Relativistic nuclear recoil effects in hyperfine splitting of hydrogenic systems

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Motivation: hydrogen hyperfine splitting

- Hydrogen ground-state HFS is one of the most precisely measured atomic transitions

$$E_{\rm hfs}^{\rm exp} = 1\,420\,405.751\,768(1)\ \text{kHz} \qquad \text{[Essen } et\ al.\ \text{1971, 1973]}$$

- The proton size is 5 orders of magnitude smaller than the atom, yet it produces ~ -33 ppm shift in the HFS
- Standing puzzle: since the 1988 calculation of Bodwin & Yennie, theory–experiment disagreement has persisted at the several-σ level
- Our goal: recalculate the $(Z\alpha)^2\, m/M\, E_F$ recoil correction using newly introduced HPQED theory and compare with Bodwin & Yennie result
- Outcome: disagreement with Bodwin & Yennie – discrepancy reduced to $\sim 2\,\sigma$
- What this $\sim 2\,\sigma$ may come from ?

Why is the recoil so hard?

- Reduced mass accounts for finite nuclear mass in the Schrödinger equation
  $\rightarrow$ not the case for the Dirac equation
- One must use the full QED theory of a two-body bound state
- NRQED (Caswell–Lepage 1986): effective Lagrangian + dim. reg. – very fruitful
- Shabaev (1985–86): exact nonperturbative formula for the point nucleus leading m/M correction
- Recent development:
  - finite nuclear size [Pachucki & Yerokhin, PRL **130**, 053002 (2023)]
  - hyperfine splitting [Pachucki, PRA **109**, 052822 (2024)]
  - higher powers in m/M – HPQED
    [Pachucki & Yerokhin, PRA **110**, 032804 (2024)]

Leading order hyperfine splitting

For a static point nuclear magnetic moment $\vec{\mu}$,

$$V_{\rm hfs}(r) = \frac{e}{4\pi}\,\vec{\mu}\cdot\vec{\alpha}\times\frac{\vec{r}}{r^3}\,, \qquad E_{\rm hfs} = \langle V_{\rm hfs}\rangle$$

For S-states (Fermi energy):

$$\boxed{E^{(4)}_{\rm hfs} \equiv E_F = \frac{8}{3}\,\frac{(Z\,\alpha)^4}{n^3}\,\frac{\mu^3}{M\,m}\,\frac{g}{2}\,\langle\vec{I}\cdot\vec{s}\rangle}$$

with reduced mass μ, nuclear g-factor $\vec{\mu} = (q/2M)\,g\,\vec{I}$, and $q = -Z\,e$.

The next-to-leading hyperfine correction splits into

$$E^{(5)}_{\rm hfs} = E^{(5)}_{\rm fns} + E^{(5)}_{\rm rec} + E^{(5)}_{\rm pol}$$

- $E^{(5)}_{\rm fns}$: finite nuclear size – Zemach correction
- $E^{(5)}_{\rm rec}$: recoil from forward two-photon-exchange Born amplitude
- $E^{(5)}_{\rm pol}$: nuclear polarizability (inelastic, not discussed here)

Zemach correction and two-photon exchange

Finite-size part involves both electric and magnetic form factors:

$$E^{(5)}_{\rm fns} = \frac{2\,Z\,\alpha\,m}{\pi^2}\int \frac{d^3k}{k^4}\left[\frac{G_E(k^2)\,G_M(k^2)}{1+\kappa} - 1\right] E_F = -2\,Z\,\alpha\,m\,r_Z\,E_F$$

with the Zemach radius

$$r_Z = \int d^3r_1\,d^3r_2\,\rho_E(r_1)\,\rho_M(r_2)\,|\vec{r}_1 - \vec{r}_2|, \qquad r_Z(p) \approx 1.054(3)\,\mathrm{fm}$$

$E^{(5)}_{\rm rec}$ comes from the two-photon exchange forward amplitude in the temporal gauge

$$E^{(5)}_{\rm hfs} = \frac{i}{2}\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{(\omega^2-k^2)^2}\left(\delta^{ik} - \frac{k^i k^k}{\omega^2}\right)\left(\delta^{jl} - \frac{k^j k^l}{\omega^2}\right) t^{ji}\,T^{kl}\,\phi^2(0)$$

Two ingredients:

- lepton tensor t^{ji} (point spin-$1/2$, computed analytically)
- nuclear Compton tensor T^{kl} – expressed in G_E, G_M in elastic approximation

Choosing the lepton mass m (not μ) in $E^{(5)}_{\rm fns}$ makes the remainder $E^{(5)}_{\rm rec}$ free of Zemach-like terms.

