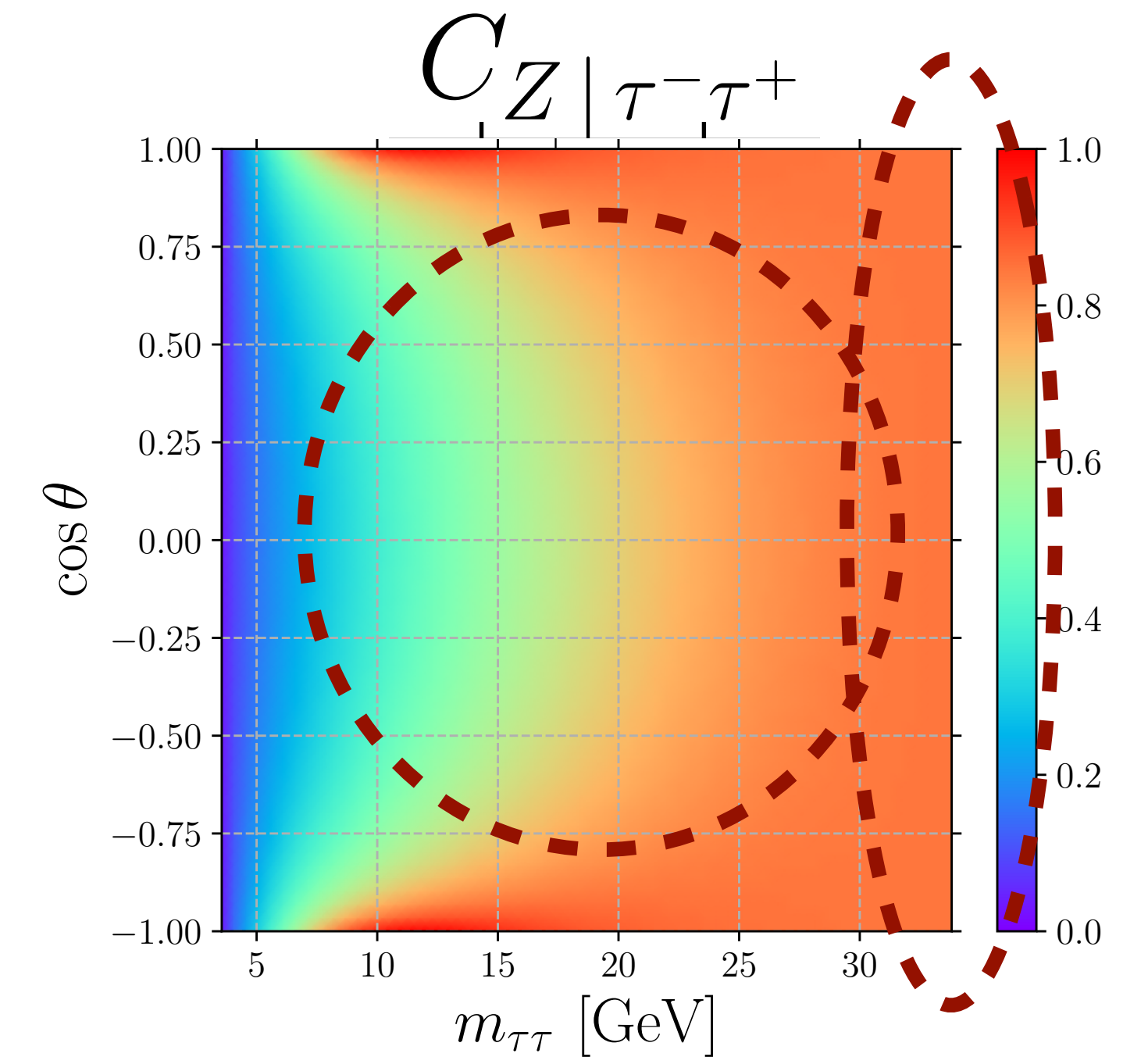
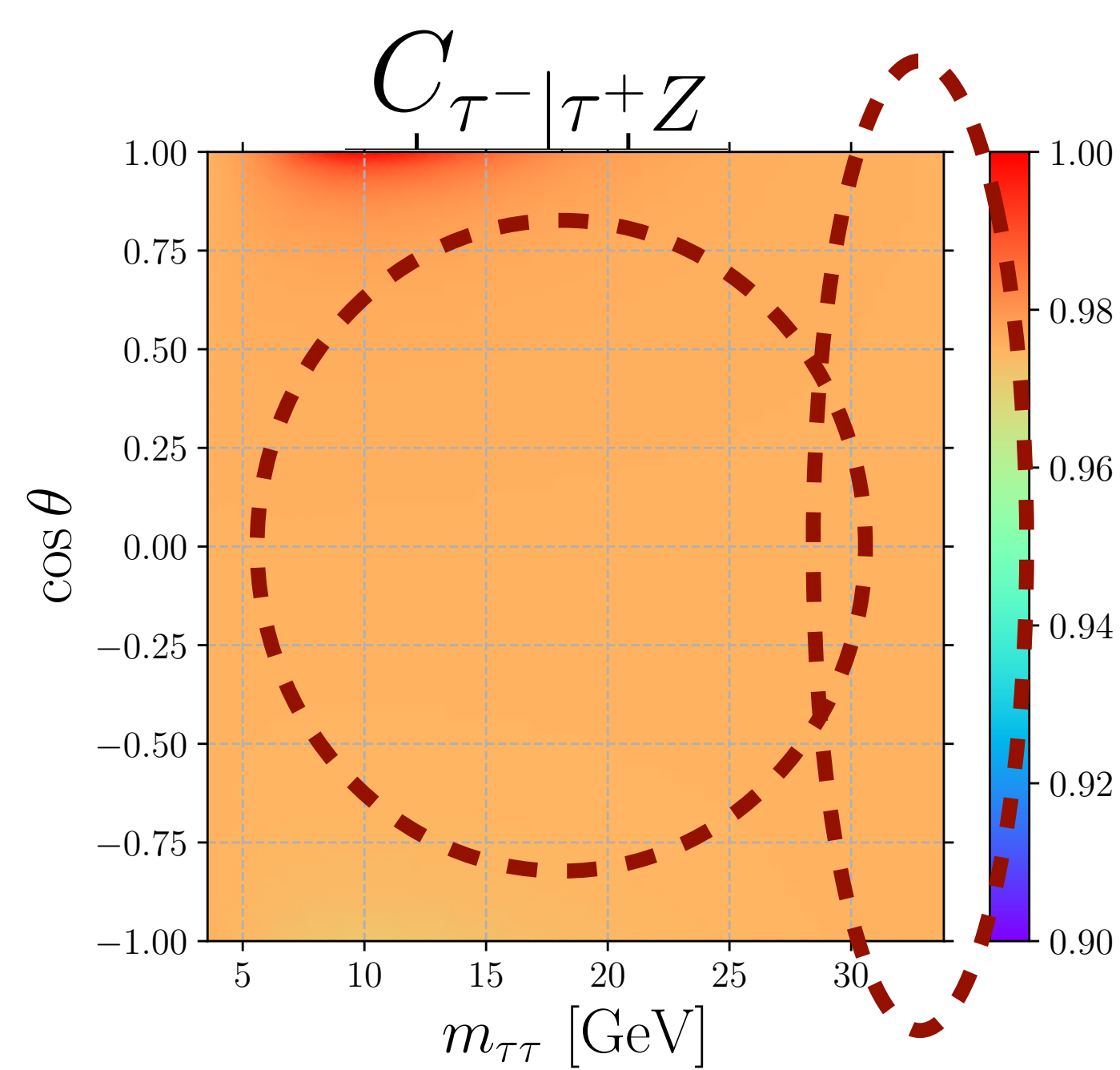
One-to-other entanglement



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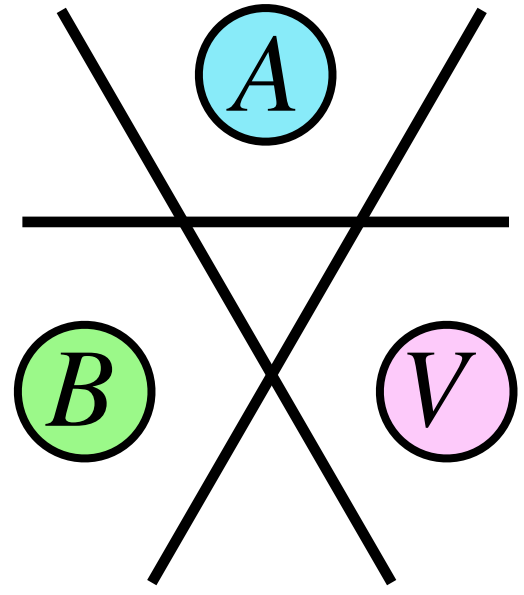
Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

$$C_{A|BV}^{(0)} = C_{B|AV}^{(0)} = \hat{s}, \quad C_{V|AB}^{(0)} = \frac{\sqrt{3}}{2} \hat{s} \frac{2m_V \sqrt{E_V^2 - k_V^2 \cos^2 \theta}}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

$$\hat{s} \equiv \frac{2|c_L c_R|}{c_L^2 + c_R^2} \simeq 0.988$$

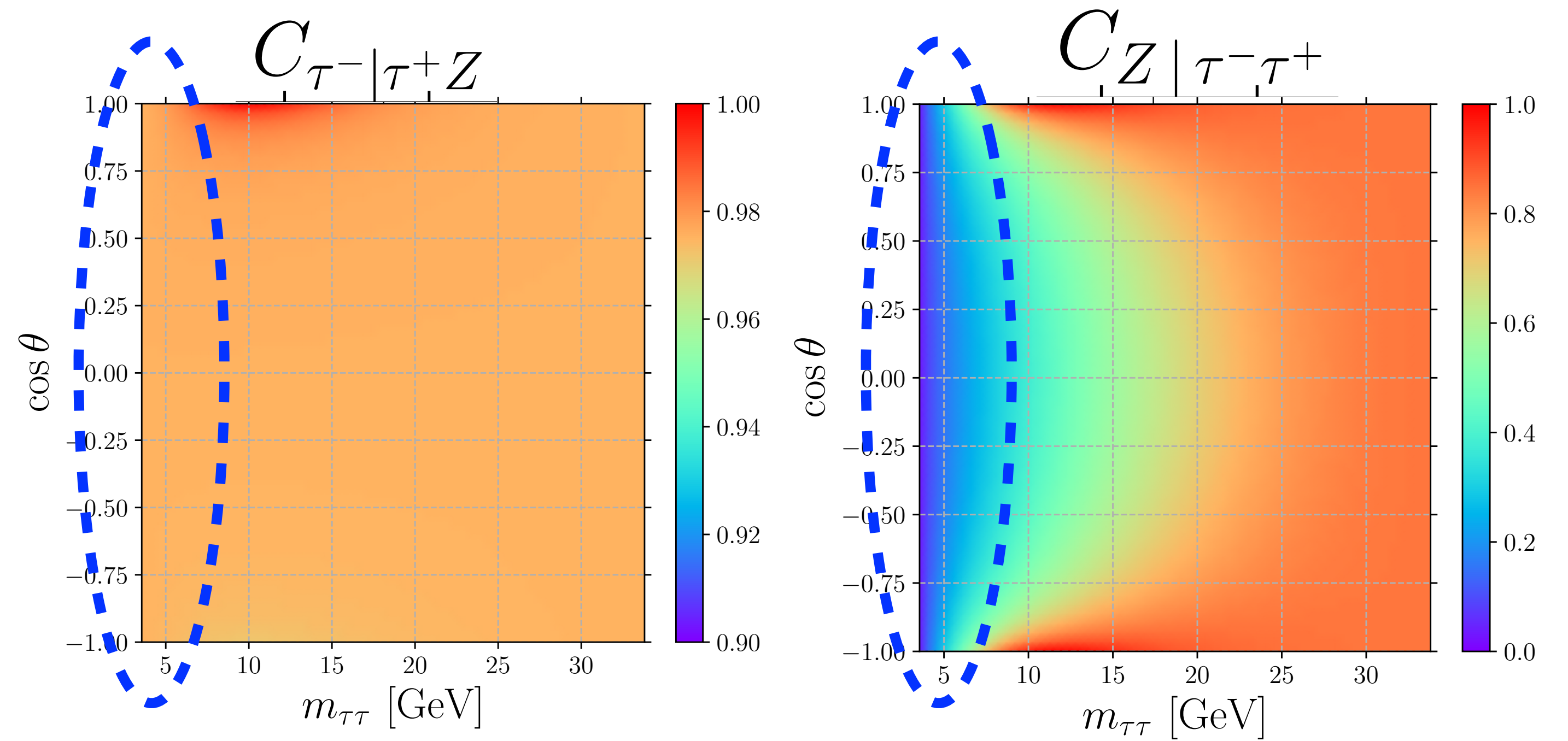
One-to-other entanglement



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Fermion threshold

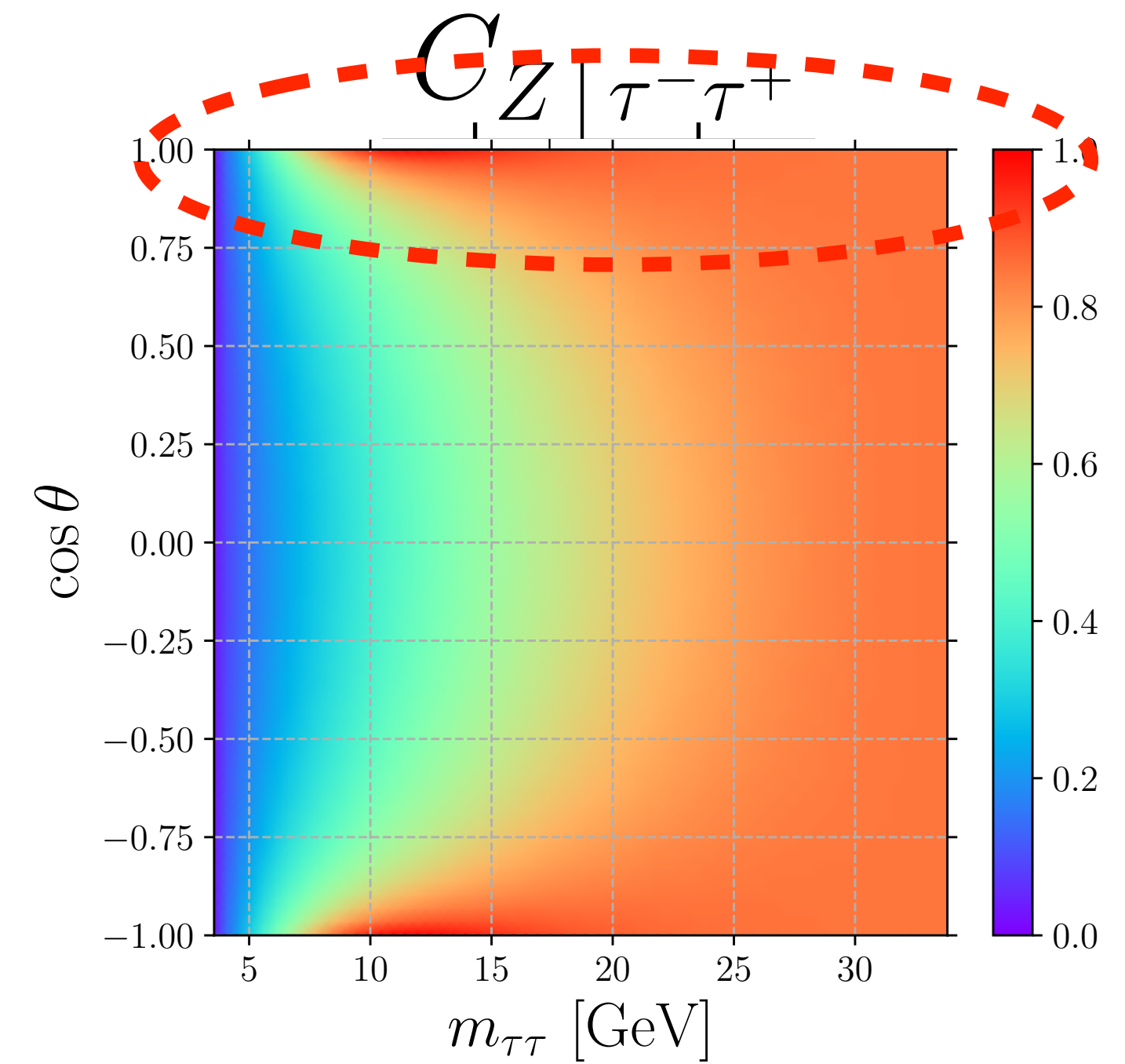
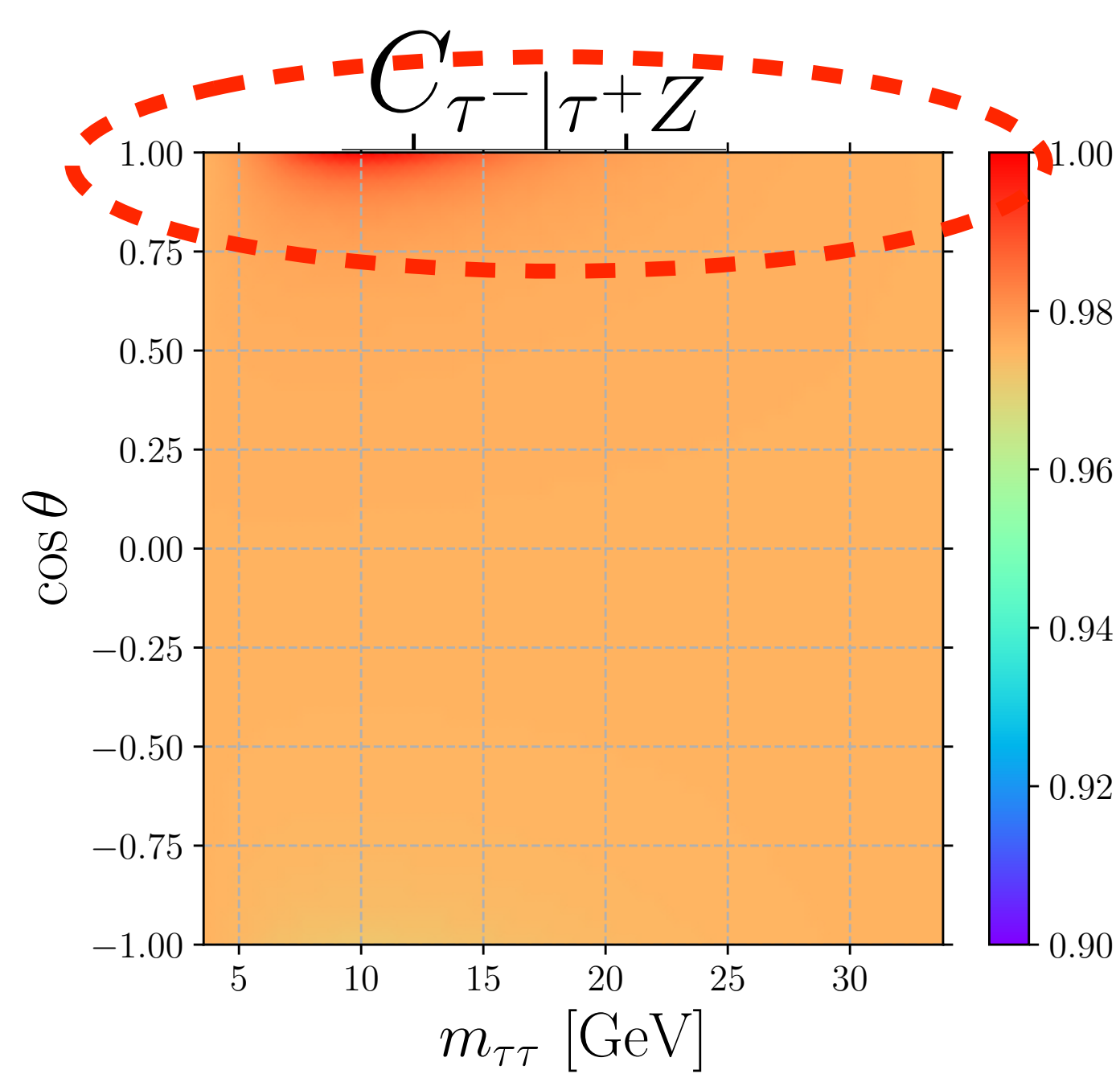
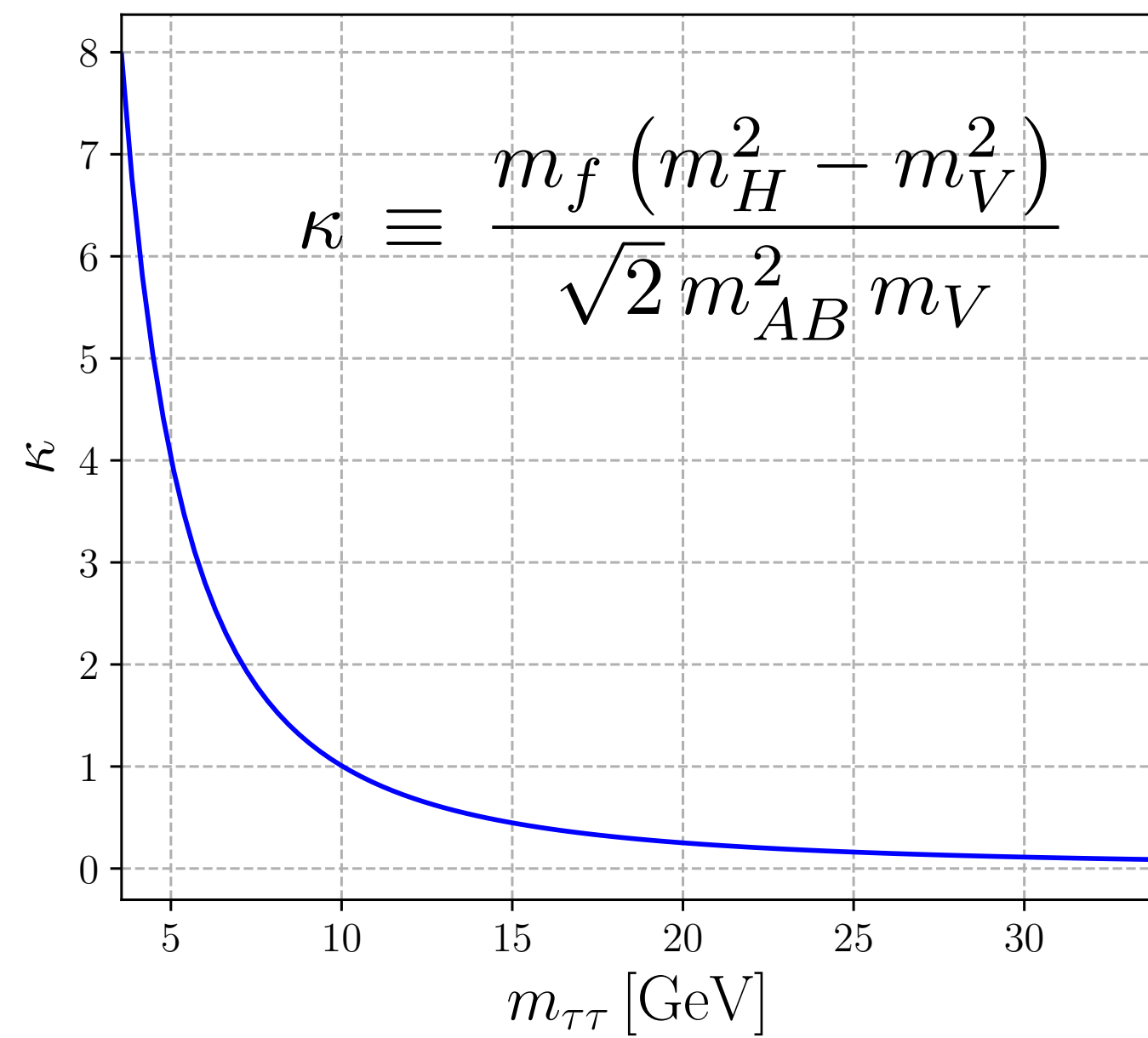
$$|\psi_{\text{thr}}\rangle = \frac{\sqrt{2}}{\mathcal{N}} \left[\begin{aligned} &-(c_V - \beta c_A) \sin \theta | - + \rangle \\ &-(c_V + \beta c_A) \sin \theta | + - \rangle \\ &+(c_A - c_V \cos \theta) | ++ \rangle \\ &+(c_A + c_V \cos \theta) | -- \rangle \end{aligned} \right] |0\rangle$$

