Energy levels of multiscale bound states from QED energy-momentum trace

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Outline

Mass (energy levels) and energy-momentum tensor $T^{\mu\nu}(x)$

Hamiltonian & EMT trace

- $|\boldsymbol{p}\rangle$ – state (fundamental or bound) with momentum $\boldsymbol{p}$
- $H = \int d^3x T^{00}(x)$ – **Hamiltonian**
- $\Longrightarrow \int d^3x \langle \boldsymbol{p}|T^{00}(x)|\boldsymbol{p}\rangle = E_p \langle \boldsymbol{p}|\boldsymbol{p}\rangle$
- **Translational invariance:** $\langle \boldsymbol{p}|T^{\mu\nu}(x)|\boldsymbol{p}\rangle = \langle \boldsymbol{p}|T^{\mu\nu}(0)|\boldsymbol{p}\rangle$
- $\Longrightarrow \langle \boldsymbol{p}|T^{00}(0)|\boldsymbol{p}\rangle = E_p \frac{\langle \boldsymbol{p}|\boldsymbol{p}\rangle}{V}$
- **Covariant normalization:** $\langle \boldsymbol{p}|\boldsymbol{p}\rangle = 2E_p V$
- $\Longrightarrow \langle \boldsymbol{p}|T^{00}(0)|\boldsymbol{p}\rangle = 2E_p^2$
- **Lorentz invariance:** $\langle \boldsymbol{p}|T^{\mu\nu}(0)|\boldsymbol{p}\rangle = 2p^\mu p^\nu, \quad \langle \boldsymbol{p}|T^\mu{}_\mu(0)|\boldsymbol{p}\rangle = 2m^2$
- $\int d^3x \langle \boldsymbol{p}|T^\mu{}_\mu(x)|\boldsymbol{p}\rangle = 2m^2 V$
- **Rest frame** $\int d^3x \langle \mathbf{0}|T^\mu{}_\mu(x)|\mathbf{0}\rangle = m\langle \mathbf{0}|\mathbf{0}\rangle$

Mass (energy levels) and energy-momentum tensor $T^{\mu\nu}(x)$

Hamiltonian & EMT trace

- **Rest frame & normalization independent expression**

$$E = \frac{\int d^3x \langle \mathbf{0}| T^\mu{}_\mu(x)|\mathbf{0}\rangle}{\langle \mathbf{0}|\mathbf{0}\rangle}, \qquad E = \frac{\int d^3x \langle \mathbf{0}| T^{00}(x)|\mathbf{0}\rangle}{\langle \mathbf{0}|\mathbf{0}\rangle}$$

- **We will use nonrelativistic normalization in calculations**
- **Mass (energy levels) of fundamental bound states can be calculated as matrix elements of QFT EMT trace!**
- **These relationships hold both perturbatively and nonperutrbatively**

Energy-momentum tensor

QED with two fermion flavors (electron and muon)

- Renormalized perturbation theory, dimensional regularization ($d = 4 - 2\epsilon$)

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}F_{0\mu\nu}F^{0\mu\nu} + \mathcal{L}_{0e} + \mathcal{L}_{0\mu}$$

$$\mathcal{L}_{0e} = \bar{\psi}_{e0}(i\partial\!\!\!/ - m_{e0})\psi_{e0} - e_0\bar{\psi}_{e0}A\!\!\!/_0\psi_{e0}$$

$$\mathcal{L}_{0\mu} = \bar{\psi}_{\mu0}(i\partial\!\!\!/ - m_{\mu0})\psi_{\mu0} - e_0\bar{\psi}_{\mu0}A\!\!\!/_0\psi_{\mu0}$$

- Symmetric EMT tensor ($D_\alpha = \partial_\alpha + ie_0A_{0\alpha}$, $\overleftarrow{D_\alpha} = \overleftarrow{\partial_\alpha} - ie_0A_{0\alpha}$)

$$T_0^{\mu\nu} = \frac{i}{4}\bar{\psi}_{e0}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\psi_{e0} + \frac{i}{4}\bar{\psi}_{\mu0}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\psi_{\mu0} - F_0^{\mu\alpha}F_{0\ \alpha}^{\nu} + \frac{1}{4}g^{\mu\nu}F_0^2$$

