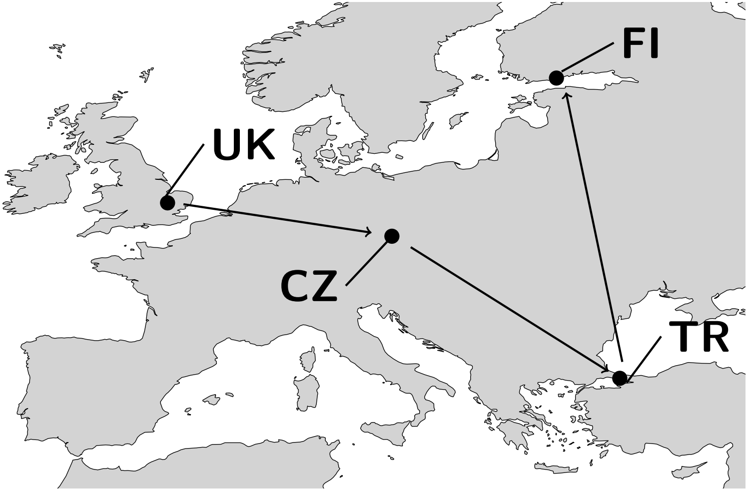
Next-generation model-building for torsion and non-metricity

Will Barker

Ongoing work with **Will Handley**, **Mike Hobson**,
Anthony Lasenby, **Carlo Marzo** and **Alessandro Santoni**

Origins

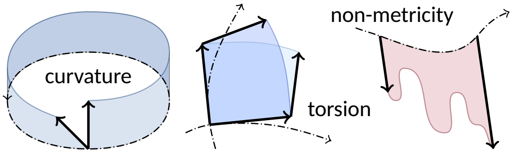


Geometric foundations

- Popular class of models from spacetime geometry **beyond curvature**:

$$\mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} \equiv 2\left(\partial_{[\mu}\Gamma_{\nu]}{}^{\rho}{}_{\sigma} + \Gamma_{[\mu|}{}^{\rho}{}_{\alpha}\Gamma_{|\nu]}{}^{\alpha}{}_{\sigma}\right)$$
$$\mathcal{T}_{\mu}{}^{\alpha}{}_{\nu} \equiv 2\Gamma_{[\mu|}{}^{\alpha}{}_{|\nu]}$$
$$\mathcal{Q}_{\lambda\mu\nu} \equiv -\partial_{\lambda} g_{\mu\nu} + 2\Gamma_{\lambda}{}^{\alpha}{}_{(\mu|} g_{\alpha|\nu)}$$

- Not a bad idea; inspired by Einstein's approach



- Capture this with a simple claim:

  "***Some fields $\mathcal{T}_{\mu}{}^{\alpha}{}_{\nu}$ and/or $\mathcal{Q}_{\lambda\mu\nu}$ have a dynamical role in our Universe.***"

- We only need two things to test this claim: **QFT** and **observational data**
- Want to set out a framework for **how**

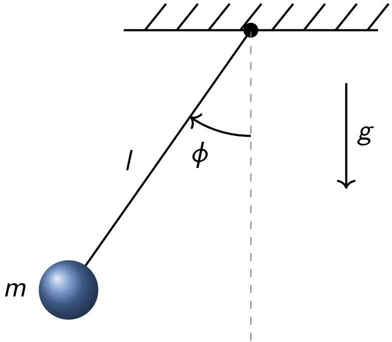
The action as a universal language

- Physical theories have an **action** $\mathcal{S}[\phi]$
- Functional of **fields** ϕ
- Function of the **couplings** θ
- Example in the simple pendulum:

$$\mathcal{S}[\phi] = \int \mathrm{d}t \left[\underbrace{\tfrac{1}{2}\, m\, l^2\, \dot{\phi}^2}_{\text{kinetic}} - \underbrace{m\, g\, l\, (1 - \cos\phi)}_{\text{potential}} \right]$$

$$= \int \mathrm{d}t \left[\underbrace{\theta_1}_{\frac{1}{2} m l^2} \underbrace{\dot{\phi}^2}_{\mathcal{O}^1} + \underbrace{\theta_2}_{-\frac{1}{2} m g l} \underbrace{\phi^2}_{\mathcal{O}^2} + \underbrace{\theta_3}_{\frac{1}{24} m g l} \underbrace{\phi^4}_{\mathcal{O}^3} + \cdots \right]$$

$$= \int \mathrm{d}t \sum_i \theta_i\, \mathcal{O}^i$$



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$$= \int \mathrm{d}t \sum_i \theta_i\, \mathcal{O}^i$$