separable AB | V

$$C_{A|BV} = C_{B|AV} = \frac{c_A^2 (1 + \beta^2 \sin^2 \theta) - c_V^2}{c_A^2 (1 + \beta^2 \sin^2 \theta) + c_V^2},$$

$$C_{V|AB} = 0$$

$$c_V \equiv \frac{1}{2}(c_R + c_L) \simeq -0.019$$



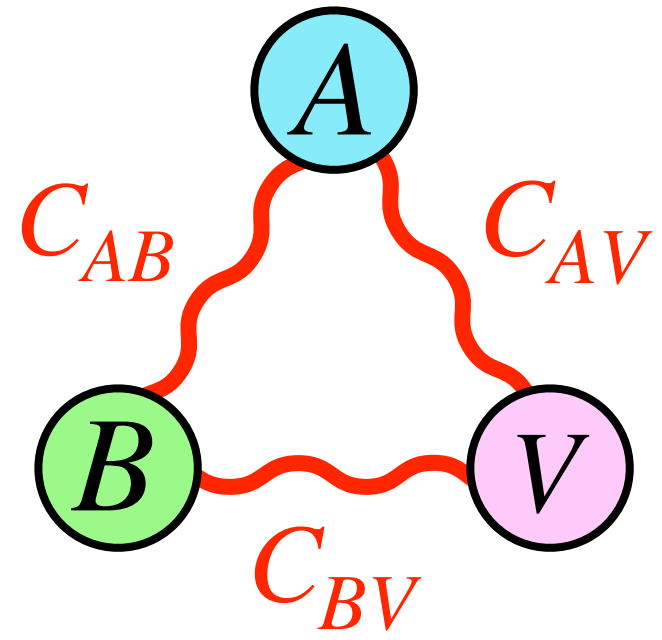
Collinear

$$|\psi\rangle \big|_{\cos\theta=+1} \propto c_L | - + \rangle | + \rangle - c_R | + - \rangle | - \rangle + \kappa (c_R | - - \rangle - c_L | + + \rangle) | 0 \rangle$$

$$\hat{s}^2 + \hat{c}^2 = 1$$

$$\underbrace{C_{A|BV} = \sqrt{\hat{s}^2 + \hat{c}^2 \left(\frac{2\kappa}{1 + \kappa^2} \right)^2}}_{\rightarrow 1 \text{ } (\kappa \rightarrow 1)}, \quad C_{B|AV} = \hat{s}, \quad C_{V|AB} = \frac{\sqrt{3}}{2} \frac{\sqrt{\hat{s}^2 + 4\kappa^2}}{1 + \kappa^2}$$

One-to-one entanglement



$$\rho_{AB} = \text{Tr}_V |\psi\rangle\langle\psi|$$

$$\rho_{AV} = \text{Tr}_B |\psi\rangle\langle\psi|$$

$$\rho_{BV} = \text{Tr}_A |\psi\rangle\langle\psi|$$

Wootters' formula for 2 x 2 **Wootters' (1998)**

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

$$\tau(\rho_{XY}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_I(|\psi_i\rangle)^2 \quad \rho_{XY} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$\tilde{C}_{XY} \equiv \sqrt{\tau(\rho_{XY})}, \quad XY \in \{AV, BV\}.$$

Osborne's formula, **[Osborne (2005)]**, to analytically compute $\tilde{C}$

Osborne's formula for a rank-2 density matrix:

Osborne (2005)

$$\tau(\rho_{XY}) = \text{Tr}(\rho \tilde{\rho}) + 2\lambda_{\min}(1 - \text{Tr} \rho^2),$$

The universal inverter $\tilde{O}$ of O is defined as

$$\tilde{O} \equiv \text{Tr}(O^\dagger) \mathbb{1}_{XY} - (\text{Tr}_Y O^\dagger) \otimes \mathbb{1}_Y - \mathbb{1}_X \otimes (\text{Tr}_X O^\dagger) + O^\dagger,$$

$\lambda_{\min}$ is the smallest eigenvalue of the following 3×3 matrix M .

$$\text{rank } \rho_{AV} = \text{rank } \rho_B \leq 2,$$

Our target density matrices ρ_{AV} and ρ_{BV} have the rank at most 2.

$$\text{rank } \rho_{BV} = \text{rank } \rho_A \leq 2$$

Let $\{|v_1\rangle, |v_2\rangle\}$ be the two eigenvectors corresponding to the two nonzero eigenvalues of the rank-2 ρ .

Define the tensor $T_{ijkl} \equiv \text{Tr}(\gamma_{ij} \tilde{\gamma}_{kl})$ from $\gamma_{ij} = |v_i\rangle\langle v_j|$. Then, the matrix M is defined by

$$\begin{aligned} M_{11} &= \frac{1}{4}T_{1221} + \frac{1}{2}T_{1122} + \frac{1}{4}T_{2112}, & M_{12} &= \frac{i}{4}(T_{1221} - T_{2112}), \\ M_{22} &= -\frac{1}{4}T_{1221} + \frac{1}{2}T_{1122} - \frac{1}{4}T_{2112}, & M_{13} &= \frac{1}{4}(T_{1121} - T_{2122} + T_{1112} - T_{1222}), \\ M_{33} &= \frac{1}{4}T_{1111} - \frac{1}{2}T_{1122} + \frac{1}{4}T_{2222}, & M_{23} &= \frac{i}{4}(T_{1121} - T_{1112} + T_{2122} - T_{1222}), \end{aligned}$$

One-to-one entanglement

Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

separable

$$C_{AV} = C_{BV} = 0$$

$$C_{AB} = \hat{s} \frac{k_V^2 \sin^2 \theta}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

$$\rho_{AB} \propto c_L^2 | - \rangle \langle - | \otimes |\Psi_+\rangle \langle \Psi_+| + c_R^2 | + \rangle \langle + | \otimes |\Psi_-\rangle \langle \Psi_-|$$

One-to-one entanglement

Bulk

$$|\psi^{(0)}\rangle = c_L | - + \rangle |\Psi_+\rangle + c_R | + - \rangle |\Psi_-\rangle$$

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$$C_{AB} = \hat{s} \frac{k_V^2 \sin^2 \theta}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

$$C_{AB}^2 = \left(\frac{2}{\sqrt{3}} C_{V|AB} \right)^2 - \frac{1}{2} [C_{A|BV}^2 + C_{B|AV}^2]$$

relation between
one-to-one and **one-to-other**

$$C_{A|BV}^{(0)} = C_{B|AV}^{(0)} = \hat{s}, \quad C_{V|AB}^{(0)} = \frac{\sqrt{3}}{2} \hat{s} \frac{2m_V \sqrt{E_V^2 - k_V^2 \cos^2 \theta}}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

One-to-one entanglement

Bulk

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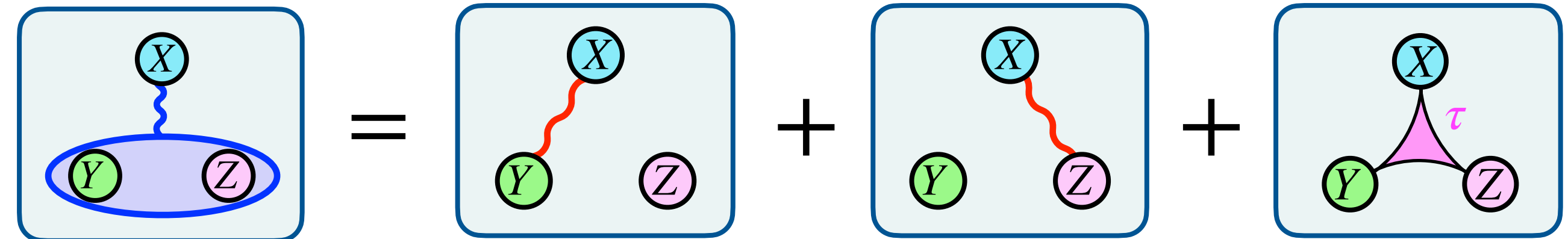
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Monogamy relation for the $2 \times 2 \times 2$ system:

[Coffman, Kundu, Wootters '99]



(for all partitions)

$$\left(\frac{2}{\sqrt{3}} C_{V|AB} \right)^2 = C_{AV}^2 + C_{BV}^2 + \tau$$

$$\frac{1}{2} C_{A|BV}^2 = \frac{1}{2} [C_{AB}^2 + C_{AV}^2 + \tau]$$

$$\frac{1}{2} C_{B|AV}^2 = \frac{1}{2} [C_{AB}^2 + C_{BV}^2 + \tau]$$

One-to-one entanglement

Bulk

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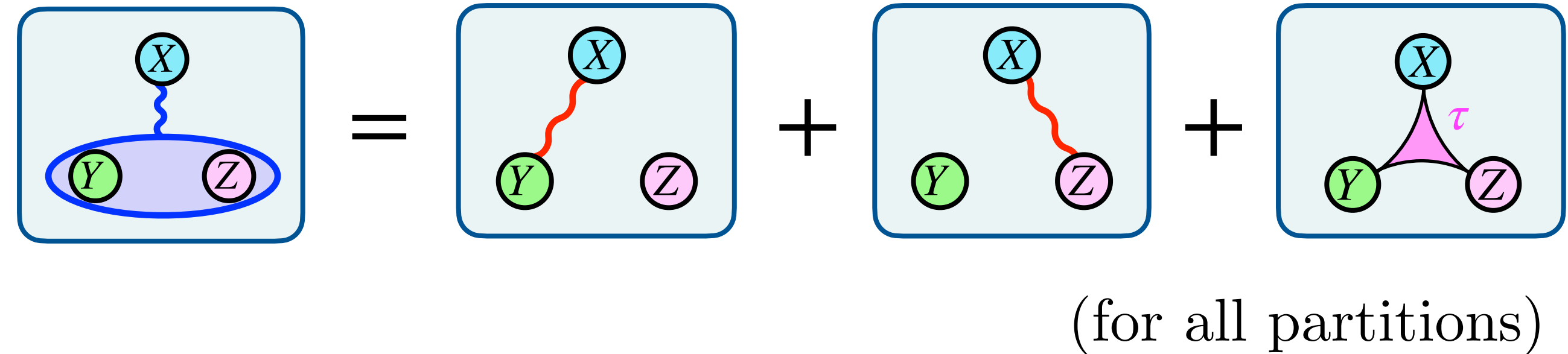
$$C_{AB} = \hat{s} \frac{k_V^2 \sin^2 \theta}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

$$C_{AB}^2 = \left(\frac{2}{\sqrt{3}} C_{V|AB} \right)^2 - \frac{1}{2} [C_{A|BV}^2 + C_{B|AV}^2]$$

$$C_{A|BV}^{(0)} = C_{B|AV}^{(0)} = \hat{s}, \quad C_{V|AB}^{(0)} = \frac{\sqrt{3}}{2} \hat{s} \frac{2m_V \sqrt{E_V^2 - k_V^2 \cos^2 \theta}}{2E_V^2 - k_V^2 (1 + \cos^2 \theta)}$$

Monogamy relation for the $2 \times 2 \times 2$ system:

[Coffman, Kundu, Wootters '99]



$$\left\{ \begin{aligned} \left(\frac{2}{\sqrt{3}} C_{V|AB} \right)^2 &= \cancel{C_{AV}^2} + \cancel{C_{BV}^2} + \cancel{\tau} \\ \frac{1}{2} C_{A|BV}^2 &= \frac{1}{2} [C_{AB}^2 + \cancel{C_{AV}^2} + \cancel{\tau}] \\ \frac{1}{2} C_{B|AV}^2 &= \frac{1}{2} [C_{AB}^2 + \cancel{C_{BV}^2} + \cancel{\tau}] \end{aligned} \right.$$

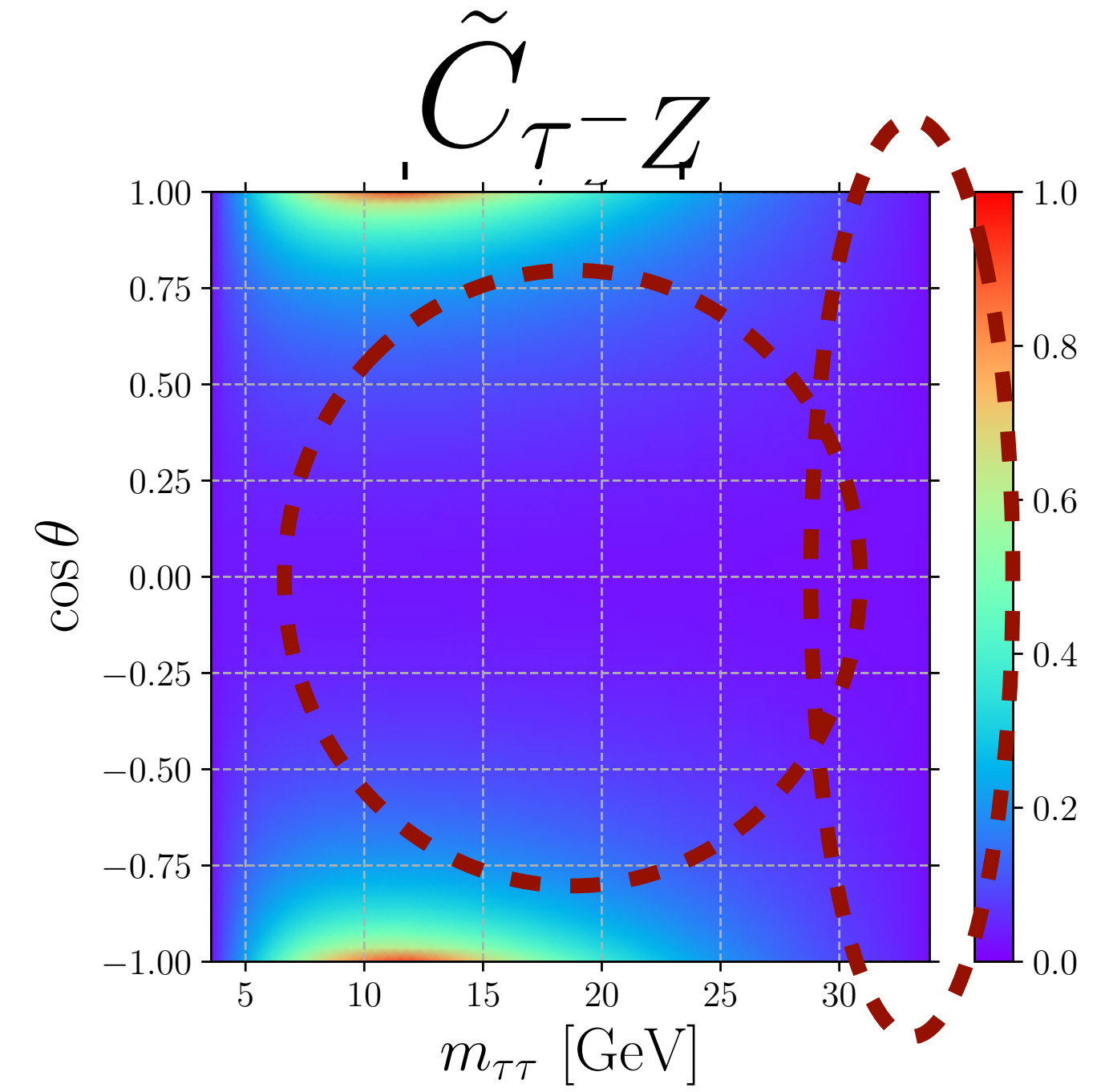
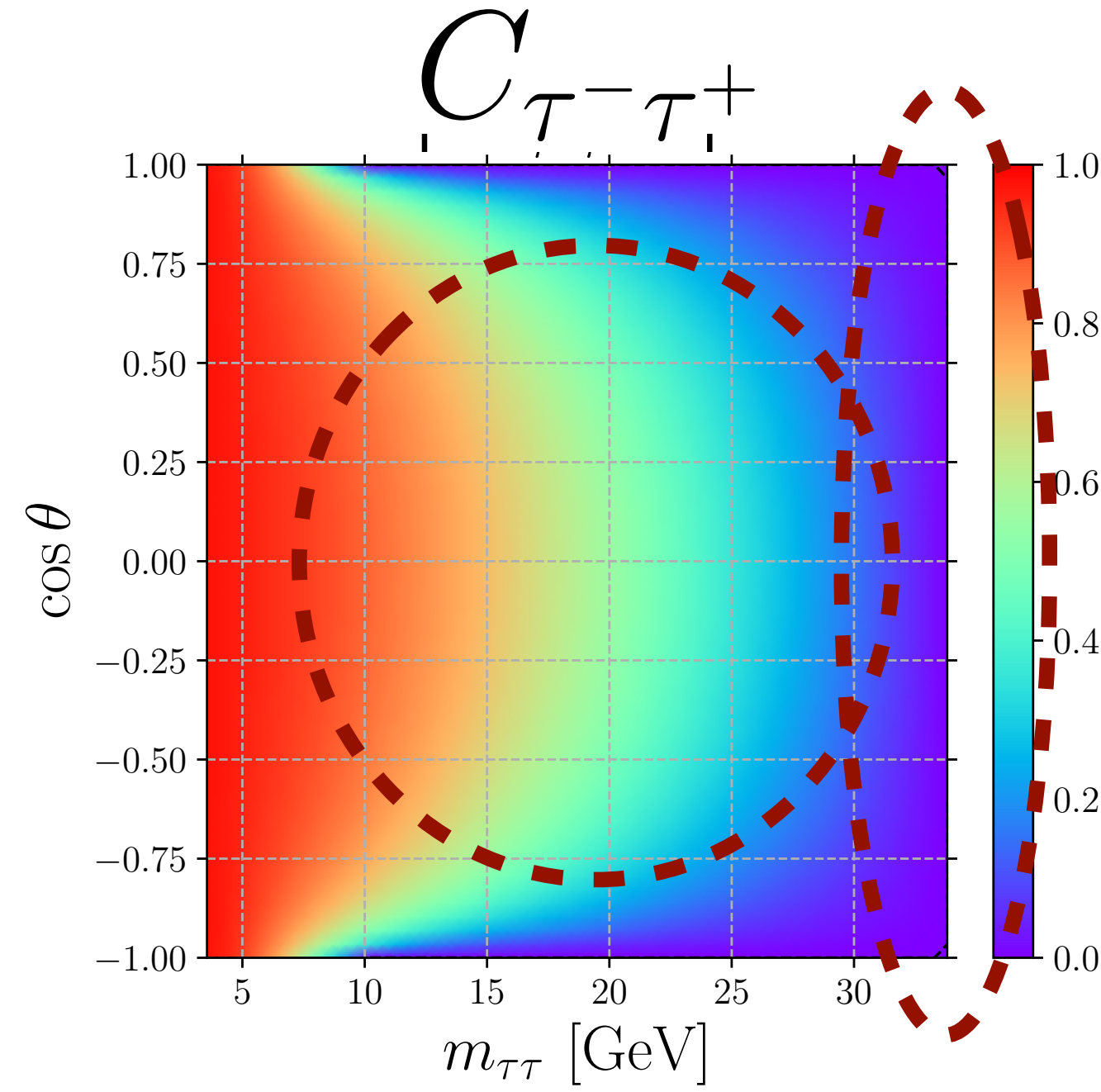
One-to-one entanglement