- **Conserved EMT tensor is not renormalized** $T_0^{\mu\nu} = [T^{\mu\nu}]_R$

QED with two fermion flavors (electron and muon)

- **EMT trace in gauge theories (QED, QCD, etc.) is anomalous**

$$T_{0\ \mu}^{\mu} = [T^{\mu}{}_{\mu}]_R = (1+\gamma_m^e)[\bar{\psi}_e m_e \psi_e]_R + (1+\gamma_m^{\mu})[\bar{\psi}_{\mu} m_{\mu} \psi_{\mu}]_R + \frac{\beta(e)}{2e}[F^2]_R$$

- **Anomalous terms:** $\frac{\beta(e)}{2e}[F^2]_R$ **and** $\gamma_m^e[\bar{\psi}_e m_e \psi_e]_R$ **and** $\gamma_m^{\mu}[\bar{\psi}_{\mu} m_{\mu} \psi_{\mu}]_R$
- **QED:**
  - One-loop beta-function: $\beta(e) = \beta^e(e) + \beta^{\mu}(e)$

$$\frac{\beta^e(e)}{2e} = \frac{\beta^{\mu}(e)}{2e} = \frac{\alpha}{6\pi}$$

  - One-loop mass anomalous dimension: $\gamma_m^e = \gamma_m^{\mu} = \frac{3\alpha}{2\pi}$
- **Aim: calculate one-loop Lamb shift in muonic hydrogen as matrix element of QED EMT trace**

Muonic hydrogen

- **One-loop EMT trace muonic hydrogen matrix element**

$$T \equiv \langle\mu| \int d^3x T^{\mu}{}_{\mu}(x)|\mu\rangle$$

$$\approx \int d^3r \langle\mu|[m_e - \delta m_e + m\gamma_m^e(e) + m_e\delta Z_{2e}]\bar{\psi}_e(\mathbf{r})\psi_e(\mathbf{r}) + \frac{\beta^e(e)}{2e}F^2(\mathbf{r})$$

$$+[m_\mu - \delta m_\mu + m\gamma_m^\mu(e) + m_\mu\delta Z_{2\mu}]\bar{\psi}_\mu(\mathbf{r})\psi_\mu(\mathbf{r}) + \frac{\beta^\mu(e)}{2e}F^2(\mathbf{r})|\mu\rangle$$

- **Tree and one-loop diagrams EMT trace diagrams**

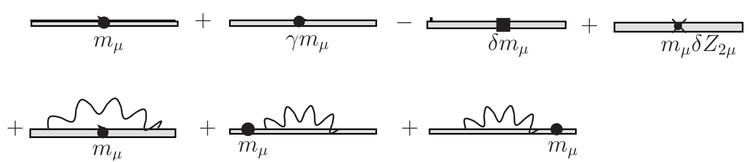


Figure: Self-energy type trace Lamb shift diagrams.

Tree and one-loop diagrams EMT trace diagrams

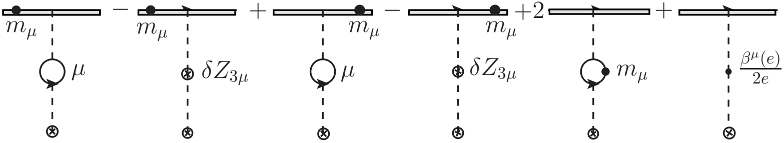


Figure: Muon vacuum polarization type trace Lamb shift diagrams.

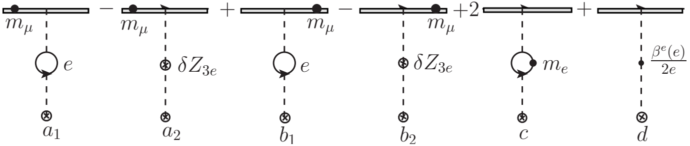


Figure: Electron vacuum polarization type trace Lamb shift diagrams.

Classic one-loop Lamb shift diagrams

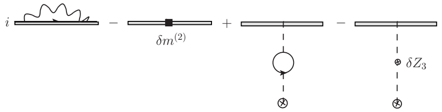


Figure: Classic one-loop Lamb shift diagrams in electronic and muonic hydrogen.