- Now consider paradigm shift:

$$\mathcal{S}[\phi] \longrightarrow \mathcal{S}(\theta)$$

- Theories ranked by **Bayesian evidence**:

$$\mathcal{Z}(D) = \int \mathrm{d}\theta\, \mathcal{L}(D|\theta)\, \pi(\theta)$$

- **Likelihood** $\mathcal{L}(D|\theta)$ of data D given $\mathcal{S}(\theta)$
- **Prior** $\pi(\theta)$ on the couplings
- How to compute the prior?

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$$= \int \mathrm{d}t \sum_i \theta_i\, \mathcal{O}^i$$

- First two terms capture essential behaviour
- Inverted pendulum means $\theta_2 > 0$ — **tachyon**
- Negative kinetic energy means $\theta_1 < 0$ — **ghost**
- **unitarity conditions** are $\theta_1 > 0$ and $\theta_2 < 0$
- Study of unitarity conditions is called **polology**

Polology

- Think of the Lagrangian in **momentum space** representation of action:

$$\mathcal{L}(\phi;\,\theta) = \phi^\dagger(k) \cdot \mathsf{O}(\theta;\,k) \cdot \phi(k) + \mathcal{O}(\phi^3)$$

- Call $\mathsf{O}(\theta;\,k)$ the **wave operator**, then $\Pi(\theta;\,k) \equiv \mathsf{O}^{-1}(\theta;\,k)$ is the **propagator**

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- **Massive states** have **spin** and **parity**
- For a **scalar** field, a **vector** field, and a **symmetric tensor** field, the basis is:

$$\phi\sim 0^+,\quad \mathcal{A}_\mu\sim 1^-\oplus 0^+,$$
$$\mathcal{H}_{\mu\nu}\sim 2^+\oplus 1^-\oplus 0^+\oplus 0^+$$

- E.g. for a **rank-three** field:

$$\mathcal{C}_{\mu\nu\rho}\sim 3^-\oplus 2^+\oplus 2^+\oplus 2^+\oplus 2^-$$
$$\oplus 2^-\oplus 1^+\oplus 1^+\oplus 1^+\oplus 1^-$$
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- Wave operator may be **singular**:

$$\mathrm{O}(\theta;\ k)\cdot \mathrm{v}(\theta;\ k) = 0$$

- Interpret as **gauge symmetries**:

$$\delta\phi \propto \mathrm{v} \implies \mathcal{L}(\phi+\delta\phi;\ \theta) = \mathcal{L}(\phi;\ \theta)$$

- Can invert **physical part** of $\mathrm{O}(\theta;\ k)$:

$$\underbrace{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathrm{O}} \to \underbrace{\frac{1}{ad-bc}\begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathrm{O}^+}$$

- (**Moore–Penrose pseudoinverse**)

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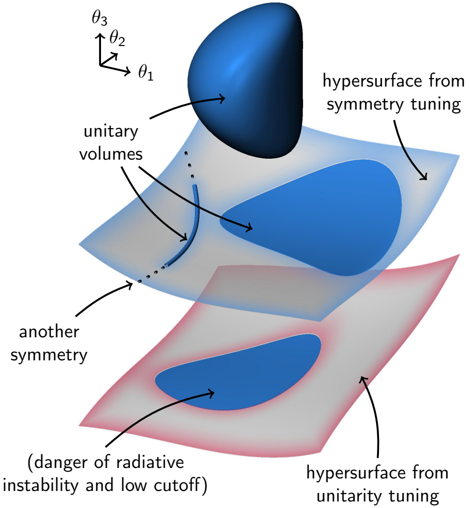