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One-to-one entanglement

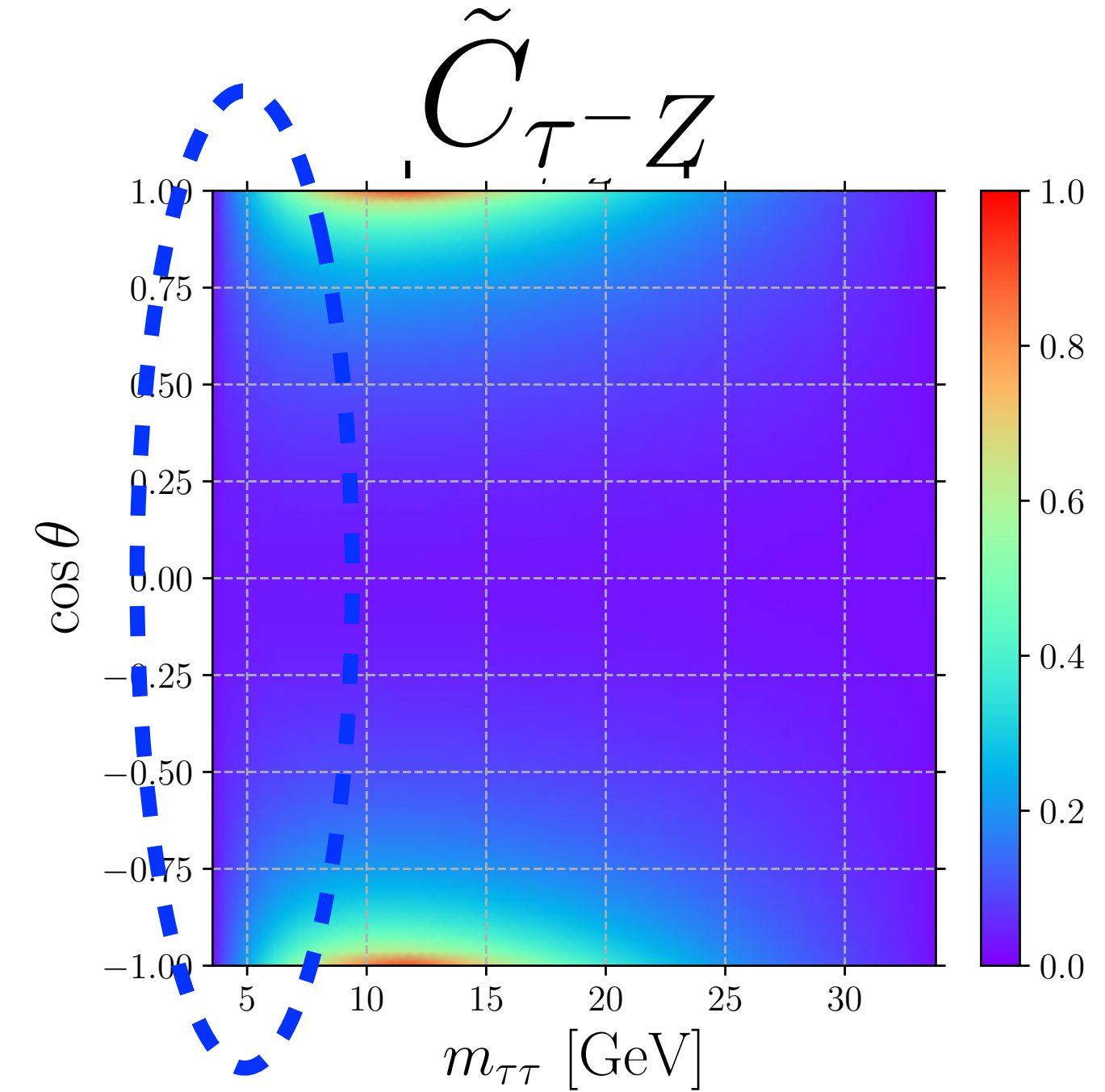
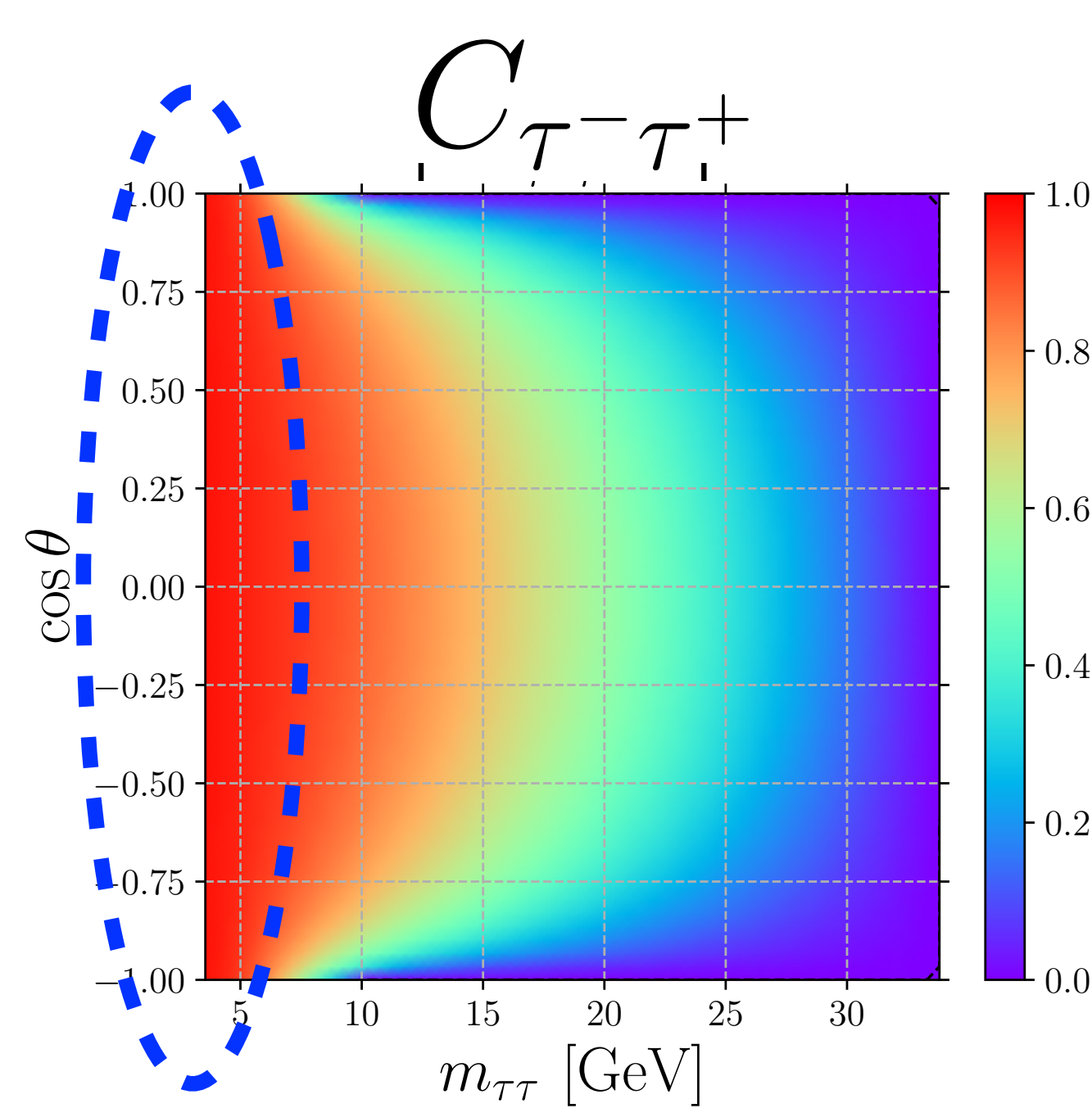
Fermion threshold

$$|\psi_{\text{thr}}\rangle = \frac{\sqrt{2}}{\mathcal{N}} \left[\begin{aligned} & - (c_V - \beta c_A) \sin \theta \, | - + \rangle \\ & - (c_V + \beta c_A) \sin \theta \, | + - \rangle \\ & + (c_A - c_V \cos \theta) \, | ++ \rangle \\ & + (c_A + c_V \cos \theta) \, | -- \rangle \end{aligned} \right] |0\rangle$$

separable AB | V

$$C_{AB} = \frac{c_A^2 (1 + \beta^2 \sin^2 \theta) - c_V^2}{c_A^2 (1 + \beta^2 \sin^2 \theta) + c_V^2}$$

$$c_V \equiv \frac{1}{2}(c_R + c_L) \simeq -0.019$$

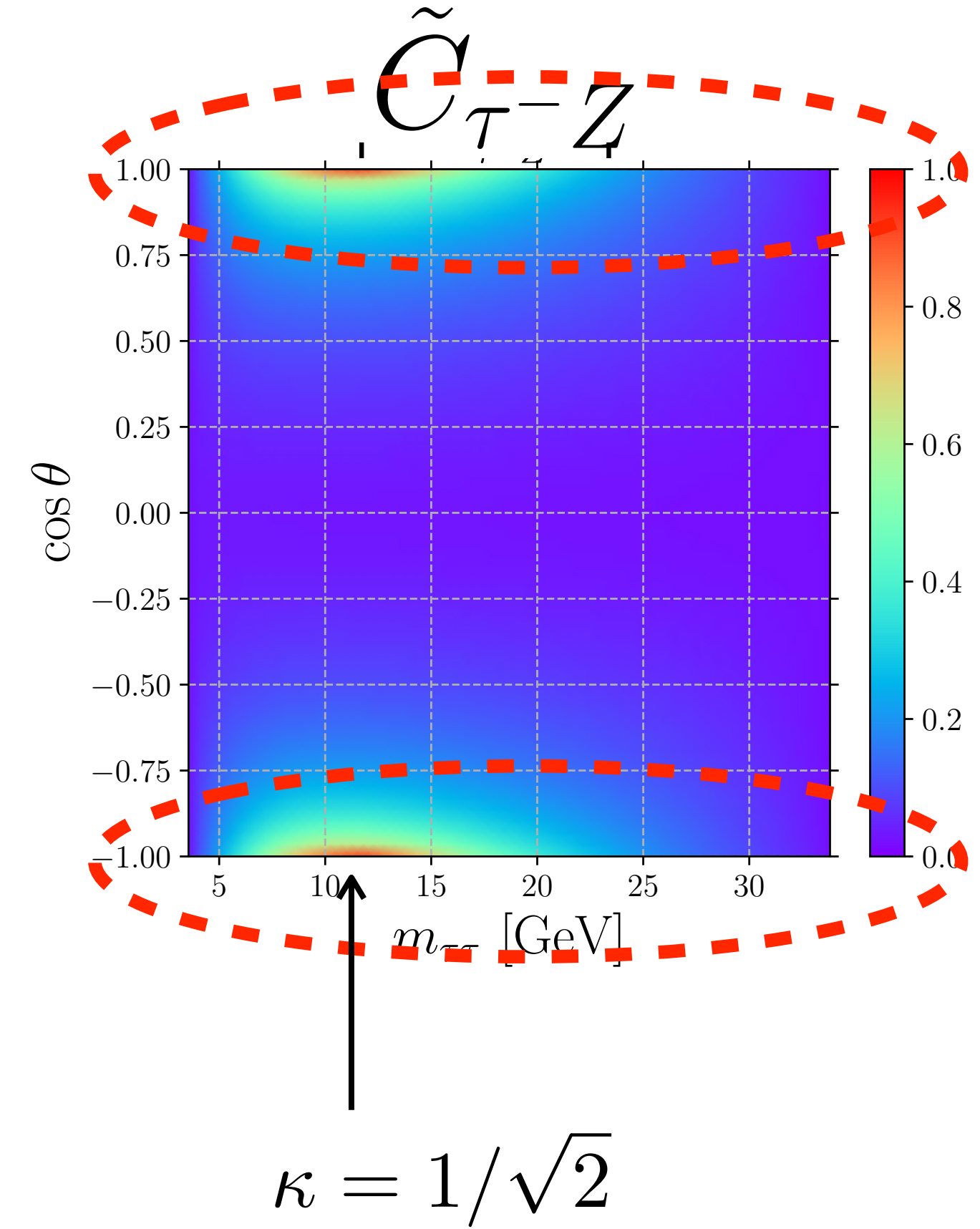
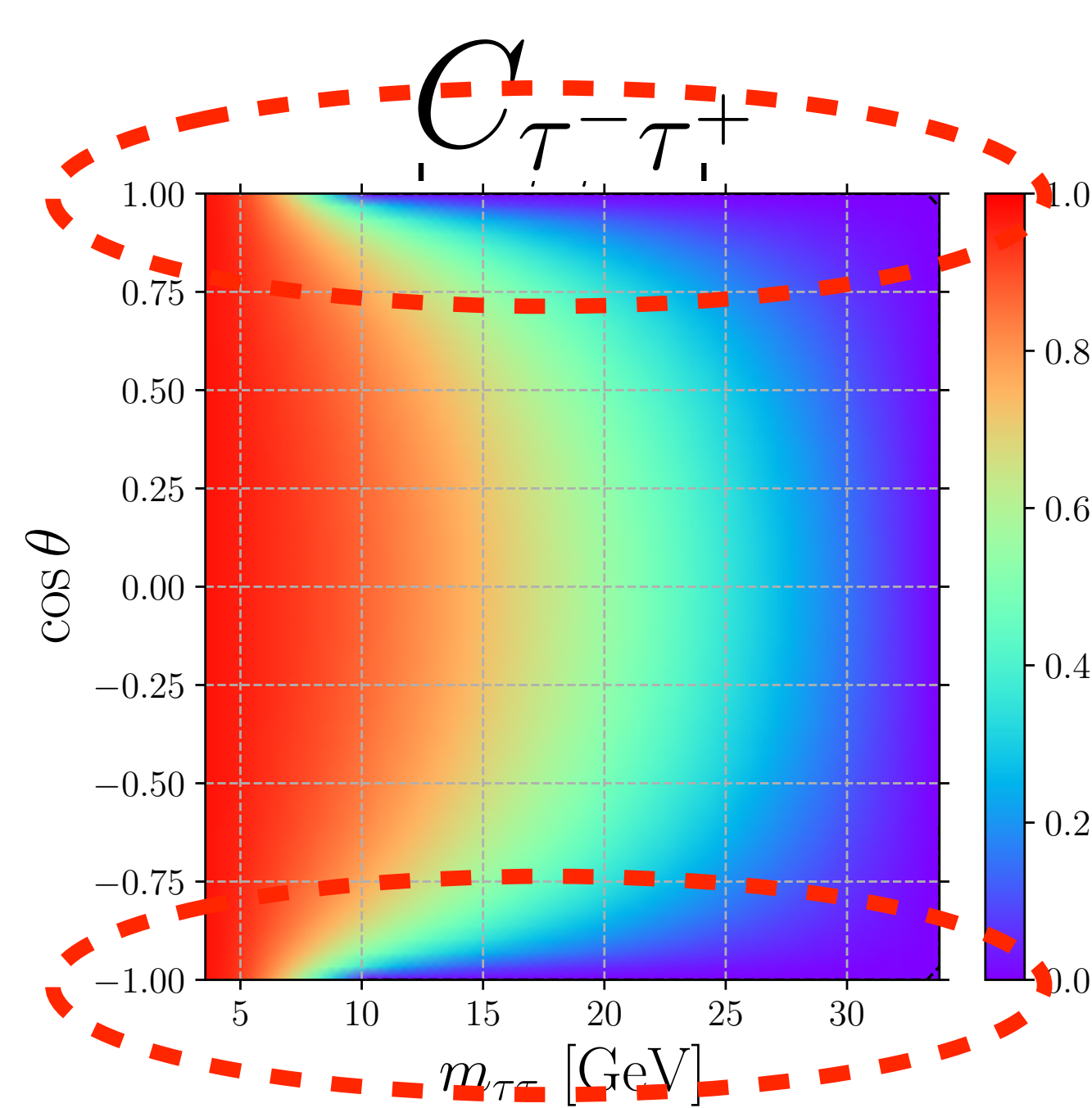
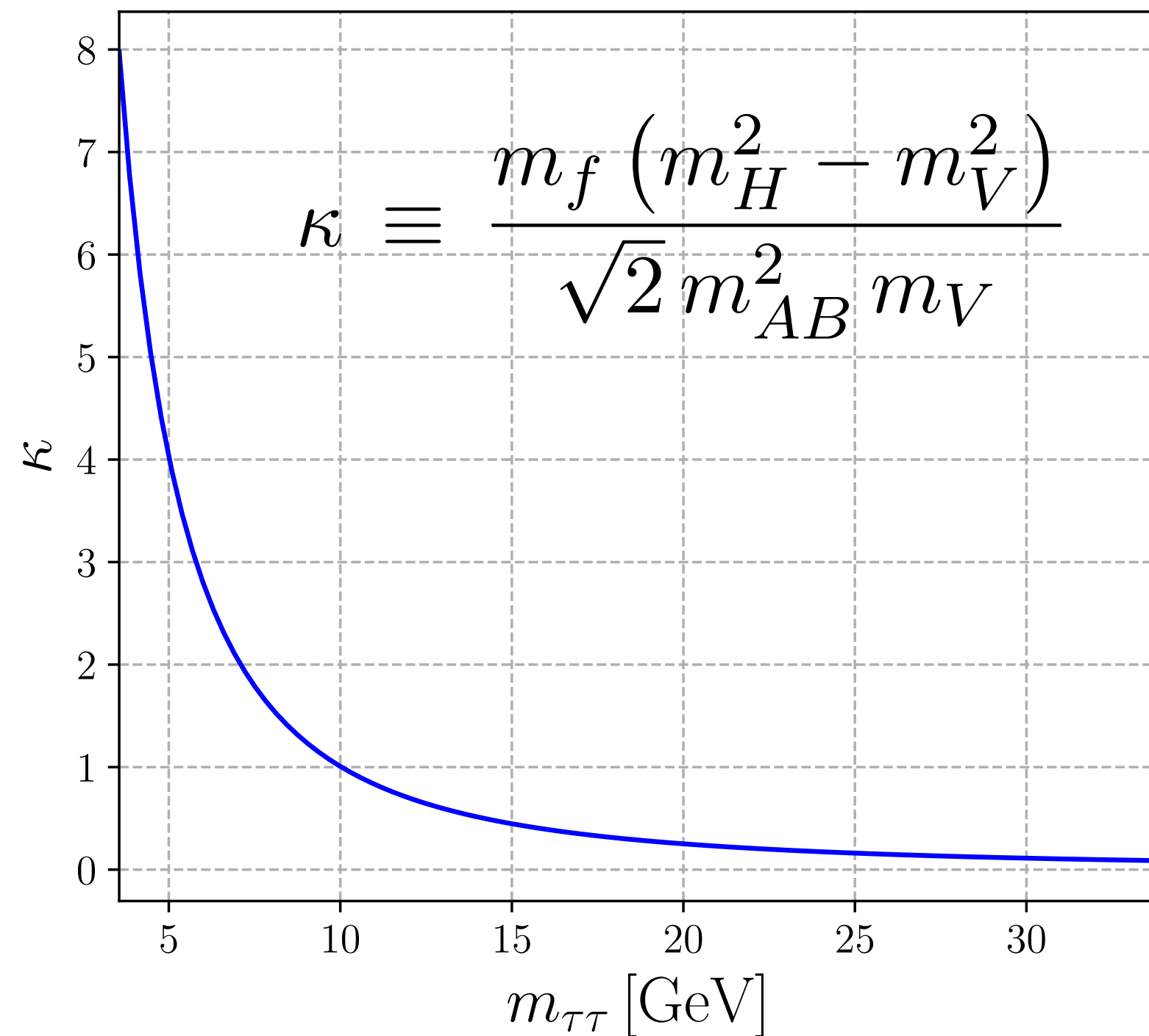


$$C_{AV} = C_{BV} = 0$$

One-to-one entanglement

$$C_{AB} = \hat{s} \max\left(0, \frac{\kappa^2 - 1}{\kappa^2 + 1}\right)$$

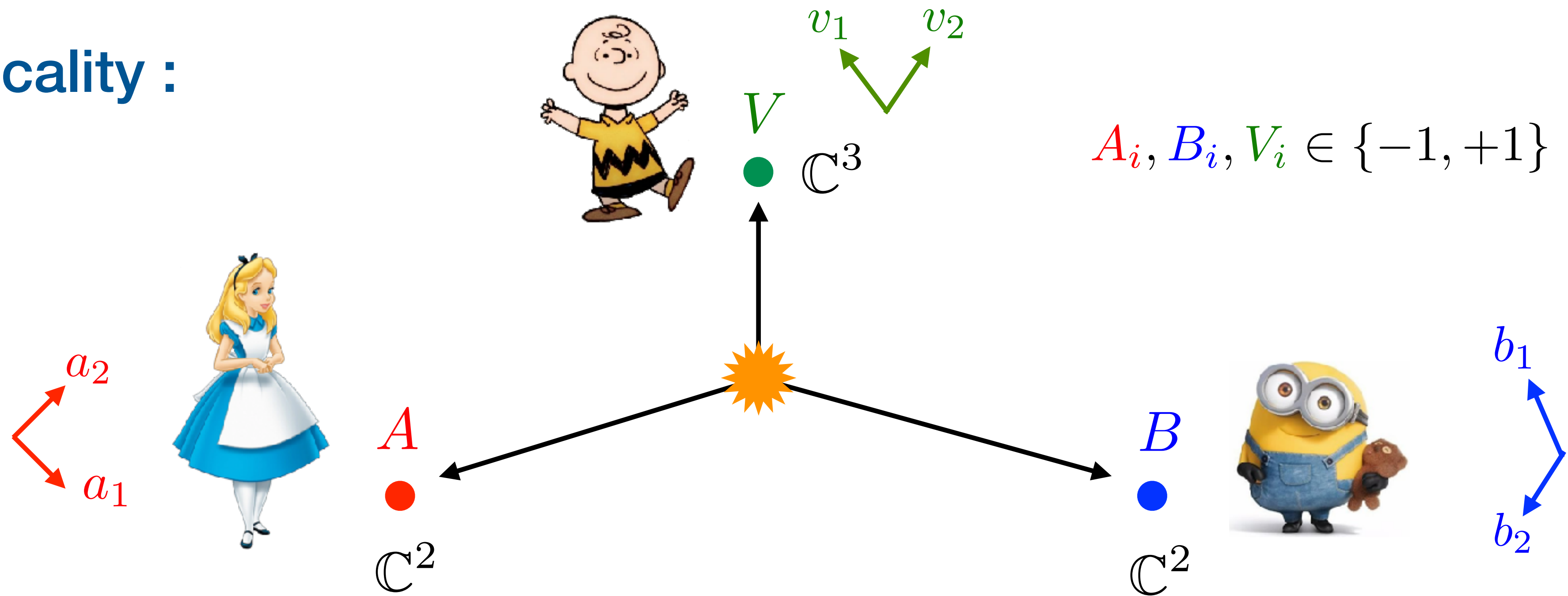
$$\tilde{C}_{AV} = \tilde{C}_{BV} = \frac{\min(2\kappa, \sqrt{1 + 2\kappa^2})}{1 + \kappa^2}$$