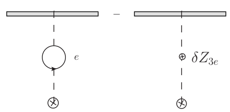


Figure: Classic electron polarization Lamb shift diagrams in muonic hydrogen.

Muonic hydrogen – classic

- **Diagrams with electron loop polarization dominate:** $\Delta E \sim \frac{\alpha(Z\alpha)^2}{n^3} m_r \sim 205$ **meV**
- **All other contributions are suppressed by** α **or** $Z\alpha$**, start with** $\Delta E \sim 1 - 2$ **meV**

Muonic hydrogen goals

- **Calculate sum of trace diagrams**
- **Find connection between two sets of diagrams**
- **Explain why sums of two different sets of diagrams coincide**

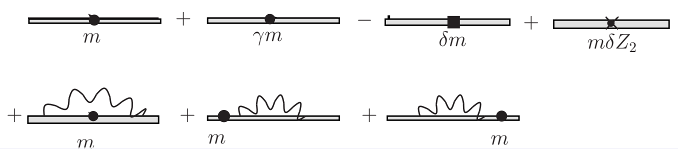
One-loop EMT trace in electronic hydrogen (ME, 2024)

- **One-loop matrix element**

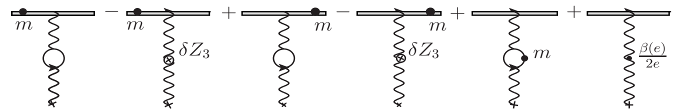
$$T \equiv \int d^3r \langle n\ell| T^{\mu}{}_{\mu}(r)|n\ell\rangle$$

$$\approx \int d^3r \langle n\ell|[m - \delta m + m\gamma_m(e) + m\delta Z_2]\bar{\psi}(r)\psi(r) + \frac{\beta(e)}{2e}F^2(r)|n\ell\rangle$$

- **One-loop perturbation theory diagrams for EMT trace**
  - **Tree and self-energy type diagrams:**



  - **Vacuum polarization type diagrams:**

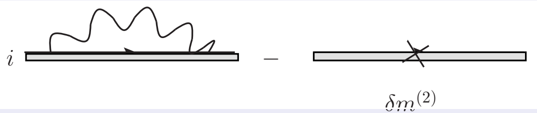


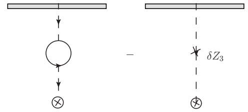
One-loop EMT trace in electronic hydrogen (ME, 2024)

- **Diagrams for trace are different from the textbook Lamb shift diagrams**
- **Two questions:**
  1. **How these diagrams reproduce Lamb shift?**
  2. **Why two different sets of diagrams produce the same result?**

Classic Lamb shift calculation

- **Furry picture: QED in external Coulomb field**
- Two sets of one-loop diagrams: self-energy and polarization





One-loop EMT trace in electronic hydrogen (ME, 2024)

- **Answers to questions above**
  1. Trace diagrams arise as logarithmic mass derivatives of the classic diagrams
  2. Subtlety: one needs to remember to differentiate the state vectors in the matrix elements!
  3. Energy levels in the electronic hydrogen are linear in the electron mass $\Longrightarrow$ logarithmic mass derivative of an energy level (matrix element of the sum of the trace diagrams) coincides with the energy level itself!

Back to muonic hydrogen

- Classic self-energy diagrams coincide with the sum self-energy type trace diagrams
- Classic muon loop diagrams coincide with the sum of muon loop trace type diagrams
- Nothing new in comparison with electronic hydrogen
- These diagrams in muonic hydrogen are strongly suppressed in comparison with the diagrams with the electron loops!

Muonic hydrogen diagrams with electron loops

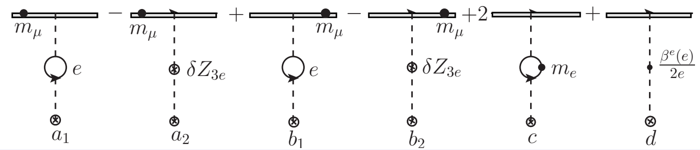


Figure: Electron vacuum polarization type trace Lamb shift diagrams.