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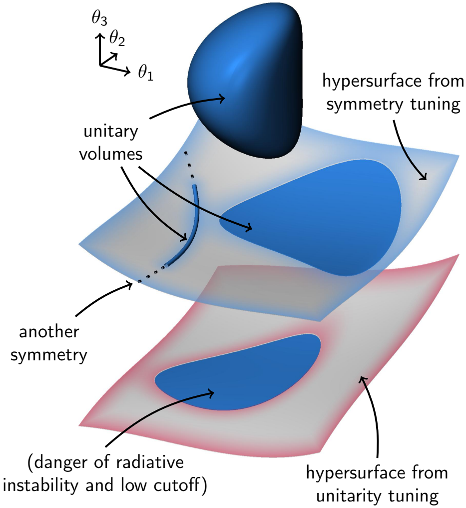
- Null vectors as **gauge symmetries**
- Symmetries protect against radiative corrections, and guide the EFT



Symbolic polology: software

- Polology can be done with **software**
- You get some crucial **science products** (symmetries, unitarity, masses)
- E.g. *PSALTer* by Marzo, Karananas, Rigouzzo and Tu:

PSALTer
Particle Spectrum for Any Tensor Lagrangian

- See also *Kummitus* by Marzo:

	$\phi^{\#1}_{0^+}$	$h^{\#1}_{0^+}$	$h^{\#2}_{0^+}$	$\mathcal{A}^{\#1}_{0^+}$
$\phi^{\#1}_{0^+}\dagger$	0	$\frac{1}{2}\sqrt{3}\,k^2\,\theta_6$	$-\frac{Y_1\,k^2}{2}$	0
$h^{\#1}_{0^+}\dagger$	$\frac{1}{2}\sqrt{3}\,k^2\,\theta_6$	$\frac{1}{2}(Y_2+Y_3\,k^2)$	$\frac{1}{2}(\sqrt{3}\,Y_4\,k^2+2\sqrt{3}\,\theta_5)$	$\frac{1}{2}\,i\sqrt{3}\,k\,\theta_6$
$h^{\#2}_{0^+}\dagger$	$-\frac{Y_1\,k^2}{2}$	$\frac{1}{2}(\sqrt{3}\,Y_4\,k^2+2\sqrt{3}\,\theta_5)$	$\frac{1}{2}(Y_5+Y_6\,k^2)$	$-\frac{1}{2}\,i\,Y_1\,k$
$\mathcal{A}^{\#1}_{0^+}\dagger$	0	$-\frac{1}{2}\,i\sqrt{3}\,k\,\theta_6$	$\frac{i\,Y_1\,k}{2}$	0

	$h^{\#1}_{1^-\,\alpha}$	$\mathcal{A}^{\#1}_{1^-\,\alpha}$
$h^{\#1}_{1^-}\dagger^{\alpha}$	$\frac{\theta_1}{2}$	$-\frac{i\,\theta_1\,k}{\sqrt{2}}$
$\mathcal{A}^{\#1}_{1^-}\dagger^{\alpha}$	$\frac{i\,\theta_1\,k}{\sqrt{2}}$	$\theta_1\,k^2$

	$h^{\#1}_{2^+\,\alpha\beta}$
$h^{\#1}_{2^+}\dagger^{\alpha\beta}$	$\frac{1}{2}(\theta_1-k^2\,\theta_3)$

Abbreviations used in matrices

$Y_1 == 2\,\theta_1-\theta_6\ \&\&\ Y_2 == \theta_1+6\,\theta_5\ \&\&\ Y_3 == 6\,\theta_2-\theta_3\ \&\&$
$Y_4 == 2\,\theta_2+\theta_4\ \&\&\ Y_5 == \theta_1+2\,\theta_5\ \&\&\ Y_6 == 2\,\theta_2+\theta_3+2\,\theta_4\ \&\&$
$\mathrm{Det}(0^+) == -\frac{1}{4}k^2(1+k^2)(2\,\theta_1{}^3+2\,\theta_1{}^2(k^2(6\,\theta_2-\theta_3)+6\,\theta_5-\theta_6)+k^2\,\theta_3\,\theta_6{}^2+2\,\theta_1\,\theta_6(k^2(\theta_3+3\,\theta_4)+\theta_6))\ \&\&$
$\mathrm{Det}(2^+) == \frac{1}{2}(\theta_1-k^2\,\theta_3)$

