Characteristic mass matrix in neutrino oscillation

S. H. Chiu
Physics Group, CGE, Chang Gung University, Taoyuan, Taiwan
chius2001@gmail.com
T. K. Kuo
Department of Physics, Purdue University, West Lafayette, IN, USA
tkkuo@purdue.edu

Abstract

The squared rephasing-invariant elements of the mixing matrix can be derived from the characteristic matrix associated with the mass matrix in a transparent way. This formulation leads to certain matter invariants and an alternative expression for the squared Jarlskog invariant J^2. The results may be applied to the analysis of neutrino oscillation in matter and the study of normalization group evolution.

Notations

- Diagonalization of a 3×3 Hermitian matrix M is governed by the characteristic polynomials,

$$\det(\lambda I - M) = \det(\lambda I - M_D) = \Pi(\lambda - \lambda_i) \quad (1)$$

- M_D is a diagonal matrix with eigenvalues λ_i, $i = (1, 2, 3)$,

$$M = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & M_{\alpha\gamma} \\ M_{\beta\alpha} & M_{\beta\beta} & M_{\beta\gamma} \\ M_{\gamma\alpha} & M_{\gamma\beta} & M_{\gamma\gamma} \end{pmatrix} = VM_DV^\dagger$$
$$= \begin{pmatrix} \Sigma W_{\alpha i}\lambda_i & \Sigma V_{\alpha i}V^*_{\beta i}\lambda_i & \Sigma V_{\alpha i}V^*_{\gamma i}\lambda_i \\ \Sigma V_{\beta j}V^*_{\alpha j}\lambda_j & \Sigma W_{\beta j}\lambda_j & \Sigma V_{\beta j}V^*_{\gamma j}\lambda_j \\ \Sigma V_{\gamma k}V^*_{\alpha k}\lambda_k & \Sigma V_{\gamma k}V^*_{\beta k}\lambda_k & \Sigma W_{\gamma k}\lambda_k \end{pmatrix} \quad (2)$$

- $W_{\alpha i} = |V_{\alpha i}|^2$ is the squared rephasing-invariant elements of the mixing matrix, with $\alpha \in (\alpha, \beta, \gamma)$
- The three sub-matrices of M with the α-th row and column deleted are denoted as M_α, with eigenvalues $\xi_i^{(\alpha)}$, $i = (1, 2)$.
- We may write

$$d_\alpha = \det M_\alpha = \xi_1^{(\alpha)} \cdot \xi_2^{(\alpha)} \quad (3)$$
$$t_\alpha = tr M_\alpha = \xi_1^{(\alpha)} + \xi_2^{(\alpha)} \quad (4)$$

- $\xi_i^{(\alpha)}$ and $(W_{\alpha i}, \lambda_i)$ are related in a compact form:

$$d_\alpha = \Sigma' W_{\alpha i}\lambda_j\lambda_k, \quad (5)$$
$$t_\alpha = \Sigma' W_{\alpha i}(\lambda_j + \lambda_k), \quad (6)$$

where Σ' indicates cyclic summation over the indices (i, j, k).

Matter invariants in neutrino oscillation

- When a neutrino propagates through a medium, it picks up an effective mass: $M \to M(A) = M + A|e\rangle\langle e|$, where $A = 2\sqrt{2}G_F n_e E$.
- (α, β, γ) are identified as (e, μ, τ)
- Since d_e and t_e are obtained by deleting the $|e\rangle\langle e|$ element of $M(A)$, they are obviously not affected by A.
- Thus, the following matter invariants hold [1]:

$$I_d = \Sigma' W_{ei}\lambda_j\lambda_k, \quad (7)$$
$$I_t = \Sigma' W_{ei}(\lambda_j + \lambda_k). \quad (8)$$