Leading recoil $E^{(5)}_{\rm rec}$: dipole parametrization

Using a dipole parametrization $\rho(Q^2) = \Lambda^4/(\Lambda^2 + Q^2)^2$ and leading order in m/Λ:

$$E^{(5)}_{\rm rec} = -\frac{Z\,\alpha}{\pi}\,\frac{m}{M}\,E_F\,\frac{2}{g}\left[\frac{g^2}{32} - \frac{25\,g}{8} - \frac{9}{8} + \left(\frac{3g^2}{16} - \frac{3g}{4} - \frac{9}{4}\right)\ln\left(\frac{m}{\Lambda}\right) + \mathcal{O}\left(\frac{m}{\Lambda}\right)^2\right]$$

Agrees with previous calculation, but

- no m/Λ term $\Rightarrow$ no r_Z in the expansion (consistent with using m in $E^{(5)}_{\rm fns}$)
- presence of $\ln\Lambda$ indicates a large momentum contribution
- for hydrogen, $\Lambda \sim 840$ MeV is not much smaller than M_p

  $\Rightarrow$ in hydrogen, $E^{(5)}_{\rm rec}$ should be computed without large-M expansion (Antognini, Lin, Meißner, PLB 835, 137575 (2022))

For the next order, $E^{(6)}_{\rm rec} \sim (Z\,\alpha)^2 m/M\,E_F$, the dominant momentum is $\sim m$
$\Rightarrow$ point-nucleus expansion is justified.

HPQED: definitions

Let us introduce the following notation:

- Dirac-Coulomb Green function

$$G(E) = [E - H_D]^{-1}$$

- Photon propagator in the temporal gauge

$$G_T^{ij}(\omega, \vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \frac{\rho(\vec{k}^2 - \omega^2)}{\omega^2 - \vec{k}^2} \left(\delta^{ij} - \frac{k^i\, k^j}{\omega^2}\right).$$

- $D^j(\omega) = -4\pi Z\alpha\, \alpha^i\, G_T^{ij}(\omega, \vec{r})$,

- Frequency dependent hyperfine interaction

$$V_{\rm hfs}(\omega, \vec{r}) = e\, \vec{\mu} \cdot \vec{\alpha} \times \vec{\nabla} D(\omega, r)\,,$$

  such that $V_{\rm hfs}(0, r) = V_{\rm hfs}(r)$,

- scalar propagator

$$D(\omega, r) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \frac{\rho(\vec{k}^2 - \omega^2)}{\omega^2 - \vec{k}^2}\,.$$

HPQED master formula

The exact HPQED formula for the relativistic recoil correction to the HFS in hydrogenic ions [K. Pachucki, PRA 109, 052822 (2024)]:

$$\boxed{E_{\rm hfsrec} = E_{\rm kin} + E_{\rm so} + E_{\rm sec}}$$

$$E_{\rm sec} = \left(\frac{4\,\pi\,Z\,\alpha}{2\,M}\,g\right)^2 \epsilon^{ijk}\,I^k \int \frac{d\,\omega}{2\,\pi}\,\frac{1}{\omega}\,\langle\phi|(\vec\alpha\times\vec\nabla)^i\,D(\omega)\,G(E_D+\omega)\,(\vec\alpha\times\vec\nabla)^j\,D(\omega)\,|\phi\rangle\,,$$

$$E_{\rm so} = -\,\frac{(g-1)}{M^2}\,\epsilon^{ijk}\,I^i \int \frac{d\,\omega}{2\,\pi}\,\omega\,\langle\phi|D^j_T(\omega)\,G(E_D+\omega)\,D^k_T(\omega)|\phi\rangle\,,$$

$$E_{\rm kin} = -\,\delta_{\rm hfs}\,\frac{i}{M}\int\frac{d\omega}{2\pi}\,\frac{1}{\omega^2}\,\langle\phi|\,[p^j(V_{\rm hfs}(\omega)) - \omega D^j_T(\omega)]G(E_D+\omega)[p^j(V_{\rm hfs}(\omega)) + \omega D^j_T(\omega)]\,|\phi\rangle$$

They come from:

$$H_{\rm nuc} = \frac{\vec\Pi^2}{2\,M} - \frac{q}{2\,M}\,g\,\vec I\cdot\vec B - \frac{q}{4\,M^2}\,(g-1)\,\vec I\cdot[\vec E\times\vec\Pi - \vec\Pi\times\vec E]$$