Lagrangian

$$\frac{1}{2}\theta_1\,h_{\alpha\beta}\,h^{\alpha\beta}+\theta_5\,h^{\alpha}{}_{\alpha}\,h^{\beta}{}_{\beta}+\theta_6\,h^{\beta}{}_{\beta}\,\partial_\alpha\mathcal{A}^\alpha-\theta_6\,h^{\beta}{}_{\beta}\,\partial_\alpha\partial^\alpha\phi-\theta_1\,\partial_\alpha\mathcal{A}^\alpha\,\partial_\beta\mathcal{A}^\beta+2\,\theta_1\,h^{\alpha\beta}\,\partial_\beta\partial_\alpha\phi-$$
$$2\,\theta_1\,h_{\alpha\beta}\,\partial^\beta\mathcal{A}^\alpha+\theta_1\,\partial_\beta\mathcal{A}_\alpha\,\partial^\beta\mathcal{A}^\alpha+\theta_2\,\partial_\beta h^{\chi}{}_{\chi}\,\partial^\beta h^{\alpha}{}_{\alpha}+\theta_3\,\partial_\alpha h^{\alpha\beta}\,\partial_\chi h_\beta{}^\chi+\theta_4\,\partial^\beta h^{\alpha}{}_{\alpha}\,\partial_\chi h_\beta{}^\chi-\frac{1}{2}\theta_3\,\partial_\chi h_{\alpha\beta}\,\partial^\chi h^{\alpha\beta}$$

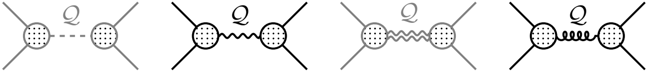
Added source term(s):	$\phi\,\rho+\mathcal{A}^\alpha\,\mathcal{J}_\alpha+h^{\alpha\beta}\,\mathcal{T}_{\alpha\beta}$		
Source constraint(s)	# constraint(s)	Covariant form	
$i\,\rho^{\#1}_{0^+}+k\,\mathcal{J}^{\#1}_{0^+} == 0$	1	$\rho == \partial_\alpha\mathcal{J}^\alpha$	
$\mathcal{J}^{\#1\alpha}_{1^-}-2\,i\,k\,\mathcal{T}^{\#1\alpha}_{1^-} == 0$	3	$\partial_\beta\partial^\alpha\mathcal{J}^\beta+2\,\partial_\chi\partial^\chi\partial_\beta\mathcal{T}^{\alpha\beta} == \partial_\beta\partial^\beta\mathcal{J}^\alpha+2\,\partial_\chi\partial_\beta\partial^\alpha\mathcal{T}^{\beta\chi}$	
Total # constraint(s):	4		
Resolved pole(s)	# polarization(s)	Square mass	Residue
$J^P=0^+$	1	$-\frac{2\,\theta_1(\theta_1{}^2+6\,\theta_1\,\theta_5-\theta_1\,\theta_6+\theta_6{}^2)}{2\,\theta_1{}^2(6\,\theta_2-\theta_3)+2\,\theta_1(\theta_3+3\,\theta_4)\,\theta_6+\theta_3\,\theta_6{}^2}$	(Hidden for brevity)
$J^P=2^+$	5	$\frac{\theta_1}{\theta_3}$	$-\frac{2}{\theta_3}$
Resolved unitarity condition(s):		$\theta_1<0\ \&\&\ \theta_3<0\ \&\&\ \theta_5<\frac{-\theta_1{}^2+\theta_1\,\theta_6-\theta_6{}^2}{6\,\theta_1}\ \&\&\ \theta_2>\frac{2\,\theta_1{}^2\,\theta_3-2\,\theta_1\,\theta_3\,\theta_6-6\,\theta_1\,\theta_4\,\theta_6-\theta_3\,\theta_6{}^2}{12\,\theta_1{}^2}$	