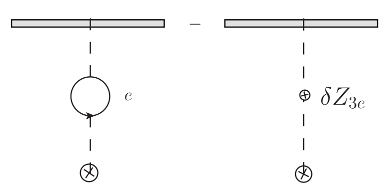


Figure: Classic electron polarization Lamb shift diagrams in muonic hydrogen.

- **Why and how sums of these diagrams coincide?**

Generic multiscale problem

- $E_n(m_1, m_2, \ldots, m_k)$ – **energy level of a bound state with a few independent mass parameters** m_i, $i = 1, 2, \ldots k$
- $E_n(m_1, m_2, \ldots, m_k)$ – **homogeneous function of the first degree (α - arbitrary parameter)**

$$E_n(\alpha m_1, \alpha m_2, \ldots, \alpha m_k) = \alpha E_n(m_1, m_2, \ldots, m_k)$$

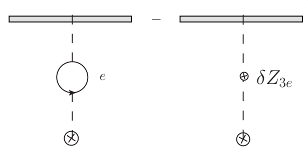
$$\frac{d}{d\alpha} E_n(\alpha m_1, \alpha m_2, \ldots, \alpha m_k) = E_n(m_1, m_2, \ldots, m_k)$$

$$\& \frac{d}{d\alpha} E_n(\alpha m_1, \alpha m_2, \ldots, \alpha m_k) = \sum_{i=1}^{i=k} \frac{\partial E_n(\alpha m_1, \alpha m_2, \ldots, \alpha m_k)}{\partial(\alpha m_i)} m_i$$

$$\Longrightarrow E_n(m_1, m_2, \ldots, m_k) = \sum_{i=1}^{i=k} \frac{\partial E_n(m_1, m_2, \ldots, m_k)}{\partial m_i} m_i$$

- **This is Euler's homogeneous function theorem**
- $\Longrightarrow$ **in muonic hydrogen electron vacuum polarization type trace Lamb shift diagrams are logarithmic derivatives of classic electron polarization Lamb shift diagrams**
- **But: now we need to differentiate with respect both to the electron and muon masses!**

Classic electron loop Lamb shift (muonic hydrogen)



- $H^e_{int} = \int d^3x \mathcal{H}^e_{int} = e\int d^3x \bar{\psi}_\mu \gamma_0 \psi_\mu A^0_{ext}(x)$ – **field Hamiltonian**
- **One-loop corrected static Coulomb field:**

$$A^0_{ext,e\ \ loop}(\boldsymbol{r}) = -\frac{Ze}{4\pi r}\frac{2\alpha}{3\pi}\int_1^\infty d\zeta e^{-2\rho\beta\zeta}\left(1+\frac{1}{2\zeta^2}\right)\frac{\sqrt{\zeta^2-1}}{\zeta^2}$$

$$\rho = m_r Z\alpha r,\ m_r = m_\mu m_p/(m_\mu + m_p),\ \beta = m_e/(m_r Z\alpha)$$

- **Schrödinger-Coulomb bound state wave functions:**

$$\psi_{n\ell m}(\boldsymbol{r}) = Y_{\ell m}(\theta,\phi)R_{n\ell}(r), \quad R_{n\ell}(r) = 2\left(\frac{m_r Z\alpha}{n}\right)^{\frac{3}{2}} f_{n\ell}(\rho_n)$$

$$f_{n\ell}(\rho_n) = \sqrt{\frac{(n-l-1)!}{n(n+l)!}}e^{-\rho_n}(2\rho_n)^l L^{2l+1}_{n-l-1}(2\rho_n), \quad \rho_n = m_r Z\alpha r/n$$

Classic electron loop Lamb shift (muonic hydrogen)

$$\Delta E^e_{VP}(n,\ell) = \langle n\ell | H^e_{int} | n\ell \rangle$$

$$= -\frac{8\alpha(Z\alpha)^2 m_r}{3\pi n^3} \int_0^\infty \rho d\rho f^2_{n\ell}(\rho_n) \int_1^\infty d\zeta e^{-2\rho\beta\zeta} \left(1 + \frac{1}{2\zeta^2}\right) \frac{\sqrt{\zeta^2-1}}{\zeta^2}$$