Collinear

$$\begin{aligned} |\psi\rangle|_{\cos\theta=+1} &\propto c_L | - + \rangle | + \rangle - c_R | + - \rangle | - \rangle + \kappa (c_R | - - \rangle - c_L | + + \rangle) | 0 \rangle \\ |\psi\rangle|_{\cos\theta=-1} &\propto c_L | - + \rangle | - \rangle - c_R | + - \rangle | + \rangle + \kappa (c_R | + + \rangle - c_L | - - \rangle) | 0 \rangle \end{aligned}$$

Bell nonlocality :



$$I_M = A_1 B_1 V_2 + A_1 B_2 V_1 + A_2 B_1 V_1 - A_2 B_2 V_2$$

[Mermin '90]

➔ Product of all four terms = $(A_1 B_1 V_1)^2 (A_2 B_2 V_2)^2$

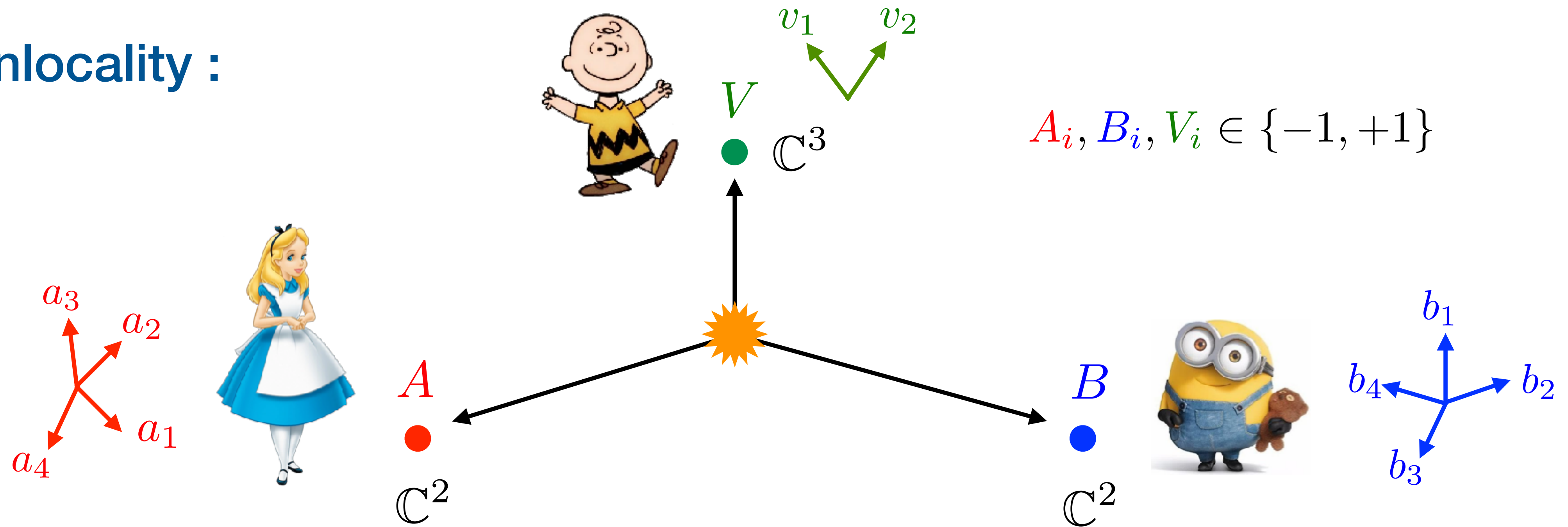
= 1 ➔ even negative terms ➔

(LHVT)

$$\langle I_M \rangle_{\text{LHVT}} \leq 2$$

$$\langle I_M \rangle_{\text{QM}} \leq 4$$

Bell nonlocality :



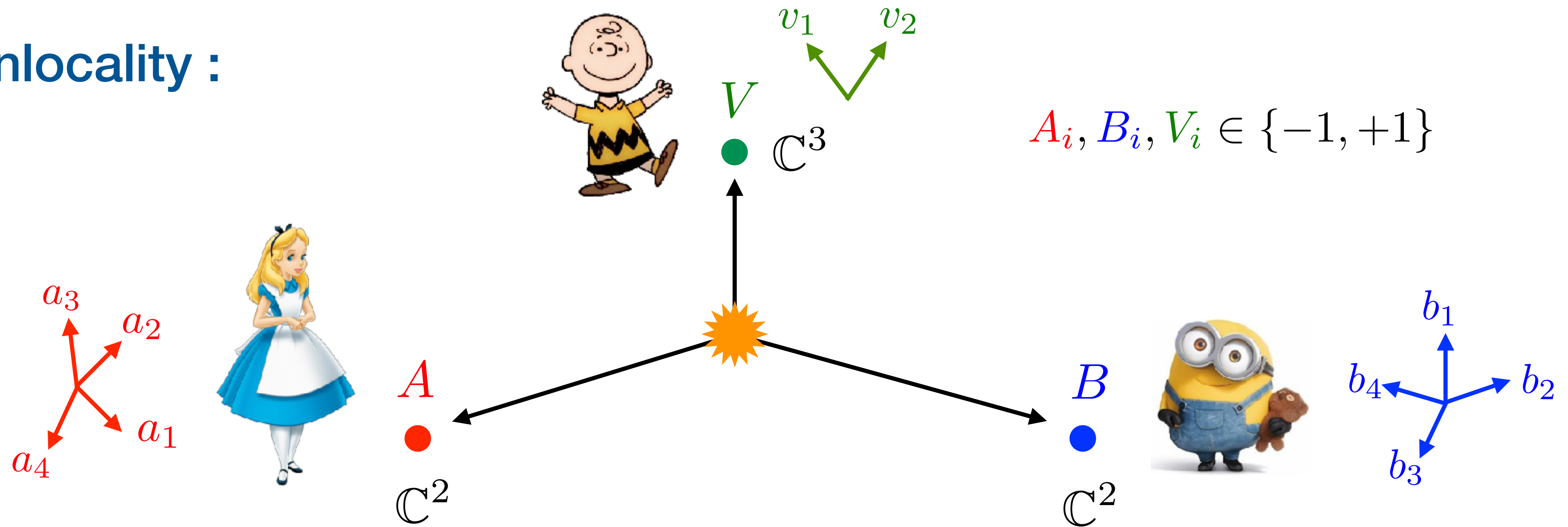
Wu, Zong (2003)

$$I_{442} = [A_1(B_1 + B_2) + A_2(B_1 - B_2)](V_1 + V_2) + [A_3(B_3 + B_4) + A_4(B_3 - B_4)](V_1 - V_2)$$

$$I'_{424} = [A_1(V_1 + V_2) + A_2(V_1 - V_2)](B_1 + B_2) + [A_3(V_3 + V_4) + A_4(V_3 - V_4)](B_1 - B_2),$$

$$I'_{244} = [B_1(V_1 + V_2) + B_2(V_1 - V_2)](A_1 + A_2) + [B_3(V_3 + V_4) + B_4(V_3 - V_4)](A_1 - A_2).$$

Bell nonlocality :



Wu, Zong (2003)

$$\begin{aligned}
 I_{442} &= [A_1(B_1 + B_2) + A_2(B_1 - B_2)](V_1 + V_2) + [A_3(B_3 + B_4) + A_4(B_3 - B_4)](V_1 - V_2) \\
 I'_{424} &= [A_1(V_1 + V_2) + A_2(V_1 - V_2)](B_1 + B_2) + [A_3(V_3 + V_4) + A_4(V_3 - V_4)](B_1 - B_2), \\
 I'_{244} &= [\underbrace{B_1(V_1 + V_2) + B_2(V_1 - V_2)}_{\pm 2}](\underbrace{A_1 + A_2}_{0}) + [\underbrace{B_3(V_3 + V_4) + B_4(V_3 - V_4)}_{\pm 2}](\underbrace{A_1 - A_2}_{0}).
 \end{aligned}$$

Bell nonlocality :

$$\langle \mathcal{I}_{442} \rangle, \langle \mathcal{I}'_{424} \rangle, \langle \mathcal{I}'_{244} \rangle \leq \begin{cases} 4 & \text{LHV, } \rho_{\text{sep}} \\ 8 & \text{QM} \\ 16 & \text{beyond QM} \end{cases}$$

Laskowski, Paterek,
Zukowski, Brukner (2004)

Wu, Zong (2003)

$$\begin{aligned} I_{442} &= [A_1(B_1 + B_2) + A_2(B_1 - B_2)](V_1 + V_2) + [A_3(B_3 + B_4) + A_4(B_3 - B_4)](V_1 - V_2) \\ I'_{424} &= [A_1(V_1 + V_2) + A_2(V_1 - V_2)](B_1 + B_2) + [A_3(V_3 + V_4) + A_4(V_3 - V_4)](B_1 - B_2), \\ I'_{244} &= [\underbrace{B_1(V_1 + V_2)}_{\pm 2} + \underbrace{B_2(V_1 - V_2)}_0] \underbrace{(A_1 + A_2)}_{\pm 2} + [B_3(V_3 + V_4) + B_4(V_3 - V_4)] \underbrace{(A_1 - A_2)}_0. \end{aligned}$$

± 2
 0

0
 ± 2

± 2
 0

0
 ± 2

Calculation in QM:

$$A_i = \vec{a}_i \cdot \vec{\sigma} \quad (i = 1, \dots, 4)$$

$$B_i = \vec{b}_i \cdot \vec{\sigma}$$

$$V_k = \pm(2|v_k\rangle\langle v_k| - \mathbb{1})$$

$$|v_k\rangle = \cos \alpha_k |+\rangle + e^{i\xi_k} \sin \alpha_k \cos \beta_k |0\rangle + e^{i\eta_k} \sin \alpha_k \sin \beta_k |-\rangle$$

$$\alpha_k, \beta_k \in [0, \frac{\pi}{2}], \quad \xi_k, \eta_k \in [0, 2\pi) \quad (k = 1, 2)$$



32 parameters for measurement settings.

Calculation in QM:

$$A_i = \vec{a}_i \cdot \vec{\sigma} \quad (i = 1, \dots, 4)$$

$$B_i = \vec{b}_i \cdot \vec{\sigma}$$

$$V_k = \pm(2|v_k\rangle\langle v_k| - \mathbb{1})$$

$$|v_k\rangle = \cos \alpha_k |+\rangle + e^{i\xi_k} \sin \alpha_k \cos \beta_k |0\rangle + e^{i\eta_k} \sin \alpha_k \sin \beta_k |-\rangle$$

$$\alpha_k, \beta_k \in [0, \frac{\pi}{2}], \quad \xi_k, \eta_k \in [0, 2\pi) \quad (k = 1, 2)$$

32 parameters for measurement settings.

Banacki, Ohta, Shaleena, KS, Horodecki, et al [2607.26122]

$$\mathcal{B}_{442} = \max_{\{\mathbf{a}, \mathbf{b}, \mathbf{v}\}} \text{Tr} [I_{442} \cdot \rho]$$

$$= 2 \max_{v_1, v_2} \left[\sum_{x=-1, +1} \sqrt{\mu_1^2(K_x) + \mu_2^2(K_x)} \right]$$

8 param.
optimization

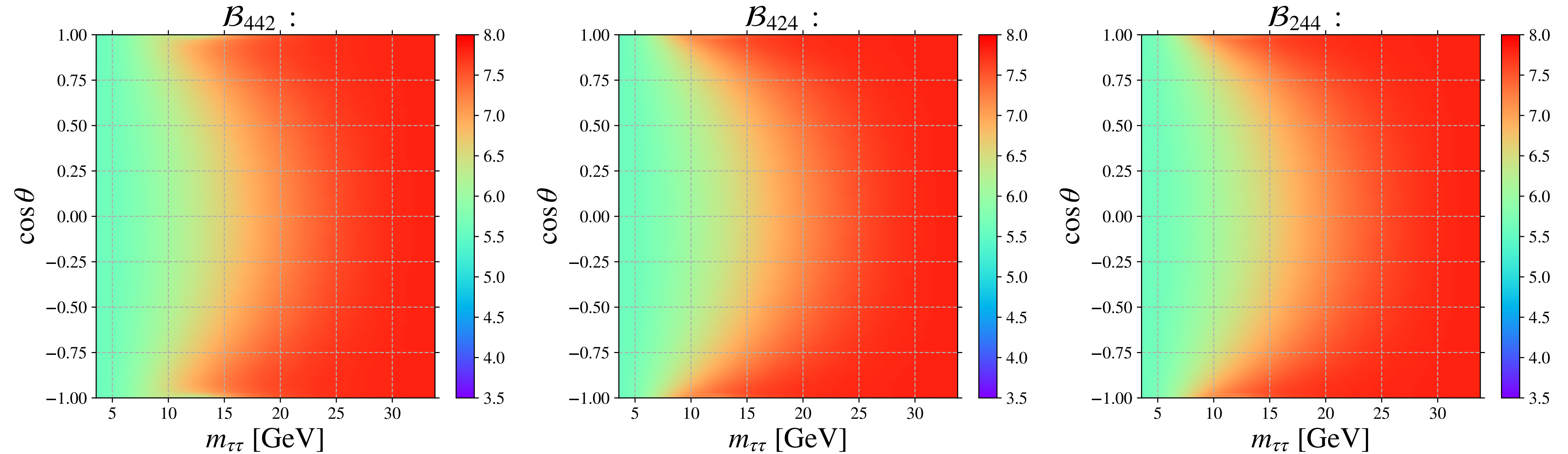
analytically optimising $\{\mathbf{a}, \mathbf{b}\}$

$$[K_{\pm}]_{ij} = \text{Tr}[(\sigma_i \otimes \sigma_j \otimes (V_1 \pm V_2)) \cdot \rho]$$

μ_1, μ_2 : the two largest eigenvalues

of $\sqrt{K_x^\dagger K_x}$

Bell nonlocality result:



- LHVT bound 4 is violated in all regions of phase space.
- Analytical expressions in some corners:

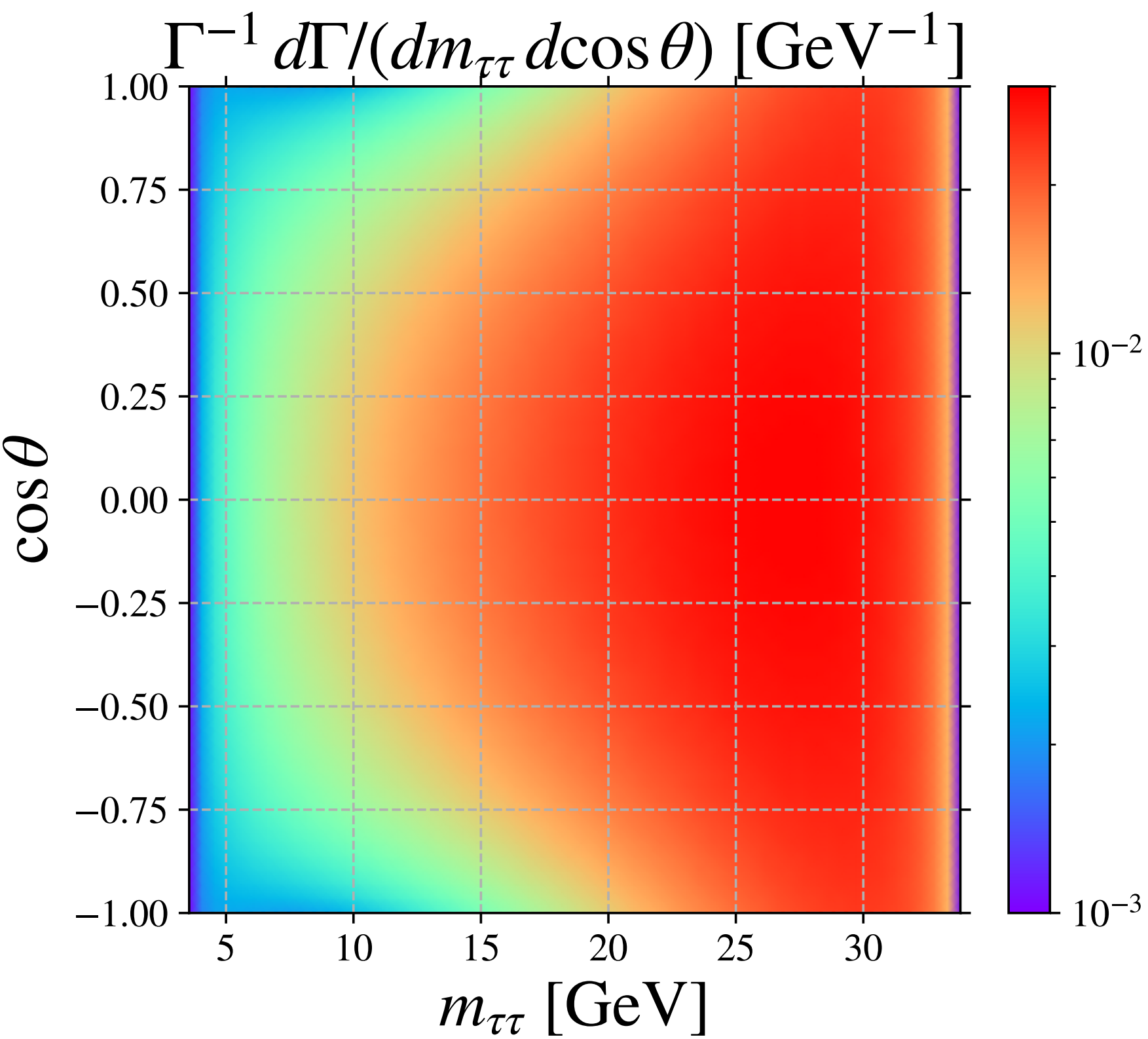
$$\hat{s} \equiv \frac{2|c_L c_R|}{c_L^2 + c_R^2} \simeq 0.988$$

Fermion threshold: $\mathcal{B}_{442}, \mathcal{B}'_{424}, \mathcal{B}'_{244} = 4\sqrt{1 + C_{AB}^2} \simeq 4\sqrt{2} \simeq 5.6$

Gauge-boson thres: $\mathcal{B}_{442} = \max \left[8\hat{s}, 4\sqrt{1 + \hat{s}^2} \right] \quad \mathcal{B}'_{424}, \mathcal{B}'_{244}, = \max \left[8\hat{s}, 4\sqrt{1 + 2\hat{s}^2} \right]$

Observability at future colliders:

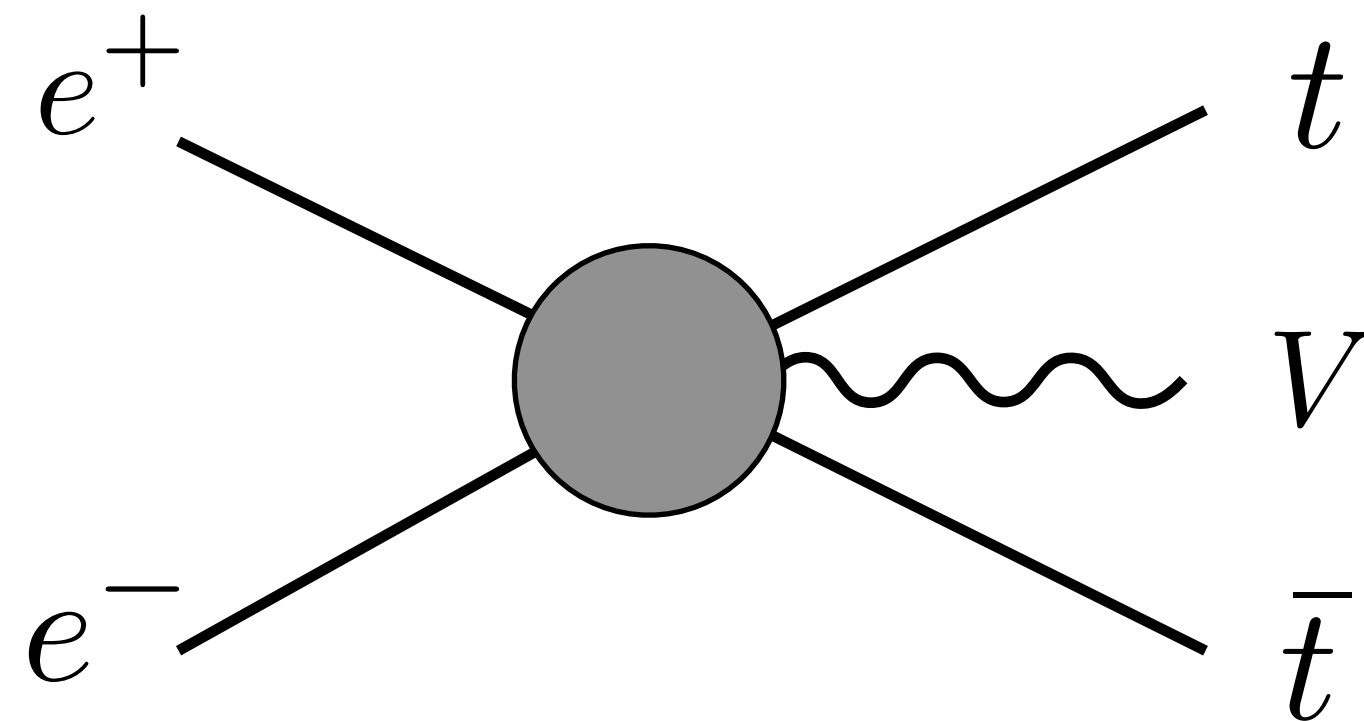
- QIS measurements seem promissing both at **HL-LHC** and **FCC-ee**.
- Dedicated studies are needed for conclusion



$$\text{BR}(h \rightarrow \tau^- \tau^+ Z) \approx \text{BR}(h \rightarrow ZZ^*) \times \text{BR}(Z^* \rightarrow \tau\tau) \sim 10^{-3}$$

Collider	$\mathcal{L}$ [ab ⁻¹]	Production	Z decay	Events [$h \rightarrow \tau^- \tau^+ Z$]
HL-LHC [13.6 TeV]	3	$\sigma(pp \rightarrow h)$ $\simeq 55$ pb	$\text{Br}(Z \rightarrow e^+ e^- / \mu^+ \mu^-)$ $\simeq 0.067$	$\mathcal{O}(10^4)$
FCC-ee [240 GeV]	5–10	$\sigma(e^+ e^- \rightarrow Zh)$ $\simeq 0.20$ pb	$Z \rightarrow \text{any}$	$\mathcal{O}(10^3)$

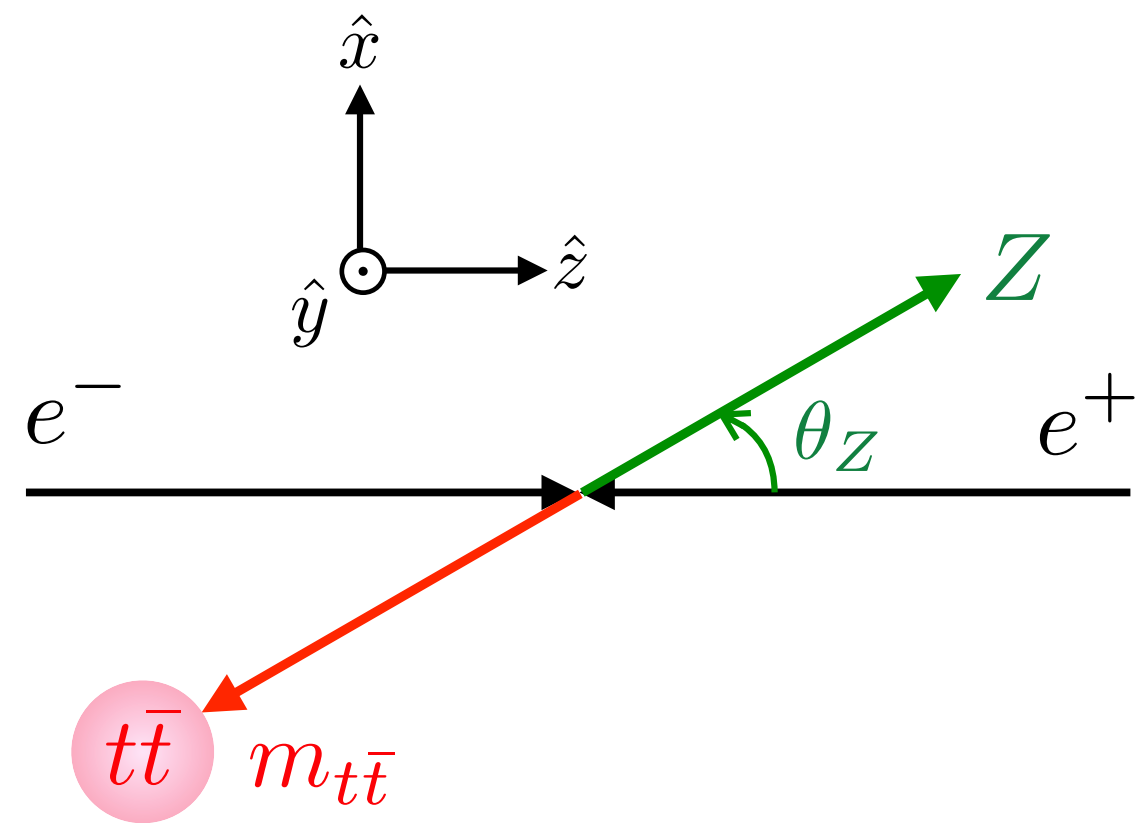
$$e^+e^- \rightarrow t\bar{t}Z$$



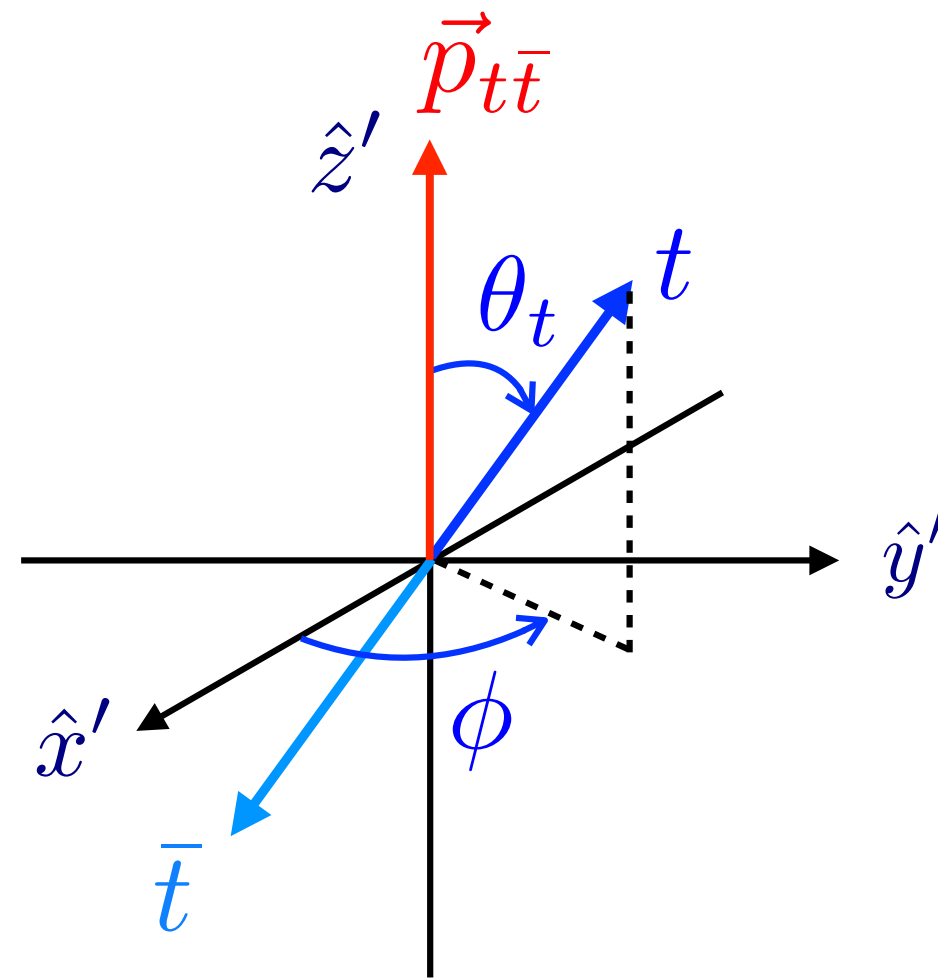
$$22 \rightarrow 223$$

Kinematics:

lab-frame

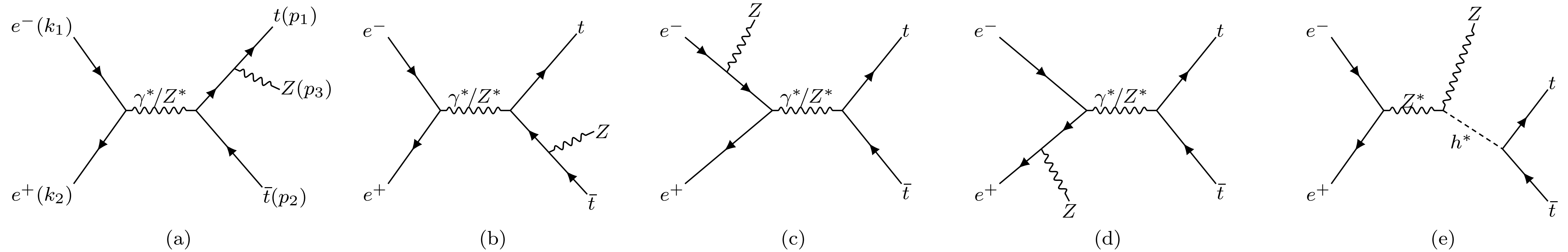


$t\bar{t}$ rest frame



- $m_{t\bar{t}}$: $t\bar{t}$ invariant mass, $m_{t\bar{t}} \in [2m_t, \sqrt{s} - m_Z]$;
- θ_Z : polar angle of Z in the lab frame, $\cos \theta_Z \in [-1, 1]$;
- θ_t : polar angle of t in the $t\bar{t}$ rest frame, $\cos \theta_t \in [-1, 1]$;
- ϕ : azimuthal angle of t in the $t\bar{t}$ rest frame, $\phi \in [0, 2\pi)$.

$$O(\textcolor{red}{t}, \textcolor{blue}{\bar{t}}, \textcolor{green}{Z}; \textcolor{violet}{\cos \theta_Z}, \textcolor{violet}{\cos \theta_t}, \textcolor{violet}{\phi}) \stackrel{\text{CP}}{=} O(\textcolor{blue}{\bar{t}}, \textcolor{red}{t}, \textcolor{green}{Z}; -\textcolor{violet}{\cos \theta_Z}, -\textcolor{violet}{\cos \theta_t}, -\textcolor{violet}{\phi})$$



Helicity amplitudes:

$$\mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} = \sum_{I=a, \dots, e} \mathcal{M}^{(I)} [\bar{u}(k_1, \lambda_1), v(k_2, \lambda_2), u(p_1, h_1), \bar{v}(p_2, h_2), \epsilon(p_3, h_3)]$$

Spin state:
(unnormalised)

$$|\tilde{\psi}_{\lambda_1, \lambda_2}\rangle = \sum_{h_1, h_2, h_3} \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} |h_1, h_2, h_3\rangle$$

Density matrix:
(unnormalised)

$$R_{(h_1, h_2, h_3), (h'_1, h'_2, h'_3)}^{\lambda_1, \lambda_2} = \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} [\mathcal{M}_{h'_1, h'_2, h'_3}^{\lambda_1, \lambda_2}]^* \quad \leftarrow \text{pure}$$

Differential cross section:

$$\frac{d\sigma_{h_1, h_2, h_3}^{\lambda_1, \lambda_2}}{d\Phi} = \frac{1}{2s} R_{(h_1, h_2, h_3), (h_1, h_2, h_3)}^{\lambda_1, \lambda_2}$$

The source of mixing:

Imperfect polarisation:

$$\rho_{e^\pm} = \frac{1}{2}(1 + \mathcal{P}_{e^\pm})|+\rangle\langle+|_{e^\pm} + \frac{1}{2}(1 - \mathcal{P}_{e^\pm})|-\rangle\langle-|_{e^\pm} \quad \mathcal{P}_{e^\pm} \in [-1, 1]$$

$t\bar{t}Z$ state:
(initial spin averaging)
(mixed)

$$R[\mathcal{P}_{e^-}, \mathcal{P}_{e^+}] = \sum_{\lambda_1, \lambda_2} P_{\lambda_1, \lambda_2} R^{\lambda_1, \lambda_2}, \quad P_{\lambda_1, \lambda_2} = \frac{1}{4}(1 + \lambda_1 \mathcal{P}_{e^-})(1 + \lambda_2 \mathcal{P}_{e^+})$$

$t\bar{t}Z$ state:
(phase-space integ.)
(mixed)

$$R_\Sigma = \int_\Sigma d\sigma \rho = \int_\Sigma \frac{d\Phi}{2s} R$$

← The spin quantization axes (helicity basis) are different from event to event.
 R_Σ is a *fictitious* state,

Afik, de Nova [2003.02280],
Cheng, Han, Low [2311.09166]

Normalised state:

$$\rho_\Sigma = \frac{R_\Sigma}{\text{Tr} R_\Sigma}$$

Separability classes:

$$q_i \geq 0, \sum_i q_i = 1$$

Fully separable:

$$\rho^{\text{fs}} = \sum_i q_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C$$

M -separable:

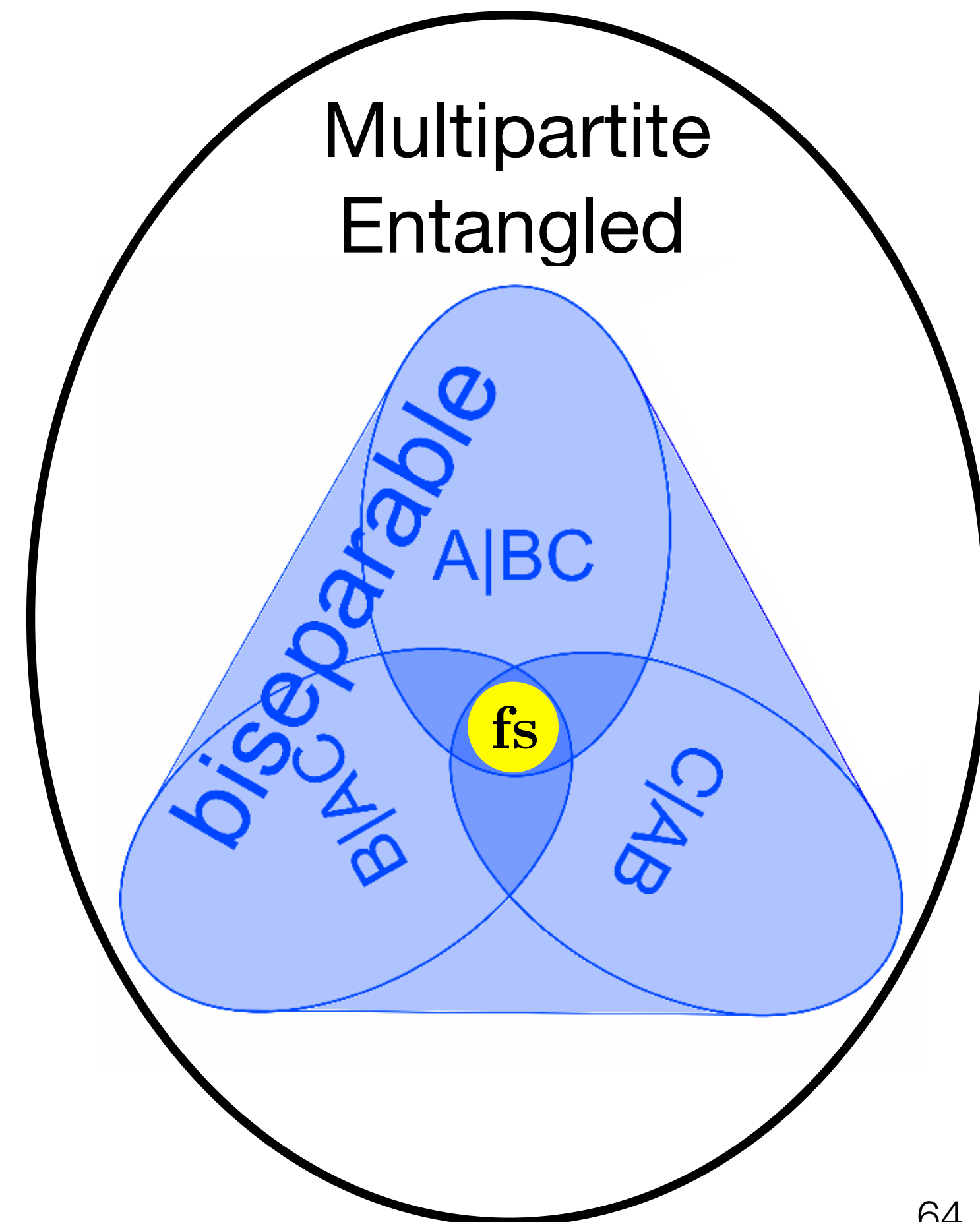
$$\rho^{M|\overline{M}} = \sum_i q_i \rho_i^M \otimes \rho_i^{\overline{M}}$$

bi-separable:

$$\rho^{\text{bs}} = q_1 \rho^{A|BC} + q_2 \rho^{B|AC} + q_3 \rho^{C|AB}$$

Genuine Multipartite
Entangled (GME):

$$\rho^{\text{GME}} \neq \rho^{\text{bs}}$$



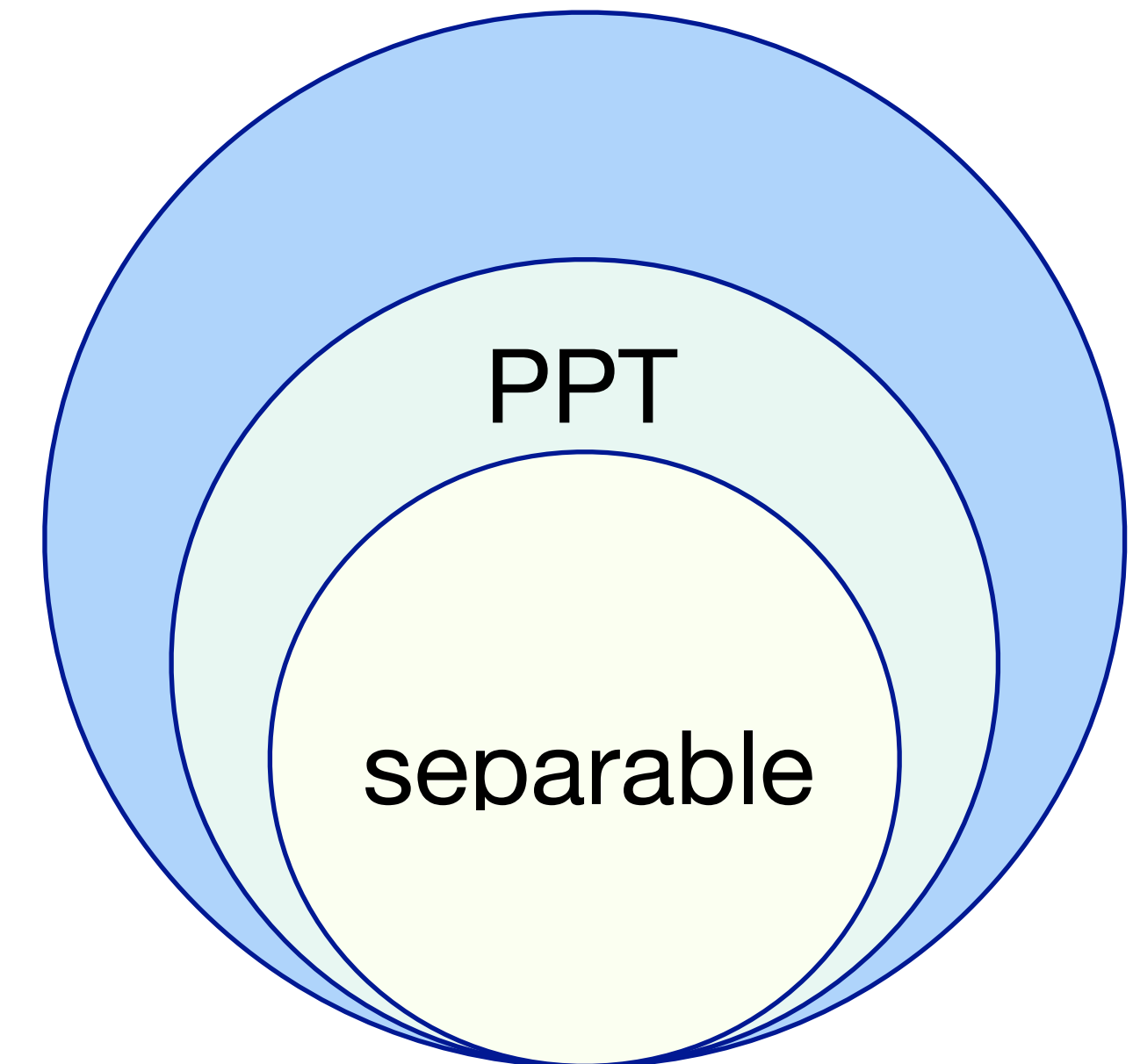
- If state is separable, $\rho^{T_B} = \sum_i p_i \rho_i^A \otimes (\rho_i^B)^T$ is Hermitian, Unit-Trace, PSD

If one finds a **negative eigenvalue** in ρ^{T_B} , the state is guaranteed to be **entangled**.