Symbolic pology: torsion and non-metricity

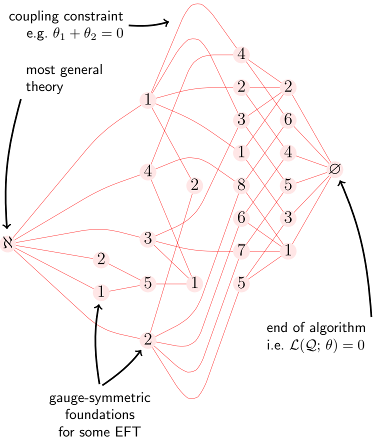
- E.g. build a theory from **non-metricity** $\mathcal{Q}_{\alpha\beta\gamma}$
- General $\mathcal{Q}_{\alpha\beta\gamma}$ theory of gravity:

$$\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[\theta_1 \mathcal{Q}_{\alpha\beta\chi}\mathcal{Q}^{\alpha\beta\chi} + \theta_2 \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta}\mathcal{Q}_{\beta}{}^{\chi}{}_{\chi} + \theta_3\partial_\beta \mathcal{Q}_{\chi}{}^{\delta}{}_{\delta}\partial^{\chi}\mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_4\partial_\chi \mathcal{Q}_{\beta}{}^{\delta}{}_{\delta}\partial^{\chi}\mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_5\partial_\alpha \mathcal{Q}^{\alpha\beta\chi}\partial_\delta \mathcal{Q}_{\beta\chi}{}^{\delta} + \theta_6\partial^\chi \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta \mathcal{Q}_{\beta\chi}{}^{\delta} + \theta_7\partial_\delta \mathcal{Q}_{\alpha\beta\chi}\partial^{\delta}\mathcal{Q}^{\alpha\beta\chi}\Big]$$

- Possible **non-metricity** particles:



✗ spin-zero ✓ spin-one ✗ spin-two ✓ spin-three



- Santoni and Marzo 2506.21662, 2505.23894

Symbolic polology: torsion and non-metricity

- E.g. build a theory from **torsion** $\mathcal{T}_{\alpha\beta\gamma}$
- General $\mathcal{T}_{\alpha\beta\gamma}$ field theory:

$$
\begin{aligned}
\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[& \theta_1 \mathcal{T}_{\alpha\beta\chi}\mathcal{T}^{\alpha\beta\chi} + \theta_2 \mathcal{T}_{\alpha\beta\chi}\mathcal{T}^{\beta\alpha\chi} + \theta_3 \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\mathcal{T}^{\chi}{}_{\beta\chi} \\
& + \theta_4 \partial_\beta \mathcal{T}^{\delta}{}_{\chi\delta}\,\partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_5 \partial_\chi \mathcal{T}^{\delta}{}_{\beta\delta}\,\partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_6 \partial_\beta \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\alpha\chi}{}^{\delta} \\
& + \theta_7 \partial_\alpha \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\beta\chi}{}^{\delta} + \theta_8 \partial_\beta \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\chi\alpha}{}^{\delta} + \theta_9 \partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta \mathcal{T}_{\chi\beta}{}^{\delta} \\
& + \theta_{10} \partial_\alpha \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}^{\delta}{}_{\beta\chi} + \theta_{11} \partial_\delta \mathcal{T}_{\alpha\beta\chi}\,\partial^\delta \mathcal{T}^{\alpha\beta\chi} + \theta_{12} \partial_\delta \mathcal{T}_{\beta\alpha\chi}\,\partial^\delta \mathcal{T}^{\alpha\beta\chi} \Big]
\end{aligned}
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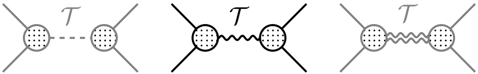

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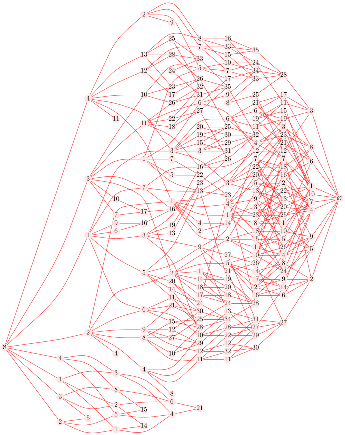
- Santoni and Marzo 2507.05349



Numerical polology: the motivation

Symbolic polology

- Closed-form analytical results
- Needs a *Mathematica* licence
- Very slow, scales badly, CPU-bound

Numerical polology

- Observational data already numerical
- Completely open-source
- Very fast, scales well, GPU-portable

Numerical polology: the ‘tuned’ algorithm

- How much physics can be obtained **numerically** per local θ-sample?