$(Z\alpha)^6\, m$ relativistic recoil correction

$$E^{(6)}_{\rm sec} = (Z\alpha)^6\, \frac{m^3}{M^2}\, g^2\, \vec{I}\cdot\vec{s}\, \frac{2}{3n^3}\Big[\frac{1}{4}\ln 2 + \frac{31}{36} + \Big(\frac{9}{8n} - \frac{1}{16n^2} - \frac{17}{16}\Big) - \frac{7}{4}\big(\gamma + \Psi(n) - \ln n + \ln(Z\alpha)\big)\Big]$$

$$E^{(6)}_{\rm so} = (Z\alpha)^6\, \frac{m^3}{M^2}\, (g-1)\, \vec{I}\cdot\vec{s}\, \frac{8}{3n^3}\Big[\frac{23}{4}\ln 2 - \frac{15}{4} + \Big(\frac{7}{8n} + \frac{11}{48n^2} - \frac{53}{48}\Big) + \frac{7}{4}\big(\gamma + \Psi(n) - \ln n + \ln(Z\alpha)\big)\Big]$$

$$E^{(6)}_{\rm kin} = (Z\alpha)^6\, \frac{m^3}{M^2}\, g\, \vec{I}\cdot\vec{s}\, \frac{4}{3n^3}\Big[-14\ln 2 + \frac{13}{2} + \Big(-\frac{1}{n} + \frac{7}{6n^2} - \frac{1}{6}\Big) - 2\big(\gamma + \Psi(n) - \ln n + \ln(Z\alpha)\big)\Big]$$

For the hydrogen $1S$ ground state, in terms of $\kappa = g/2 - 1$:

$$\boxed{E^{(6)}_{\rm rec}(1S) = \Big[\frac{65}{18} + \frac{13}{18}\kappa + \frac{31}{36}\kappa^2 - \Big(8 + 2\kappa - \frac{1}{4}\kappa^2\Big)\ln 2 - \Big(2 + 2\kappa + \frac{7}{4}\kappa^2\Big)\ln(Z\alpha)\Big]\,(Z\alpha)^2\,\frac{m}{M}\,\frac{E_F}{1+\kappa}}$$

Comparison with Bodwin & Yennie (1988)

Our new result for $1S$

$$E^{(6)}_{\rm rec}(1S) = \left[\frac{65}{18} + \frac{13}{18}\kappa + \frac{31}{36}\kappa^2 - \left(8 + 2\kappa - \frac{1}{4}\kappa^2\right)\ln 2 - \left(2 + 2\kappa + \frac{7}{4}\kappa^2\right)\ln(Z\alpha)\right](Z\alpha)^2\,\frac{m}{M}\,\frac{E_F}{1+\kappa}$$

Bodwin & Yennie [Phys. Rev. D **37**, 498 (1988)]:

$$E^{(6)}_{\rm BY}(1S) = \left[\frac{65}{18} + \frac{43}{72}\kappa + \frac{31}{36}\kappa^2 - \left(8 + 5\kappa + \frac{11}{4}\kappa^2\right)\ln 2 - \left(2 + 2\kappa + \frac{7}{4}\kappa^2\right)\ln(Z\alpha)\right](Z\alpha)^2\,\frac{m}{M}\,\frac{E_F}{1+\kappa}$$

- Differences are only in the κ and κ^2 coefficients
- For $g = 2$ ($\kappa = 0$) the two formulas agree – as required by independent checks for muonium
- Numerically for the proton ($\kappa \approx 1.793$):

$$\delta^{(2)}_{\rm rel,rec}(\rm new) = 0.575\ \rm ppm \quad vs. \quad \delta^{(2)}_{\rm rel,rec}(\rm BY) = 0.464\ \rm ppm$$

- Could not identify the error in Bodwin & Yennie, but $E_{\rm sec}$ verified by direct numerical evaluation of the HPQED formula [Hevler, in preparation]

Updated theory of hydrogen ground-state HFS

$$E_{\text{hfs}} = E_F(1+\delta), \quad E_F = 1\,418\,840.091 \text{ kHz}, \quad E_{\text{hfs}}^{\text{exp}} = 1\,420\,405.751\,766(3) \text{ kHz}$$