$\Rightarrow$ **necessary** condition, it is also **sufficient** for $(d_A, d_B) = (2,2), (2,3), (3,2)$.

⌘ If $\rho^{T_B} \geq 0$, the state ρ is called
“**Positive-Partial-Transpose (PPT)**”.

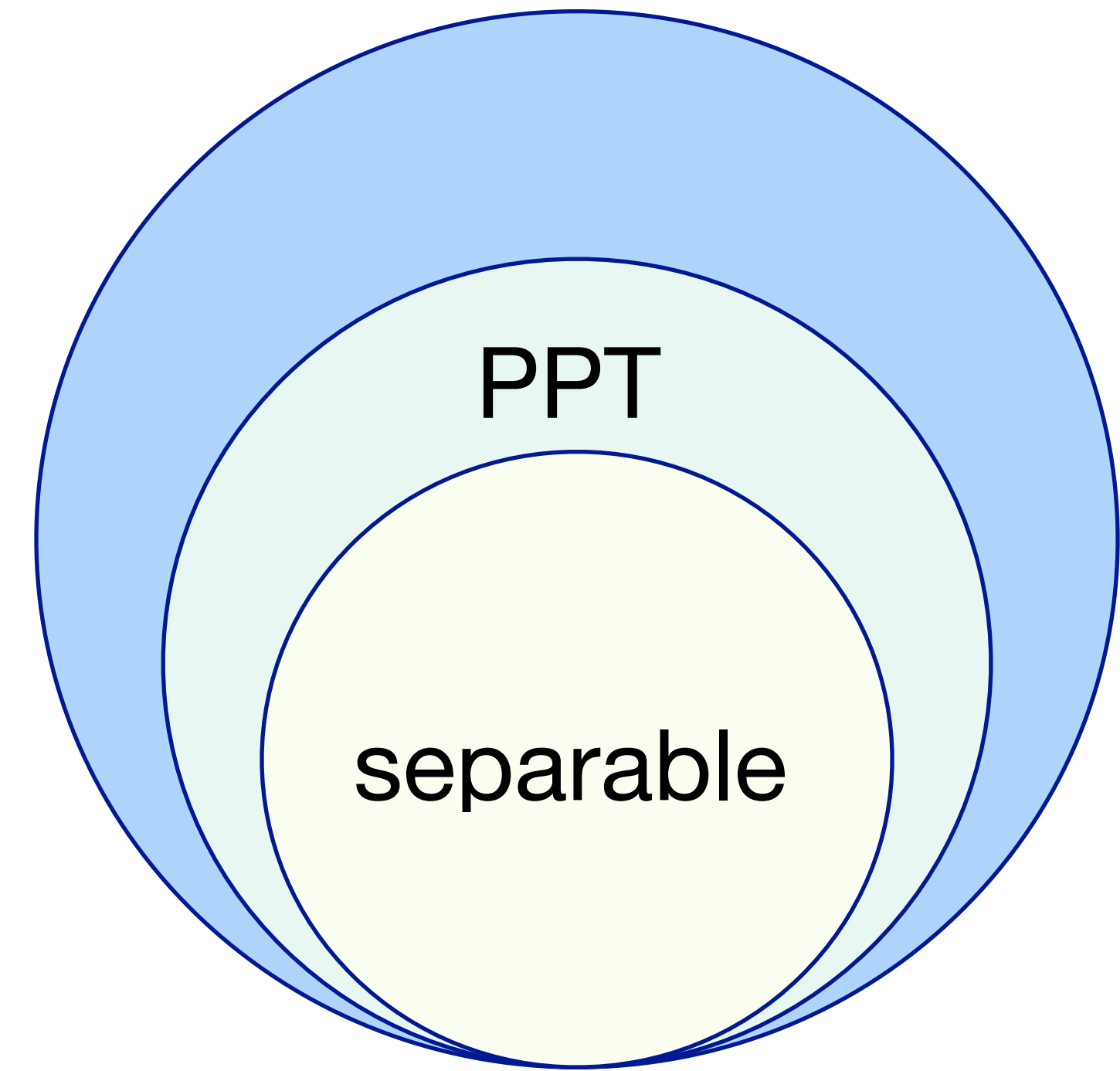
⌘ **All separable states are PPT.**



Negativity:

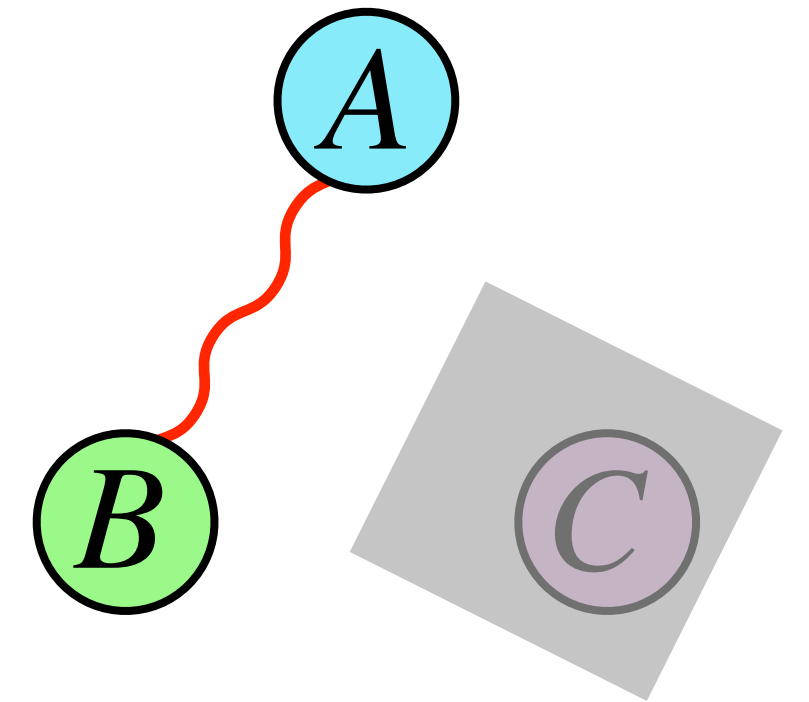
$$N(\rho) = \frac{\|\rho^{T_B}\|_1 - 1}{2} = \sum_{\mu: \lambda_\mu < 0} |\lambda_\mu|$$

λ_μ are the eigenvalues of ρ^{T_B}



⌘ Negativity is a **faithful** ENT measure only for **2 x 2** and **2 x 3** systems.

One-to-one entanglement



Reduced states:

$$\rho_{t\bar{t}} = \text{Tr}_Z[\rho] \quad (2 \times 2),$$

$$\rho_{tZ} = \text{Tr}_{\bar{t}}[\rho] \quad (2 \times 3),$$

$$\rho_{\bar{t}Z} = \text{Tr}_t[\rho] \quad (2 \times 3),$$

Negativities:

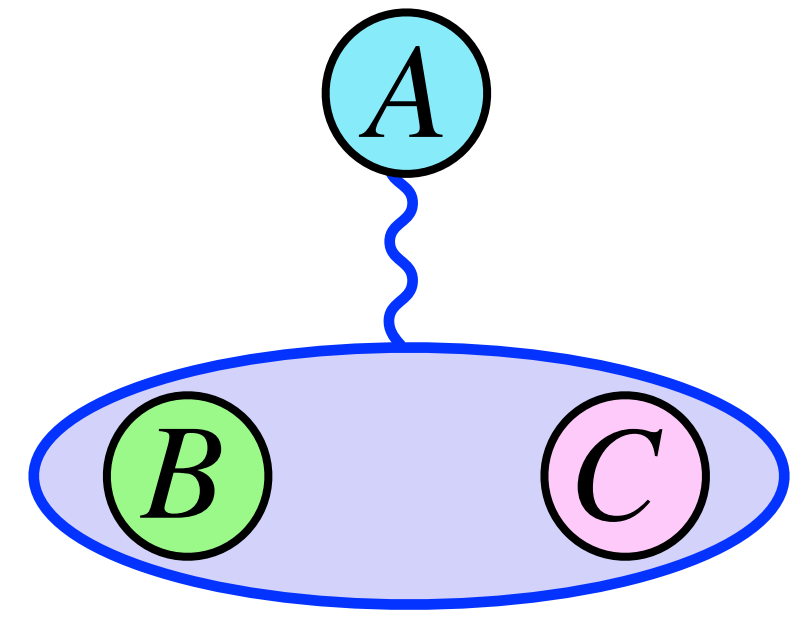
$$N_{t\bar{t}} \equiv N_t(\rho_{t\bar{t}}),$$

$$N_{tZ} \equiv N_Z(\rho_{tZ}),$$

$$N_{\bar{t}Z} \equiv N_Z(\rho_{\bar{t}Z}).$$

$$\text{(separable)} \quad 0 \leq N_{t\bar{t}}, N_{tZ}, N_{\bar{t}Z} \leq \frac{1}{2} \quad \text{(maximally entangled)}$$

One-to-other entanglement



Negativities:

$$N(Z|t\bar{t}) \equiv N_Z(\rho) \quad (3 \times 4), \quad N(t|\bar{t}Z) \equiv N_t(\rho) \quad (2 \times 6), \quad N(\bar{t}|tZ) \equiv N_{\bar{t}}(\rho) \quad (2 \times 6).$$

$$\begin{array}{ll} \text{(PPT)} & 0 \leq N(Z|t\bar{t}) \leq 1 \\ & 0 \leq N(t|\bar{t}Z), N(\bar{t}|tZ) \leq \frac{1}{2} \end{array} \quad \text{(maximally entangled)}$$

GME certification Jungnitsch, Moroder, Gühne [1010.6049]

- GME state cannot be a PPT mixture:

$$\rho^{\text{PPT-mix}} = p_A \rho_A^{\text{PPT}} + p_B \rho_B^{\text{PPT}} + p_C \rho_C^{\text{PPT}}$$

- Consider the *fully decomposable* operator

$$\mathcal{W} = P_M + Q_M^{T_M} \quad P_M \geq 0, \quad 0 \leq Q_M \leq 1$$

(for simultaneously all M)

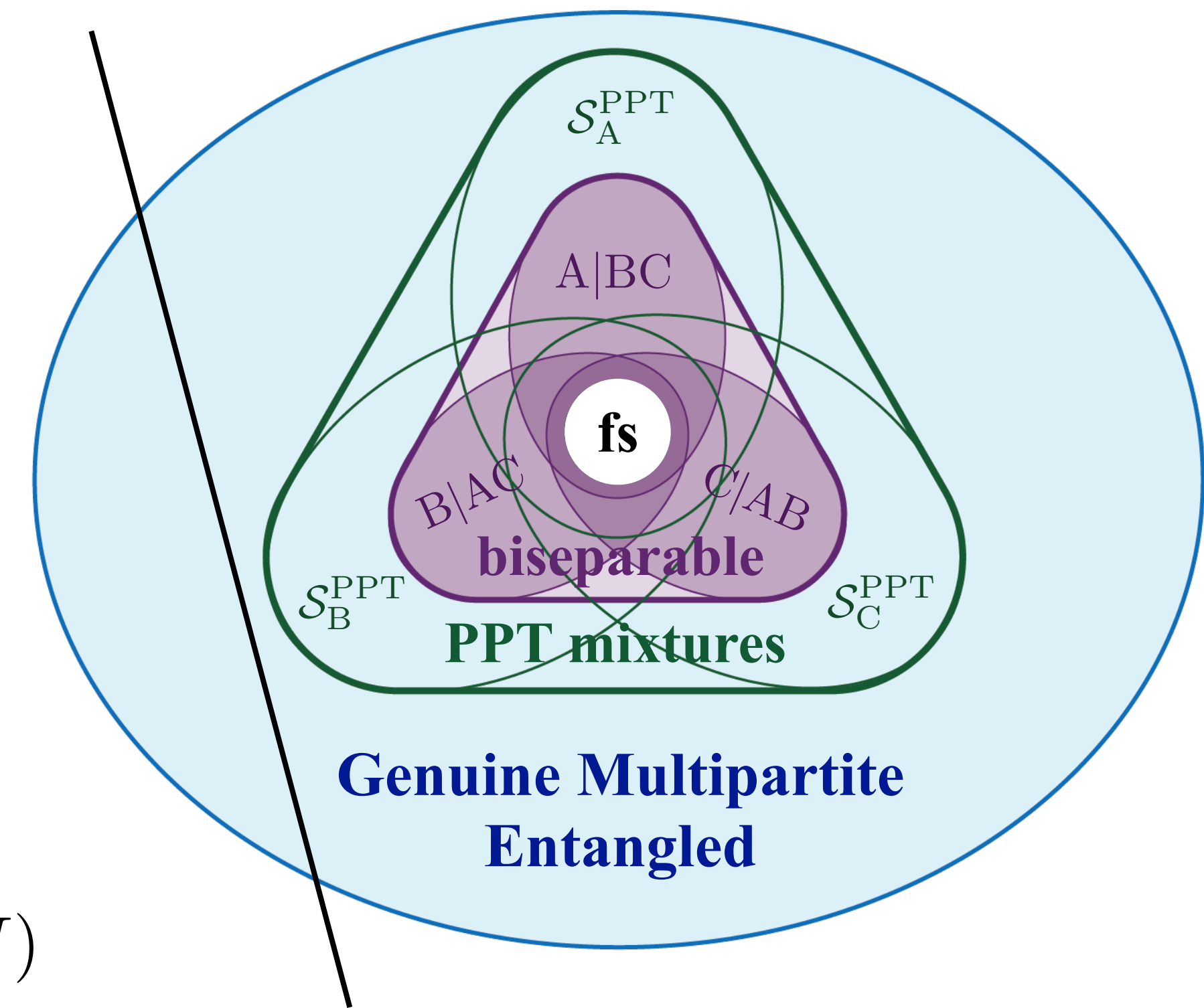
- For all M-PPT states,

$$\text{Tr}[\rho_M^{\text{PPT}} \mathcal{W}] = \text{Tr}[\rho_M^{\text{PPT}} P_M] + \text{Tr}[(\rho_M^{\text{PPT}})^{T_M} Q_M] \geq 0$$

- For all PPT mixtures,

$$\text{Tr}[\rho^{\text{PPT-mix}} \mathcal{W}] = p_A \text{Tr}[\rho_A^{\text{PPT}} \mathcal{W}] + p_B \text{Tr}[\rho_B^{\text{PPT}} \mathcal{W}] + p_C \text{Tr}[\rho_C^{\text{PPT}} \mathcal{W}] \geq 0$$

$\text{Tr}[\rho \mathcal{W}] < 0 \implies \rho \text{ is GME}$



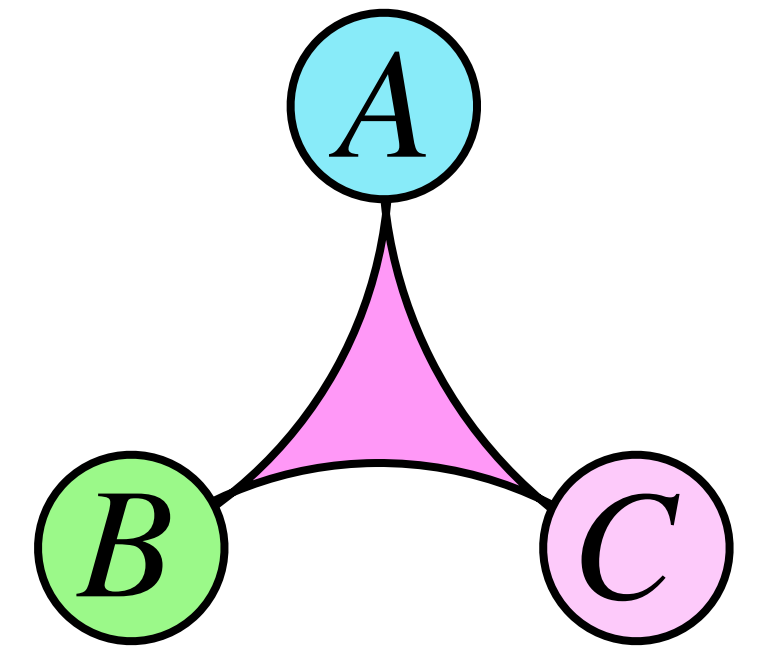
$$\mathcal{W} = P_M + Q_M^{T_M}$$

GME measure

- Genuine Multipartite Negativity (GMN) is defined as

$$\mathcal{N}_G(\rho) \equiv - \min_{\{P_M, Q_M\}} \text{Tr}[\rho \mathcal{W}]$$

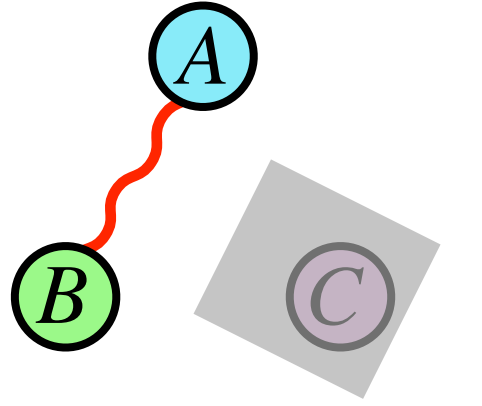
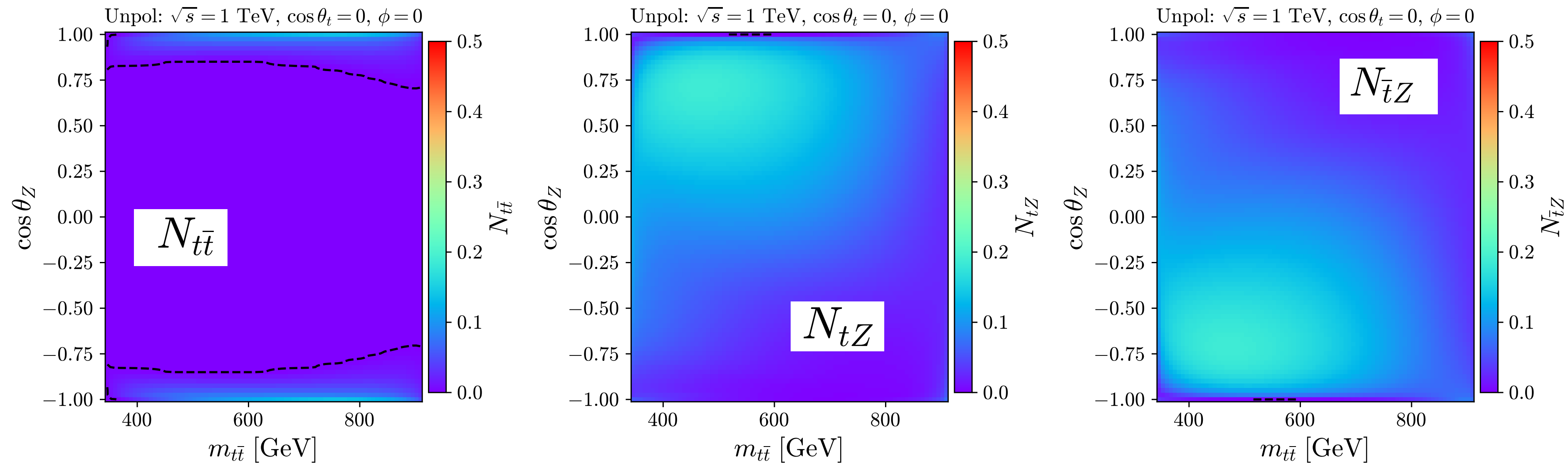
Jungnitsch, Moroder,
Gühne [1010.6049]