Numerical polology: the 'tuned' algorithm

- How much physics can be obtained **numerically** per local θ-sample?

- Assume poly expansion per J^P block, solve $\mathrm{O}_{J^P}(\theta;k)\cdot \mathrm{v}_{a_{J^P}}(\theta;k)=0$ for $\mathrm{v}_{a_{J^P}}(\theta;k)$ order-by-order in k:

$$\mathrm{O}_{J^P}(\theta;k)=\sum_n \mathrm{O}^{(n)}_{J^P}(\theta)k^n,\quad \mathrm{v}_{a_{J^P}}(\theta;k)=\sum_n \mathrm{v}^{(n)}_{a_{J^P}}(\theta)k^n$$

- Build gauge-regularised wave operator:

$$\mathrm{O}^{\mathrm{reg}}_{J^P}(\theta;k)\equiv \mathrm{O}_{J^P}(\theta;k)+\sum_a \mathrm{v}_{a_{J^P}}(\theta;k)\cdot \mathrm{v}^{\dagger}_{a_{J^P}}(\theta;k)$$

- Expect $\det \mathrm{O}^{\mathrm{reg}}_{J^P}(\theta;k)\sim\prod\left(k^2-m(\theta)^2\right)$
- I.e. a **poly eigenvalue problem** in k^2
- Numerically require $m(\theta)^2>0$

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- How much physics can be obtained **numerically** per local θ-sample?
- All **masses** and **tachyon diagnostics**

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Numerical pology: the 'tuned' algorithm

- How much physics can be obtained **numerically** per local θ-sample?
- All **masses** and **tachyon diagnostics**

- We usually use the **pole residue**:

$$\operatorname*{Res}_{k^2\mapsto m(\theta)^2}\left(P\,\mathrm{tr}\,\mathrm{O}^{+}_{J^P}(\theta;k)\right)>0$$

- Build matrix $\mathrm{V}^{m(\theta)}_{J^P}$ whose columns are the $\mathrm{v}_{a_{J^P}}(\theta;m(\theta))$, then:

$$\mathrm{w}^{m(\theta)}_{s_{J^P}}\equiv\left[1-\mathrm{V}^{m(\theta)}_{J^P}\left(\mathrm{V}^{m(\theta)\dagger}_{J^P}\mathrm{V}^{m(\theta)}_{J^P}\right)^{-1}\mathrm{V}^{m(\theta)\dagger}_{J^P}\right]\cdot\mathrm{u}^{m(\theta)}_{s_{J^P}}$$

- Closed-form numerical expression for the residue itself:

$$Z_{s_{J^P}}(\theta)=2P\left[\sum_n n\left|m_{s_{J^P}}(\theta)\right|^{n-2}\mathrm{w}^{m(\theta)\dagger}_{s_{J^P}}\cdot\mathrm{O}^{(n)}_{J^P}(\theta)\cdot\mathrm{w}^{m(\theta)}_{s_{J^P}}\right]^{-1}$$

Numerical polology: the 'tuned' algorithm

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- All **masses** and **tachyon diagnostics**
- All **ghost diagnostics**

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$$\mathrm{w}^{m(\theta)}_{s_{J^P}}\equiv\left[1-\mathrm{V}^{m(\theta)}_{J^P}\left(\mathrm{V}^{m(\theta)\dagger}_{J^P}\mathrm{V}^{m(\theta)}_{J^P}\right)^{-1}\mathrm{V}^{m(\theta)\dagger}_{J^P}\right]\cdot\mathrm{u}^{m(\theta)}_{s_{J^P}}$$