Term	Value	Ref. & comments
a_e	0.001 159 652	CODATA22
$\delta^{(2)}$	−0.000 016 340	Eides
$\delta^{(3)}$	−0.000 007 099	Eides, Patkos et al.
$\delta^{(4)}$	−0.000 000 121	Eides, Yerokhin
$\delta^{(1)}_{\text{fns}}$	−0.000 039 835(113)	Zemach, Lin et al. 2022
$\delta^{(1)}_{\text{rec}}$	0.000 005 291(17)	Antognini, Lin, Meißner 2022
$\delta^{(1)}_{\text{pol}}$	0.000 001 090(310)	Ruth et al. 2024
$\delta^{(2+)}_{\text{rel,fns}}$	−0.000 000 029	this work
$\delta^{(2)}_{\text{rad,fns}}$	−0.000 000 609	Karshenboim 1997
$\delta^{(2)}_{\text{rel,rec}}$	**0.000 000 575**	this work (vs BY: 0.464 ppm)
$\delta^{(2)}_{\text{rad,rec}}$	0.000 000 072	Maroń, Pachucki (2026)
$\delta^{(2)}_{\mu\text{vp}}$	0.000 000 072	Karshenboim 1997
$\delta^{(2)}_{\text{hvp}}$	0.000 000 061	Hagelstein et al. (2026)
δ_{weak}	0.000 000 058	Eides 1996
δ_{theo}	0.001 102 838(330)	this work
δ_{exp}	0.001 103 480	
$\delta_{\text{exp}} - \delta_{\text{theo}}$	0.000 000 642(330)	2 σ discrepancy

2σ discrepancy: pointing at the proton structure

- Sum of all QED and recoil contributions:

$$\delta_{\rm theo} = 0.001\,102\,838(330)$$

vs experiment $\delta_{\rm exp} = 0.001\,103\,480$ – $2\,\sigma$ difference

- The discrepancy is dominated by uncertainty in nuclear-structure part $\delta^{(1)}_{\rm nuc} = \delta^{(1)}_{\rm fns} + \delta^{(1)}_{\rm rec} + \delta^{(1)}_{\rm pol}$:

$$\delta^{(1)}_{\rm nuc}(\text{theory}) = -33.45(33)\ \text{ppm}, \quad \delta^{(1)}_{\rm nuc}(\text{exp.}) = -32.81\ \text{ppm}$$

$\Rightarrow$ likely originates from the proton polarizability $\delta^{(1)}_{\rm pol}$

- Pure-QED side now appears under control at the required level

State dependence: specific difference D_{21} introduced by SGK

$D_{21} \equiv 8\,E_{\rm hfs}(2S) - E_{\rm hfs}(1S)$ – proton structure largely cancels.

Our result:

$$\delta D_{21}(\text{new}) = (Z\,\alpha)^2 \,\frac{m}{M}\, E_F \left[2\ln 2 - \frac{19}{8} + \left(-\frac{7}{2}\ln 2 + \frac{73}{32}\right)\frac{g-1}{g} + \left(\frac{7}{8}\ln 2 - \frac{145}{128}\right) g\right]$$

differs from previous result [Karshenboim, Ivanov, PLB 524, 259 (2002)] by

$$\Delta_{21} = (Z\,\alpha)^2 \,\frac{m}{M}\, E_F \left(\frac{5}{4} - 2\ln 2\right)\frac{g-2}{g}$$

D_{21} [Hz]	H	He^+
$D_{21}^{\rm the}(\text{old})$	48 954.1(2.3)	−1 190 068.(64)
Δ_{21}	−3.6	60.
$D_{21}^{\rm the}$	48 950.5(2.3)	−1 190 008.(64)
$D_{21}^{\rm exp}$	48 959.2(6.8)	−1 189 979.(71)
$D_{21}^{\rm exp} - D_{21}^{\rm the}$	8.7(7.2)	29.(96)

He^+: excellent agreement – a sensitive test of long-range spin-dependent interactions.
H: improved $E_{\rm hfs}(2S)$ measurement would be welcome.

Summary and conclusions

- Recalculated $(Z\alpha)^2\, m/M\, E_F$ relativistic recoil correction to HFS in hydrogenic systems using HPQED theory
- Result for $1S$ disagrees with Bodwin & Yennie (1988) in the κ, κ^2 coefficients of $\ln 2$ and the constant piece; agrees in the limit $g = 2$
- $E_{\rm sec}$ and $E_{\rm so}$ verified numerically from the exact HPQED formula
- Hydrogen ground-state HFS: theory–experiment discrepancy reduced to $\sim 2\,\sigma$, likely from $\delta^{(1)}_{\rm pol}$ (proton polarizability)
- Specific difference D_{21}: excellent agreement for He^+; for H, a new measurement of $E_{\rm hfs}(2S)$ would be valuable
- Upcoming μH HFS measurements (CREMA, FAMU) + the H/μH Δ test will sharpen the test of fundamental interactions
- Next step: direct numerical evaluation of full HPQED recoil HFS to all orders in $Z\alpha$, including finite nuclear size – in progress [Hevler, in preparation]

Thank you for your attention!