- GMN (non-negative function) is a GME measure:
 - ◆ $\mathcal{N}_G(\rho) = 0$ iff ρ is PPT-mix (which includes all bi-separable states)
 - ◆ non-increasing under LOCC
- The general upper bound: $\mathcal{N}_G(\rho) \leq \min \left[N(Z|t\bar{t}), N(t|\bar{t}Z), N(\bar{t}|tZ) \right]$

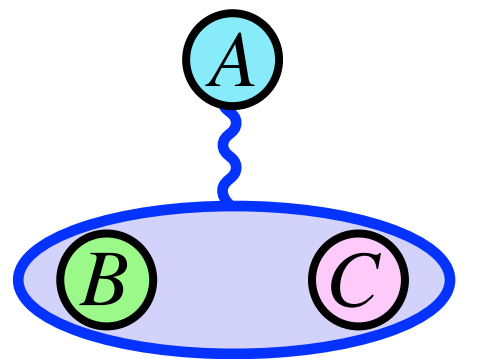
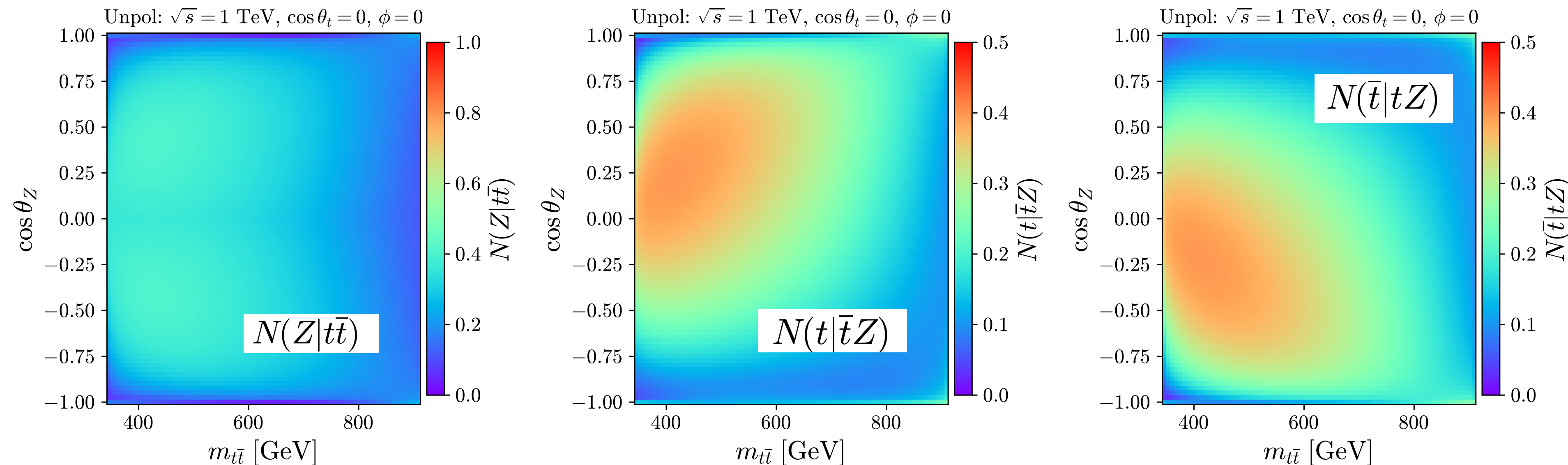
$$\text{(PPT-mixture)} \quad 0 \leq \mathcal{N}_G(\rho) \leq \frac{1}{2} \quad \text{(maximally GME)}$$

One-to-one @ phase-space points: $\cos \theta_t = 0, \phi = 0$



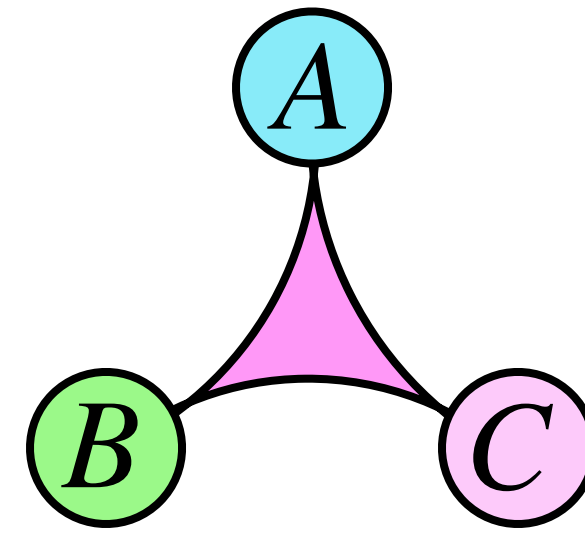
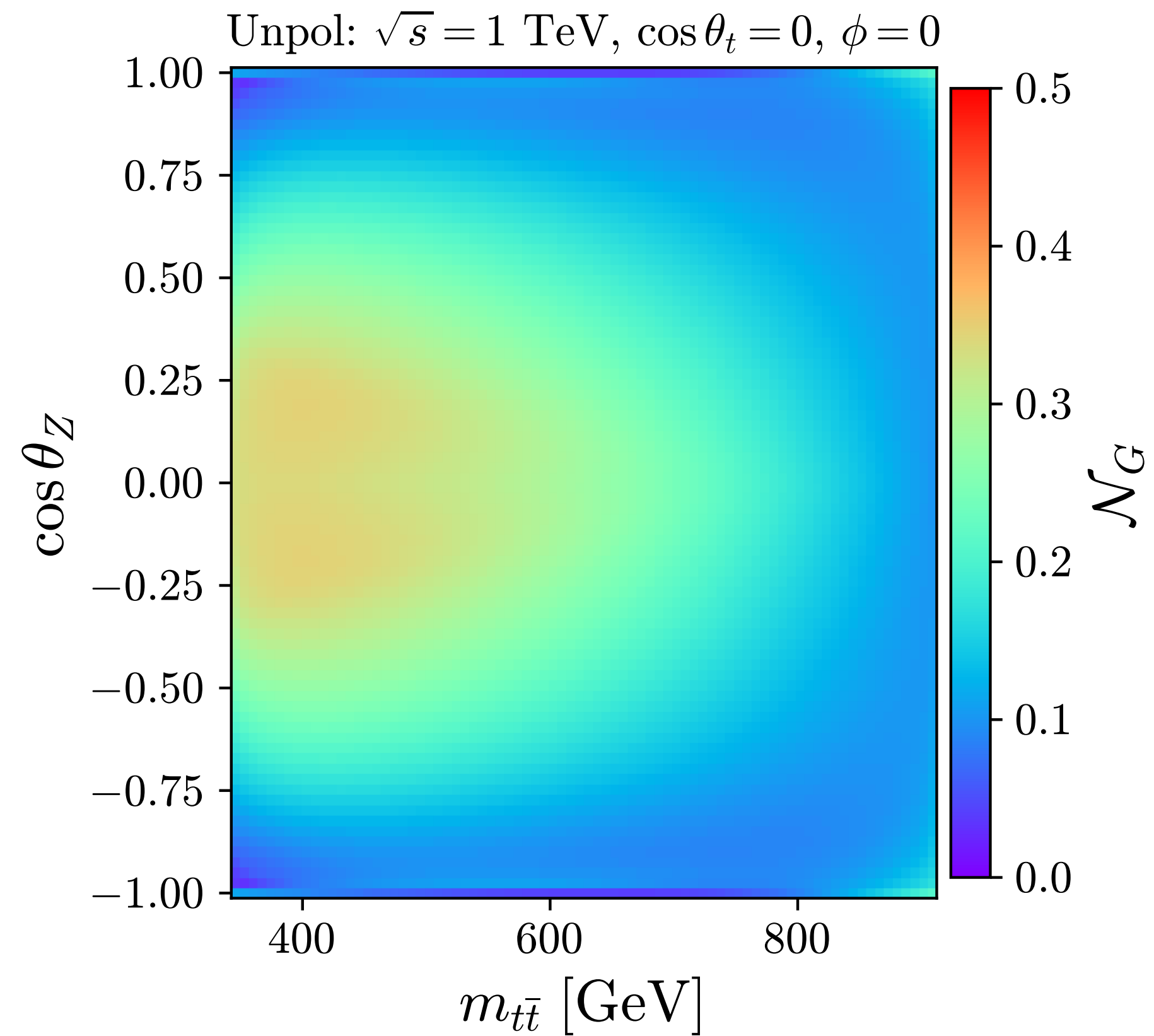
- Small
- **Separable** in some region

One-to-other @ phase-space points:



- **~ max. entangled** in some region

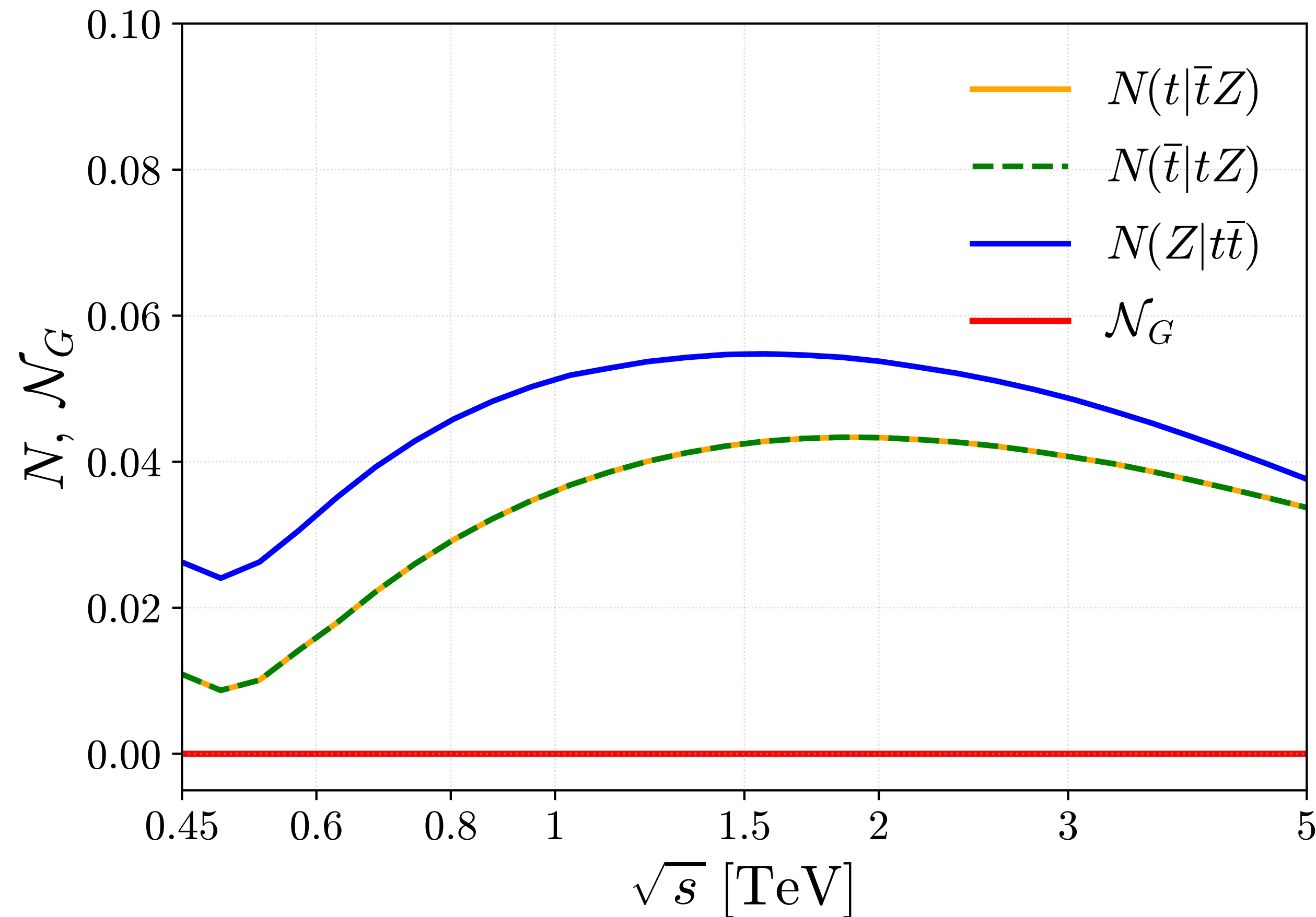
GME @ phase-space points



- Large GME @ central threshold region

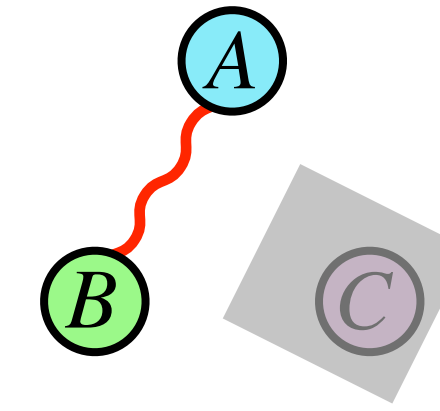
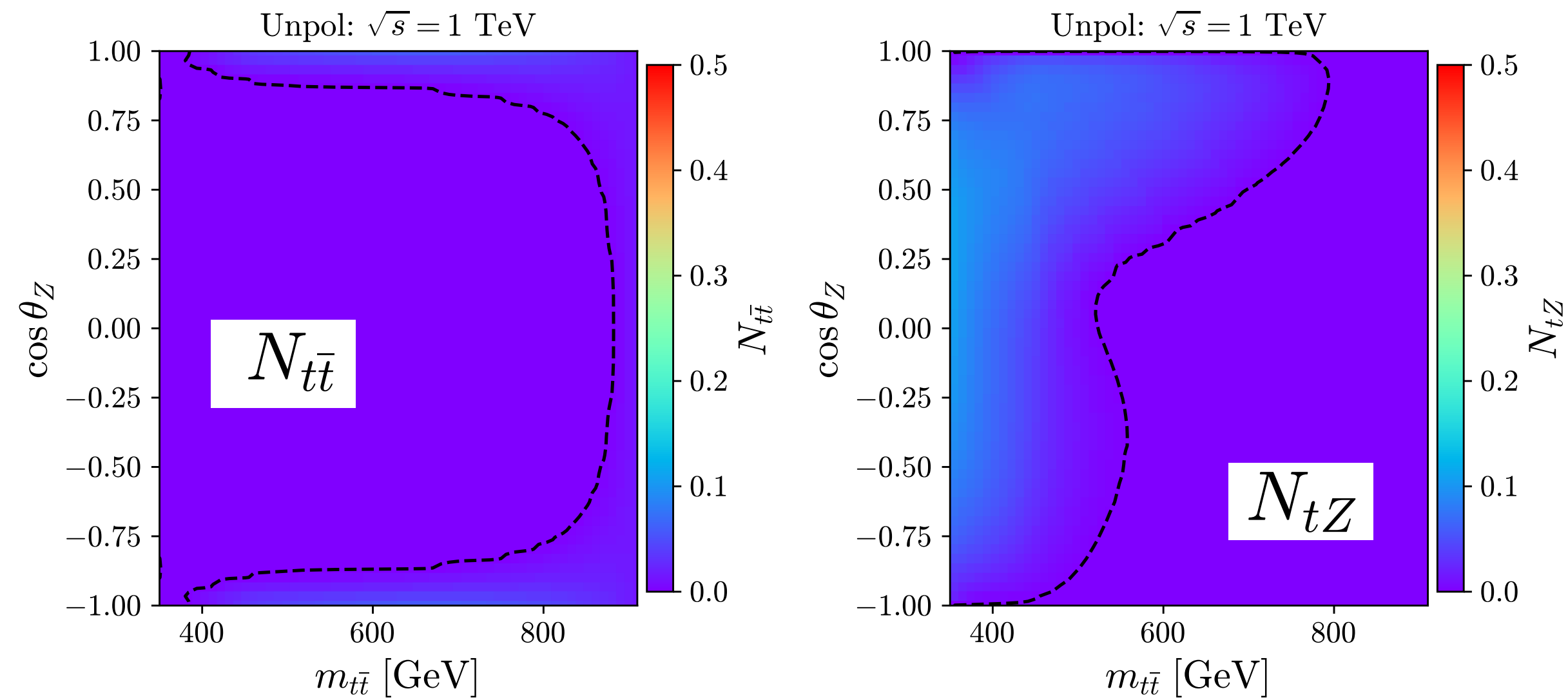
After the *phase-space integration* (event averaging),

$$N_{t\bar{t}} = N_{tZ} = N_{\bar{t}Z} = 0 \quad \text{for all } \sqrt{s}$$



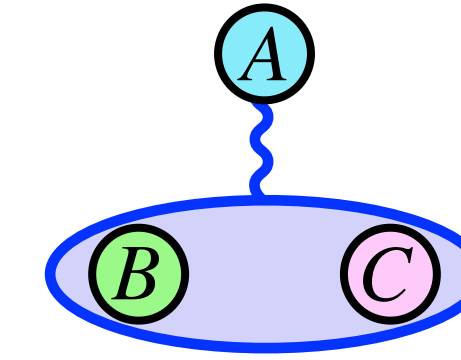
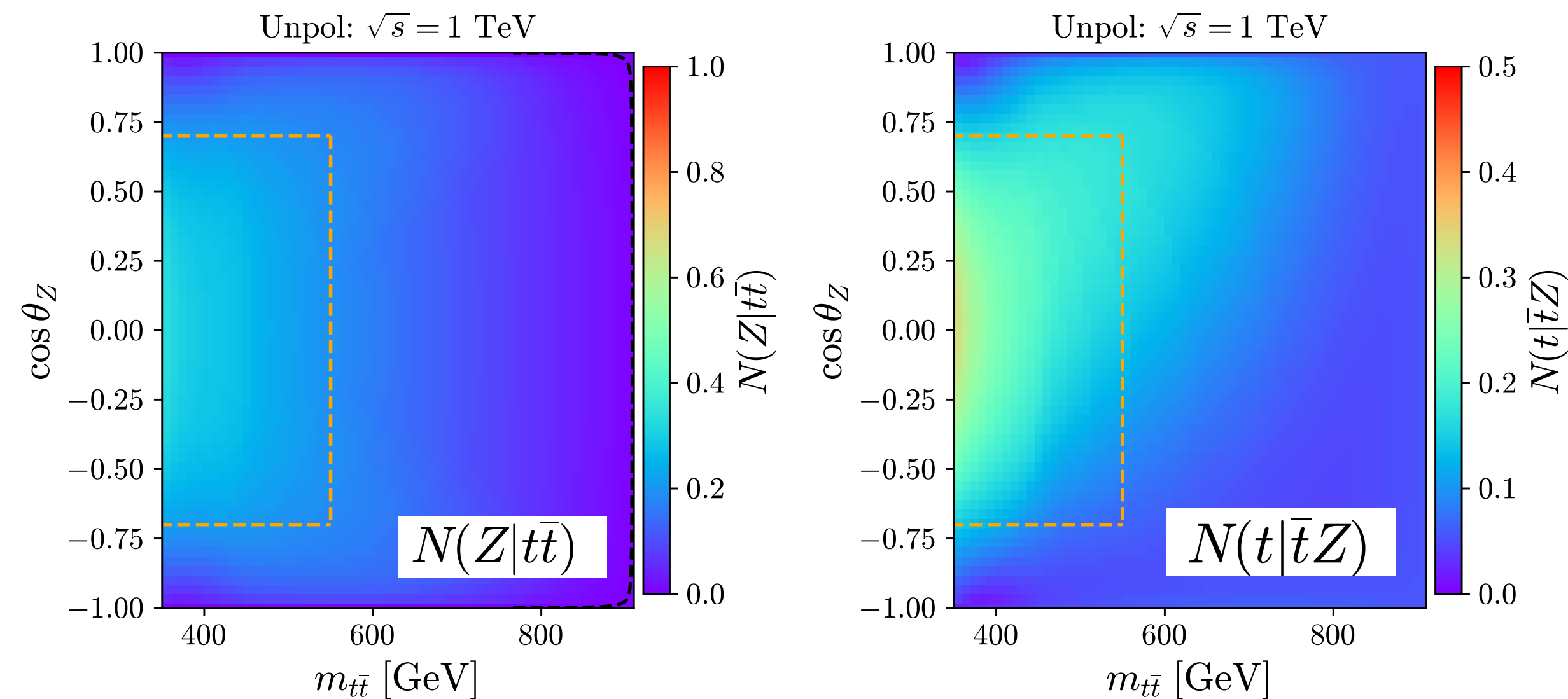
- Entanglement is *washed out* due to the *phase-space integration*.