- **Numerically ($n=2$)**

$$\Delta E^e_{VP}(n=2,\ell=0) = -219.5839\ldots \text{ meV}$$

$$\Delta E^e_{VP}(n=2,\ell=1) = -14.5765\ldots \text{ meV}$$

$$\Delta E^e_{VP}(n=2,\ell=0) - \Delta E^e_{VP}(n=2,\ell=1) = -205.0073\ldots \text{ meV}$$

Trace diagrams

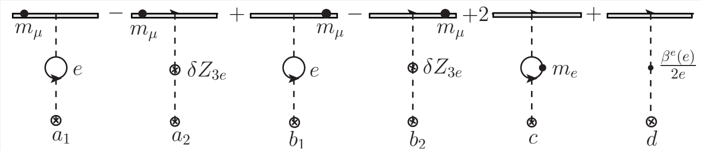
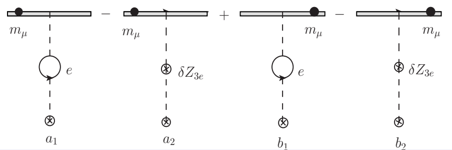


Figure: Electron vacuum polarization type trace Lamb shift diagrams.

Trace diagrams calculations

Scalar vertex $m_\mu \bar{\psi}_\mu \psi_\mu$ and sidewise polarization loop



$$\Delta E_a = \Delta E_b = e\int d^3r d^3r' \psi_n(\boldsymbol{r}) A^0_{ext,e\ loop}(\boldsymbol{r})[-iG_r(\boldsymbol{r},\boldsymbol{r}',E_n)]m\gamma_0\psi_n(\boldsymbol{r}')$$

- **Reduced Dirac-Coulomb Green's function**

$$G_r(E,\boldsymbol{r},\boldsymbol{r}') = \left\langle \boldsymbol{r} \left| \left(\frac{i}{E-H}\right)' \gamma_0 \right| \boldsymbol{r}' \right\rangle = \left\langle \boldsymbol{r} \left| \sum_{k\neq n} \frac{i|k\rangle\langle k|}{E-E_k}\gamma_0 \right| \boldsymbol{r}' \right\rangle$$

- **Perturbed wave function $\widetilde{\psi}_n$**

$$\widetilde{\psi}_n(\boldsymbol{r}) = \int d^3r' [-iG_r(\boldsymbol{r},\boldsymbol{r}',E_n)]m\gamma_0\psi_n(\boldsymbol{r}')$$

$$\Longrightarrow \Delta E_a = \Delta E_b = e\int d^3r \psi_n(\boldsymbol{r}) A^0_{ext,e\ loop}(\boldsymbol{r})\widetilde{\psi}_n(\boldsymbol{r})$$

Scalar vertex $m_\mu \bar{\psi}_\mu \psi_\mu$ and sidewise polarization loop

- **Leading nonrelativistic approximation**

$$\widetilde{\psi}_{n\ell}(\boldsymbol{r}) = \left(\frac{3}{2} + r\frac{d}{dr}\right)\psi_{n\ell}(\boldsymbol{r})$$

$$\Longrightarrow \widetilde{f}_{n\ell}(\rho_n) = \left(\frac{3}{2} + r\frac{d}{dr}\right) f_{n\ell}(\rho_n$$

- **Energy shift**

$$\Delta E_{a,b}(n,\ell) = -\frac{8\alpha(Z\alpha)^2 m_r}{3\pi n^3}\int_0^\infty \rho d\rho f_{n\ell}(\rho_n)\widetilde{f}_{n\ell}(\rho_n)$$
$$\times \int_1^\infty d\zeta e^{-2\rho\beta\zeta}\left(1+\frac{1}{2\zeta^2}\right)\frac{\sqrt{\zeta^2-1}}{\zeta^2}$$

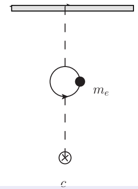
- **Numerically ($n = 2$)**

$$\Delta E_{a,b}(n=2,\ell=0) = -235.5288\ldots \text{ meV}$$

$$\Delta E_{a,b}(n=2,\ell=1) = -27.2887\ldots \text{ meV}$$

$$\Delta E_{a,b}(n=2,\ell=0) - \Delta E_{a,b}(n=2,\ell=1) = -208.2401\ldots \text{ meV}$$