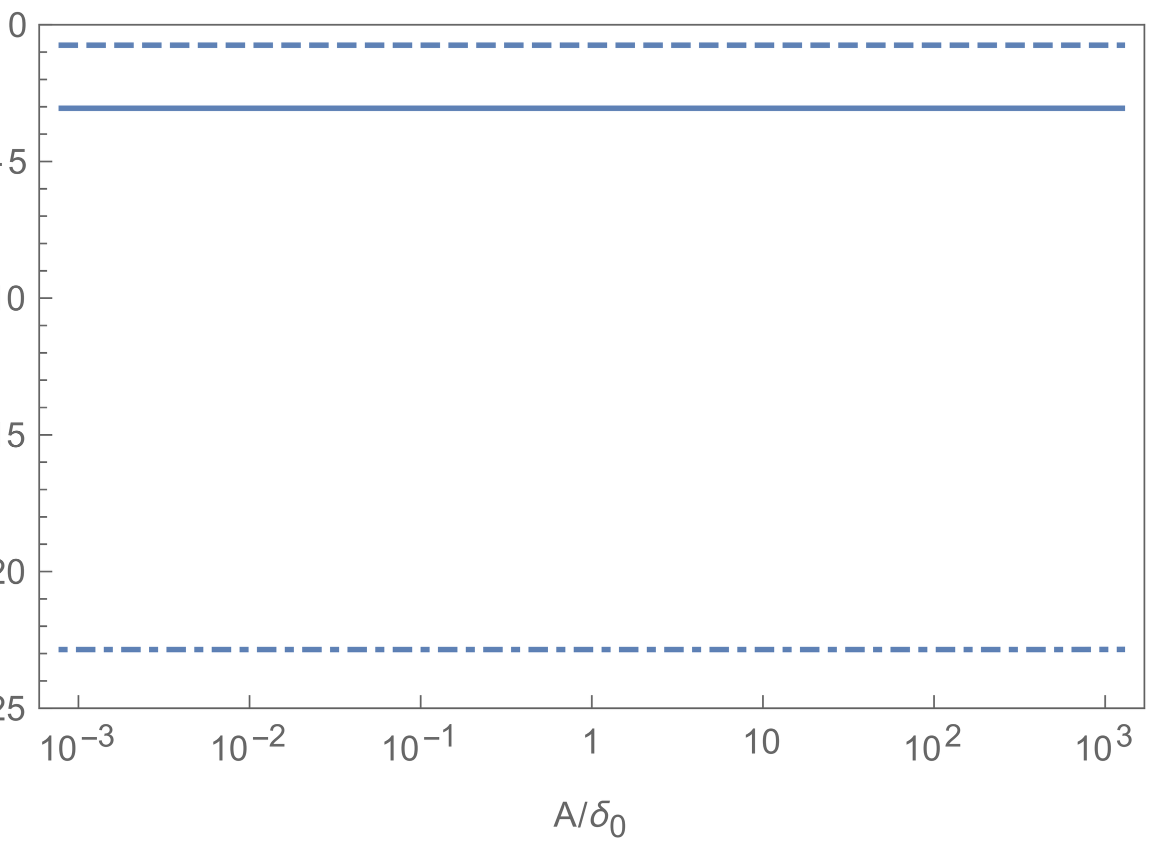
- Along this line, several invariants consisting of $(W_{\alpha i}, \lambda_i)$ can be identified:

$$I_1 = \Sigma' W_{\tau i}(\lambda_j + \lambda_k) - \Sigma' W_{\mu i}(\lambda_j + \lambda_k), \quad (9)$$

$$I_2 = \Sigma'(W_{\mu i}W_{\mu j} - W_{\tau i}W_{\tau j})(\lambda_i - \lambda_j)^2\lambda_k, \quad (10)$$

$$I_3 = \Sigma'(-W_{ei}W_{ej})(\lambda_i - \lambda_j)^2\lambda_k. \quad (11)$$

- The numerical check of the invariance for I_1/δ_0 (solid), I_2/δ_0^3 (dashed), and I_3/δ_0^3 in matter is shown below. Here $\delta_0 = m_2^2 - m_1^2$ in vacuum.



An alternative expression of J^2

In a general way, we write J^2 explicitly in terms of $W_{\alpha i}$ and λ_i:

$$J^2 = \frac{|M_{\alpha\beta}M_{\beta\gamma}M_{\gamma\alpha}|^2 - \frac{1}{4}(\det M - \Sigma M_{\alpha\alpha}d_\alpha + 2\Pi M_{\alpha\alpha})^2}{(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2} \quad (12)$$

- Diagonal elements $M_{\alpha\alpha}$ are related to $(W_{\alpha i}, \lambda_i)$ in Eq. (2).
- Off-diagonal elements are also related to $(W_{\alpha i}, \lambda_i)$

$$|M_{\alpha\beta}|^2 = d_\gamma - M_{\alpha\alpha}M_{\beta\beta} \quad (13)$$

Outlook

- Our main results relate the parameters in the flavor space $(\det M, d_\alpha, t_\alpha)$ with those of the eigenvalue space (λ_i) and the mixing parameters $(W_{\alpha i})$.
- Updated future precision measuments of δm_{ij}^2 and the mixing angles θ_{ij} may be applied directly to the analysis of J^2 through this approach.
- Similar formulation may also apply to future studies related to the normalization group equations in flavor physics.

References

[1] S. H. Chiu and T. K. Kuo, “Hermitian matrix diagonalization and its symmetry properties,” *Advances in High Energy Physics, Volume 2024, 3681297 (2024).*