One-to-one: θ_t, ϕ integrated



- Small
- **Separable** in a large region

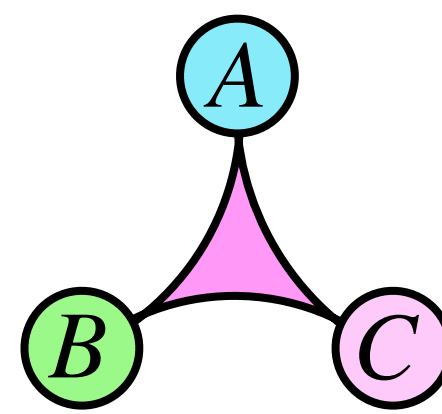
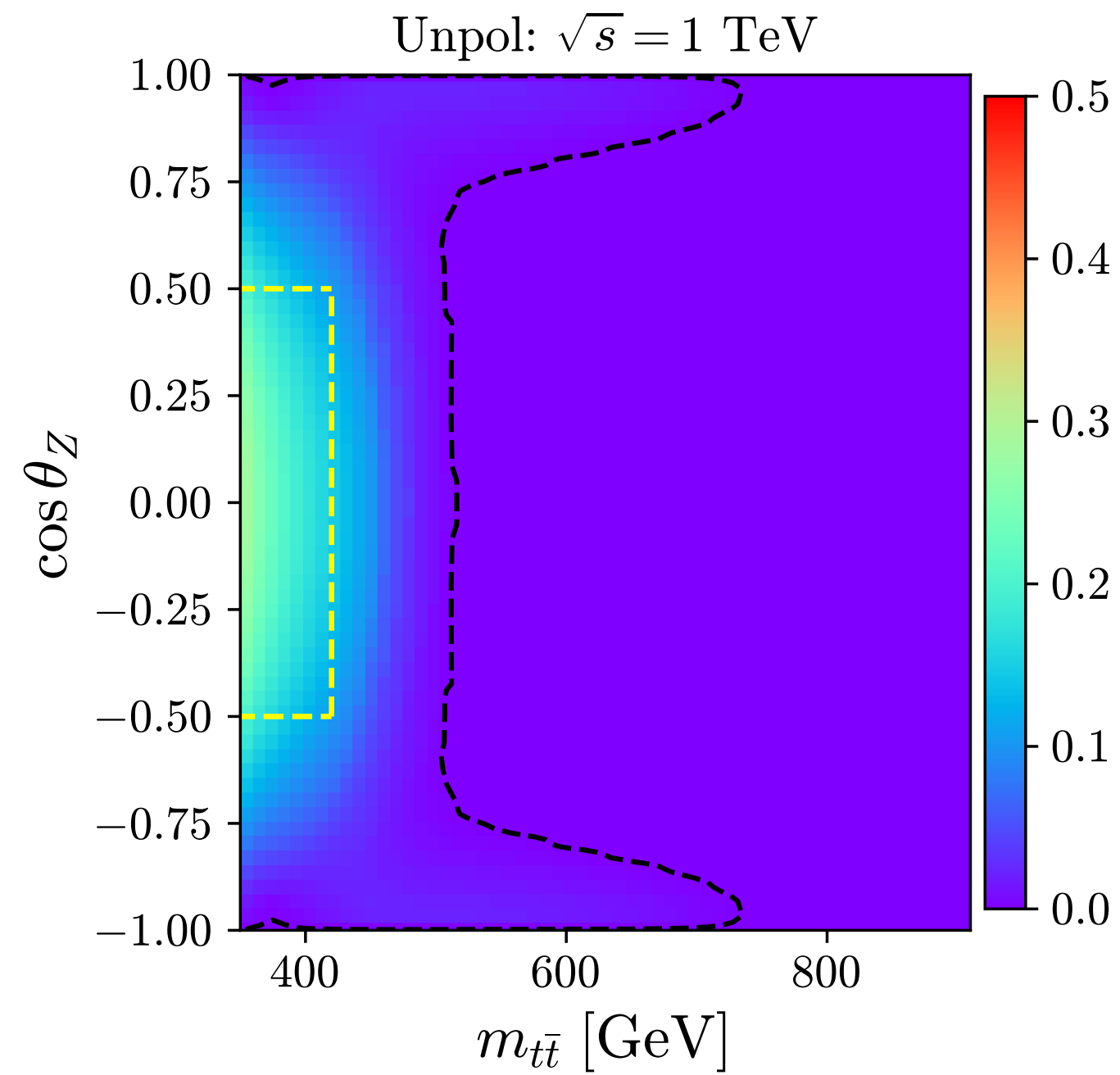
One-to-other: θ_t, ϕ integrated



- Large in the central threshold region

$$\Sigma_{1|2} : \quad |\cos \theta_Z| < 0.7 \quad \text{and} \quad m_{t\bar{t}} < 550 \text{ GeV}$$

GMN: θ_t, ϕ integrated

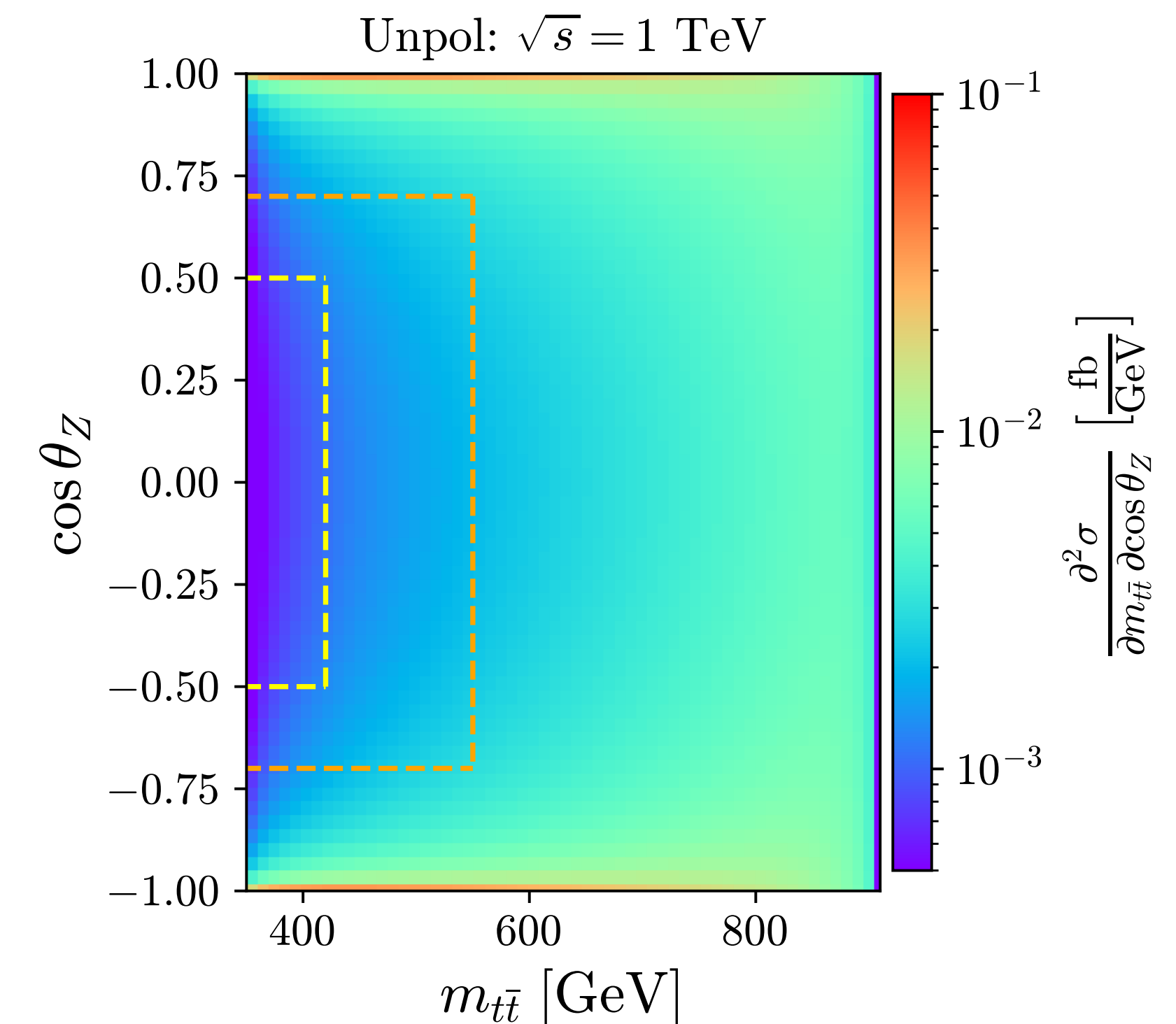


- Sizeable at the central threshold region

$$\Sigma_{\text{GME}} : \quad |\cos \theta_Z| < 0.5 \quad \text{and} \quad m_{t\bar{t}} < 420 \text{ GeV}$$

- The cross section is suppressed at the central threshold region

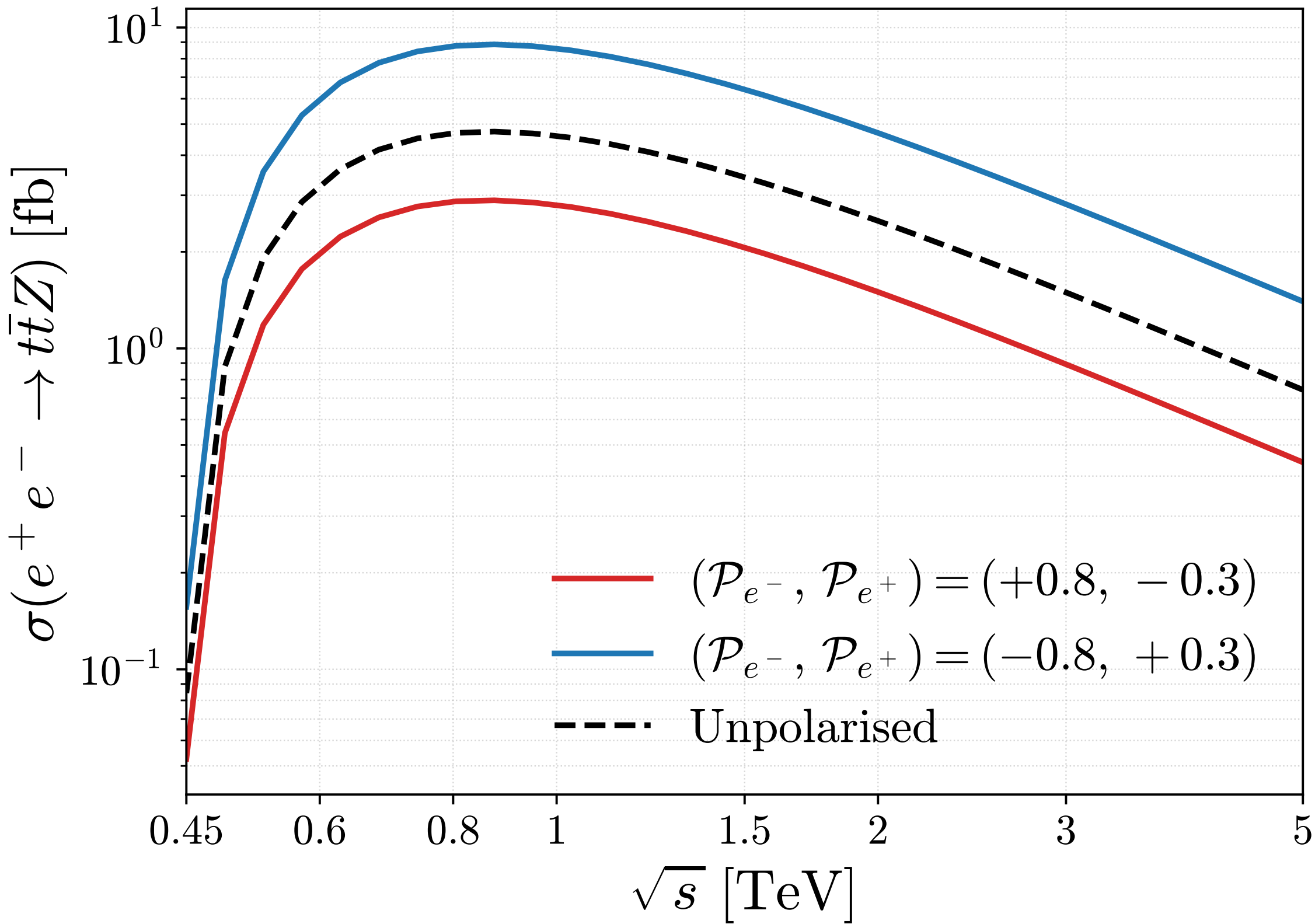
Differential cross section



The effect of polarisation:

- Pol− beam configuration *enhances* the cross section

	Unpol	Pol+	Pol−
$(\mathcal{P}_{e^-}, \mathcal{P}_{e^+})$	$(0, 0)$	$(+0.8, -0.3)$	$(-0.8, +0.3)$



	$\sigma_{1 2}$ [fb]	σ_{GME} [fb]	$N(Z t\bar{t})$	$N(t \bar{t}Z)$	$\mathcal{N}_G$
Unpol	0.41	0.054	0.199	0.136	0.156
Pol+	0.27	0.036	0.216	0.150	0.169
Pol−	0.74	0.099	0.244	0.170	0.181

A naive statistical estimate:

$$\text{BR} = [\text{BR}(t \rightarrow b\ell\nu)]^2 \cdot \text{BR}(Z \rightarrow \ell^+\ell^-) \simeq 0.01$$

$$n_\Sigma = \sigma_\Sigma \cdot \text{BR} \cdot \mathcal{L} \quad : \text{ \# of events}$$

$$\delta_v = c_{\text{sys}} \cdot \frac{v_{\text{max}}}{\sqrt{n_\Sigma}} \quad : \text{uncertainty on } v \in \{N(Z|t\bar{t}), N(t|\bar{t}Z), N(\bar{t}|tZ), \mathcal{N}_G\}.$$

$$\sigma_v \equiv \frac{v_\Sigma}{\delta_v} = \frac{v_\Sigma}{v_{\text{max}}} \sqrt{\sigma_\Sigma \cdot \text{BR} \cdot \mathcal{L}} \quad : \text{the significance of the measurement for } v$$

Result:

	$N(Z t\bar{t})$			$N(t \bar{t}Z)$			$\mathcal{N}_G$		
significance, luminosities	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$	σ_v	$\mathcal{L}_{1\sigma}$	$\mathcal{L}_{2\sigma}$
Unpol	1.1	6.2	24.8	1.5	3.3	13.2	0.65	19.1	76.4
Pol+	1.0	7.9	31.7	1.4	4.1	16.4	0.57	24.6	98.4
Pol−	1.9	2.3	9.1	2.6	1.2	4.7	1.01	7.8	31.1

σ_v : signif. at 8 ab^{-1}

$\mathcal{L}_{1\sigma}$: lumi. required for 1σ

$\mathcal{L}_{2\sigma}$: lumi. required for 2σ

lumi unit = ab

- $\sim 2\sigma$ certification for **one-to-other** entanglement
- $\sim 1\sigma$ certification for **GME**
- **Optimisation** of event selection will **improve the sensitivity** further

Conclusion:

- Colliders produce genuinely tripartite systems; $\rightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$
- The tripartite system has a structure that is **absent** in the bipartite system; **one-to-one vs one-to-other** entanglement, **monogamy**, **GME** and **tripartite nonlocality**.

$$h \rightarrow \tau^- \tau^+ Z \quad (\text{pure state})$$

- Expanding in the fermion mass, the spin state admits a **simple expression** in the kinematical corners.
- The bulk region is GHZ-type ($2 \otimes 2 \otimes 2$), while the collinear region is genuinely $2 \otimes 2 \otimes 3$.
- Experimental prospect is promising $\sim 10^4$ (10^3) events at **HL-LHC** (**FCC-ee**).

$$e^+ e^- \rightarrow t \bar{t} Z \quad (\text{mixed state})$$

- The state depends on 5 kinematical param. A partial integration mixes states with different quantisation axes $\rightarrow$ a **fictitious state**.
- one-to-one is small, while one-to-other and GME are sizeable in the central threshold region.
- Beam polarisation enhances the rate: $\sim 2\sigma$ for **one-to-other**, $\sim 1\sigma$ for **GME**

Thank you for listening

$\mathbf{s}$: spin direction of the decaying particle
 $\mathbf{x}$: direction of decay product x (unit vector)

$$\frac{1}{\Gamma} \frac{d\Gamma}{d(\mathbf{x} \cdot \mathbf{s})} = \frac{1 + \alpha_x \cdot (\mathbf{x} \cdot \mathbf{s})}{2}$$

$\alpha_x \in [-1, +1]$: **spin analyzing power**

$$\tau^- \rightarrow \pi^- \nu : \alpha \simeq 1$$

$$\tau^- \rightarrow \rho^- \nu : \alpha \simeq 0.45$$

$$\tau^- \rightarrow e^- \nu \bar{\nu} : \alpha \simeq -0.33$$

$$t \rightarrow b \nu \ell^+ : \alpha \simeq 1$$

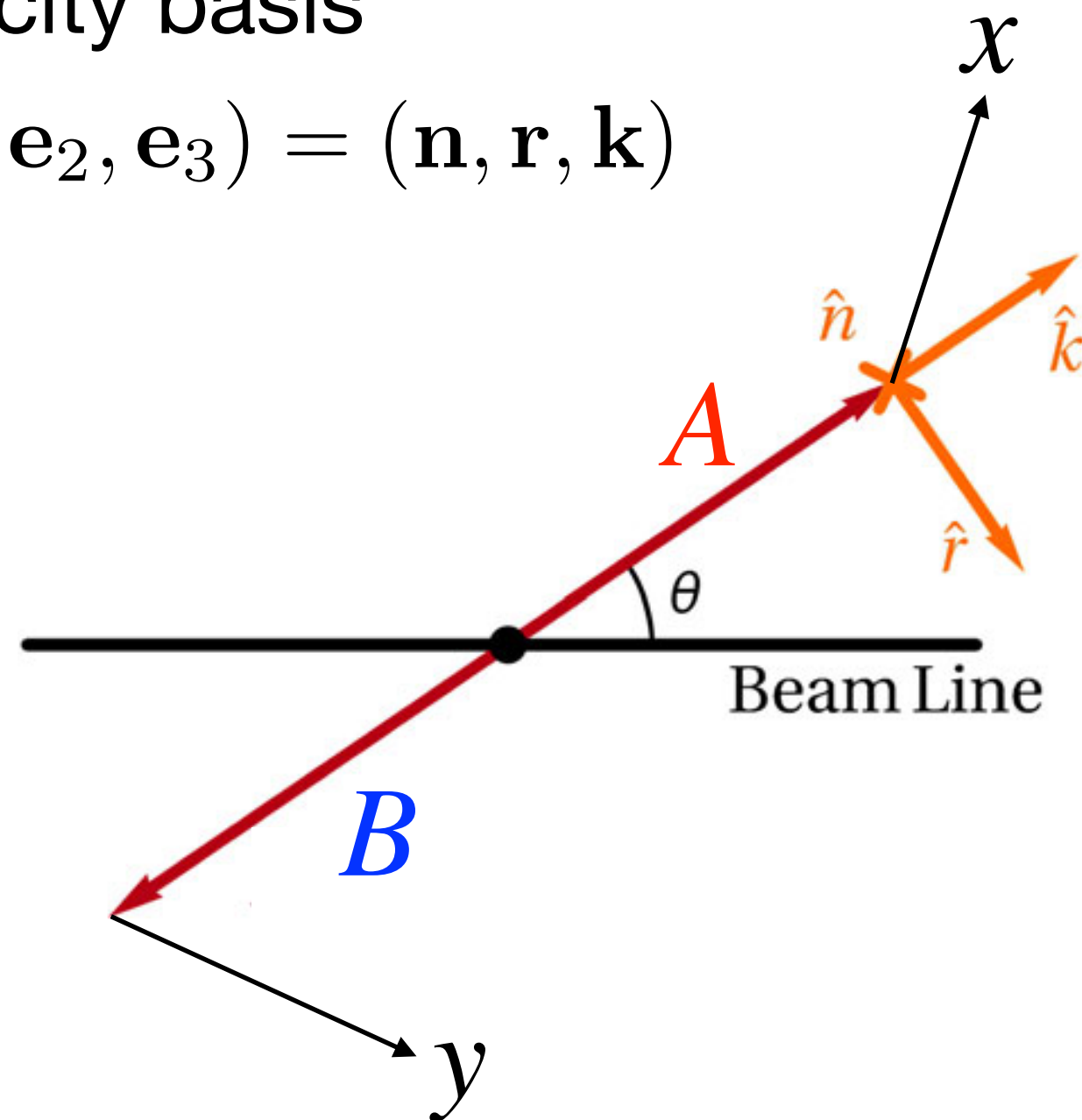
$$t \rightarrow b \nu \ell^+ : \alpha \simeq 0.39$$

$$t \rightarrow b u \bar{d} : \alpha \simeq -0.32$$

$$t \rightarrow b u \bar{d} : \alpha \simeq 0.97$$

helicity basis

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{n}, \mathbf{r}, \mathbf{k})$$



$$B_i = \langle \mathbf{s}^A \cdot \mathbf{e}_i \rangle = \frac{3}{\alpha_x} \langle \mathbf{x} \cdot \mathbf{e}_i \rangle$$

observable

$$\bar{B}_i = \langle \mathbf{s}^B \cdot \mathbf{e}_i \rangle = \frac{3}{\alpha_y} \langle \mathbf{y} \cdot \mathbf{e}_i \rangle$$

$$C_{ij} = \langle (\mathbf{s}^A \cdot \mathbf{e}_i)(\mathbf{s}^B \cdot \mathbf{e}_j) \rangle = \frac{9}{\alpha_x \alpha_y} \langle (\mathbf{x} \cdot \mathbf{e}_i)(\mathbf{y} \cdot \mathbf{e}_j) \rangle$$

$$\rho = \frac{1}{4} \left(\mathbf{1}_{AB} + B_i \cdot [\sigma^i \otimes \mathbf{1}_B] + \bar{B}_i \cdot [\mathbf{1}_A \otimes \sigma^i] + C_{ij} \cdot [\sigma^i \otimes \sigma^j] \right)$$