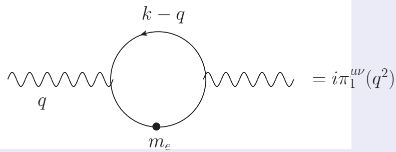
Scalar vertex $m_e\bar{\psi}_e\psi_e$ insertion in electron polarization loop



$$\Delta E_c(n,j,m) = 2e\int d^3r\,\psi^\dagger_{n,j,m}(\boldsymbol{r})\psi_{n,j,m}(\boldsymbol{r})A^0_m(\boldsymbol{r})$$

- **First we calculate polarization loop with scalar vertex insertion:**

$$i\pi_1^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\pi_1(q^2)$$



Scalar vertex $m_e\bar{\psi}_e\psi_e$ insertion in electron polarization loop

- **Polarization loop**

$$\pi_1(q^2) = \frac{\alpha}{3\pi}i + i\frac{e^2}{\pi^2}\left[m_e^2 \frac{1-\frac{4m_e^2\tanh^{-1}\left(\sqrt{\frac{\boldsymbol{q}^2}{4m_e^2+\boldsymbol{q}^2}}\right)}{\sqrt{\boldsymbol{q}^2(4m_e^2+\boldsymbol{q}^2)}}}{2\boldsymbol{q}^2} - \frac{1}{12}\right]$$

$$= \frac{\alpha}{3\pi}i + i\frac{e^2}{\pi^2}\widetilde{\pi}_1(-\boldsymbol{q}^2)$$

- **Radiatively corrected Coulomb field**

$$A_m^0(\boldsymbol{r}) = Ze\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{i\pi_1(-\boldsymbol{q}^2)}{\boldsymbol{q}^2}$$

$$= -Ze\int\frac{d^3q}{(2\pi)^3}\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{r}}}{\boldsymbol{q}^2}\left(\frac{\alpha}{3\pi} + \frac{e^2}{\pi^2}\widetilde{\pi}_1(-\boldsymbol{q}^2)\right)$$

$$= -\frac{Ze\alpha}{12\pi^2 r} - \frac{Ze^3}{\pi^2}\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{\widetilde{\pi}_1(-\boldsymbol{q}^2)}{\boldsymbol{q}^2}$$

$$\equiv A_{m1}^0(\boldsymbol{r}) + A_{m2}^0(\boldsymbol{r})$$

Scalar vertex $m_e\bar{\psi}_e\psi_e$ insertion in electron polarization loop

- **Energy shift**

$$\Delta E_c = 2e\int d^3r\psi^\dagger_{n,j,m}(\boldsymbol{r})\psi_{n,j,m}(\boldsymbol{r})A^0_{m1}(\boldsymbol{r}) + 2e\int d^3r\psi^\dagger_{n\ell}(\boldsymbol{r})\psi_{n\ell}(\boldsymbol{r})A^0_{m2}(\boldsymbol{r})$$
$$\equiv \Delta E_{c1} + \Delta E_{c2}$$

- **Consider first ($q = m_r Z\alpha\widetilde{q}$)**

$$\Delta E_{c2}(n\ell) = -32\alpha(Z\alpha)\int d^3r\psi^\dagger_{n\ell}\psi_{n\ell}(\boldsymbol{r})\int\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{\widetilde{\pi}_1(-\boldsymbol{q}^2)}{\boldsymbol{q}^2}$$

$$= -\frac{64\alpha(Z\alpha)^2 m_r}{\pi^2 n^3}\int d\rho\rho f^2_{n\ell}(\rho_n)\int_0^\infty d\widetilde{q}\frac{\sin(\widetilde{q}\rho)}{\widetilde{q}}\left[\beta^2\frac{1-\frac{4\beta^2\tanh^{-1}\left(\sqrt{\frac{\widetilde{\boldsymbol{q}}^2}{4\beta^2+\widetilde{\boldsymbol{q}}^2}}\right)}{\sqrt{\widetilde{\boldsymbol{q}}^2(4\beta^2+\widetilde{\boldsymbol{q}}^2)}}}{2\widetilde{\boldsymbol{q}}^2} - \frac{1}{12}\right]$$

Scalar vertex $m_e\bar{\psi}_e\psi_e$ insertion in electron polarization loop

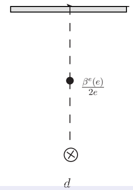
- **Numerically ($n = 2$)**

$$\Delta E_{c2}(n=2,\ell=0) = 251.4737\ldots \ \mathrm{meV}$$

$$\Delta E_{c2}(n=2,\ell=1) = 40.0008\ldots \ \mathrm{meV}$$

$$\Delta E_{c2}(n=2,\ell=0) - \Delta E^{e}_{c2}(n=2,\ell=1) = 211.4729\ldots \ \mathrm{meV}$$

Anomalous term $\frac{\beta_e(e)}{2e}F^2$ in the Coulomb line



- $\frac{\beta_e(e)}{2e}F^2 \Longrightarrow$ **two-prong vertex** $4\frac{\beta_e}{2e}(g_{\mu\nu}q^2 - q_\mu q_\nu)$ **in momentum space** $\Longrightarrow$ **insertion** $4\frac{\beta_e}{2e} = \frac{2\alpha}{3\pi}$ **in the Coulomb line**

Anomalous term $\frac{\beta_e(e)}{2e}F^2$ in the Coulomb line

$$\Longrightarrow \Delta E_d = \frac{\beta_e(e)}{2e}(\pi Z\alpha)\int d^3r\psi^\dagger_{njm}(\boldsymbol{r})\frac{1}{r}\psi_{njm}()r$$

$$= \frac{2\alpha(Z\alpha)}{3}\int d^3r\psi^\dagger_{njm}(\boldsymbol{r})\frac{1}{r}\psi_{njm}(\boldsymbol{r})$$

- **Recall ΔE_{c1} – matrix element of the leading term in low-momentum expansion of $\pi_1(q^2)$**

$$\Delta E_{c1} = -\frac{2\alpha(Z\alpha)}{3}\int d^3r\psi^\dagger_{njm}(\boldsymbol{r})\frac{1}{r}\psi_{njm}(\boldsymbol{r})$$

$$= -\frac{\beta_e(e)}{2e}(\pi Z\alpha)\int d^3r\psi^\dagger_{njm}(\boldsymbol{r})\frac{1}{r}\psi_{njm}(\boldsymbol{r})$$

$$\Longrightarrow \Delta E_d = -\Delta E_{c1}$$

Anomalous term cancels exactly!

- **Cancellation is not by chance: $\delta Z_3 = \Pi^{(2)}_{reg}(0)$ and $\pi_1(q^2)$ – logarithmic mass derivative of $\Pi^{(2)}_{reg}(q^2)$. But**

$$m\frac{d\delta Z_3}{dm} = -\mu\frac{d\delta Z_3}{d\mu} = \frac{2\beta(e)}{e} \approx \frac{2\alpha}{3\pi}$$

$\Longrightarrow$ **cancellation of anomalous contribution ΔE_d**

Sum of trace type diagrams

$$\Delta E(n=2,\ell=0) = 2\Delta E_{a,b}(n=2,\ell=0) + \Delta E_{c2}(n=2,\ell=0) = -219.5839\ldots \text{ meV}$$

$$\Delta E(n=2,\ell=1) = 2\Delta E_{a,b}(n=2,\ell 1) + \Delta E^{e}_{c2}(n=2,\ell=1) = -14.5765\ldots \text{ meV}$$

$$\Delta E(n=2,\ell=0) - \Delta E(n=2,\ell=1) = -205.0073\ldots \text{ meV}$$

- **Coincides with the classic result**

$$\Delta E^{e}_{VP}(n=2,\ell=0) - \Delta E^{e}_{VP}(n=2,\ell=1) = -205.0073\ldots \text{ meV}$$

Summary

- Multiscale bound state energy levels can be calculated as a matrix element of the EMT trace
- Perturbation theory diagrams for the trace do not coincide with the classic perturbation theory diagrams
- Equality of contributions of two type of diagrams was expected from the trace anomaly and Lorentz invariance
- Trace diagrams arise as logarithmic mass derivatives (with respect to all masses in the problem) of the classic perturbation theory diagrams
- Technically equality of contributions of two different sets of diagrams arises because energy levels are homogeneous functions of the first degree in masses

Thank you!