- Closed-form numerical expression for the residue itself:

$$Z_{s_{J^P}}(\theta)=2P\left[\sum_n n\left|m_{s_{J^P}}(\theta)\right|^{n-2}\mathrm{w}^{m(\theta)\dagger}_{s_{J^P}}\cdot\mathrm{O}^{(n)}_{J^P}(\theta)\cdot\mathrm{w}^{m(\theta)}_{s_{J^P}}\right]^{-1}$$

Numerical polology: the 'tuned' algorithm

- How much physics can be obtained **numerically** per local θ-sample?
- All **masses** and **tachyon diagnostics**
- All **ghost diagnostics**

- The **mass dimension** of θ_i cannot be asserted ab initio!
- The particle spectrum drives the θ_i dimensions
- Let $[\theta_i] = d_i$, with $[k] = [m(\theta)] = 1$. Rescaling the mass unit:

$$\sum_i d_i \frac{\partial \ln m(\theta)}{\partial \ln \theta_i} = 1$$

- Seek integer d_i satisfying this for all $m(\theta)$
- Fix $\min_i d_i = 0$

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- Unitarity is boolean; soften with a 'fuzzy' penalty $\mathcal{U}$ on the sick side only:

$$\log \mathcal{L}(\text{QFT}|\theta) = \sum_{J,P} \sum_{s_{JP}} \Big[\mathcal{U}\Big(Z_{s_{JP}}(\theta) \Big) + \mathcal{U}\Big(- \Big| \Im m_{s_{JP}}(\theta) \Big| \Big) \Big]$$

- Plateau at zero in the healthy region, smoothly negative in the sick region even when singular:

$$\mathcal{U}(x) \equiv \begin{cases} 0, & x > 0, \\ -\dfrac{2\,|x|^u}{1+|x|^{2u}}, & x \leq 0 \end{cases}$$

- Parameter u sets the compression scale

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<barker@fzu.cz>

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- Null vectors or **vanishing eigenvalues** of $\mathrm{O}_{J^P}(\theta;k)$ are **gauge symmetries**
- Otherwise use **singular values** $\sigma_{a_{J^P}}(\theta)$ of the Toeplitz $\bar{\mathrm{O}}_{J^P}(\theta)$ built from the $\mathrm{O}^{(n)}_{J^P}(\theta)$
- Steepest descent on a chosen $\sigma_{a_{J^P}}(\theta)$ walks θ onto a gauge surface; gradient by singular-value Hellmann–Feynman:

$$\frac{\partial\sigma_{a_{J^P}}(\theta)}{\partial\theta_i}=\Re\left[\bar{\mathrm{u}}^{\dagger}_{a_{J^P}}\cdot\frac{\partial\bar{\mathrm{O}}_{J^P}(\theta)}{\partial\theta_i}\cdot\bar{\mathrm{v}}_{a_{J^P}}\right]$$

- Trajectories end at **gauge surfaces** in θ-space where gauge symmetries emerge

Numerical polology: the 'tuned' algorithm

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- All **ghost diagnostics**
- All θ **mass-dimensions**
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- All **gauge symmetries** and **technical naturalness**

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- Trajectories end at **gauge surfaces** in θ-space where gauge symmetries emerge

Numerical polology: textbook examples

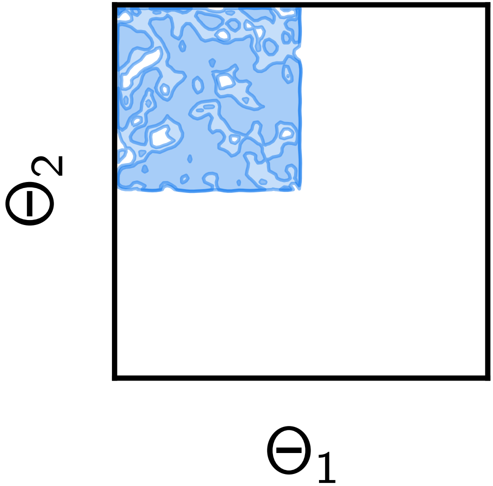
- Simple example of **Fierz–Pauli**:

$$\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[\theta_1\Big(\tfrac{1}{2}\partial_\beta\mathcal{H}\partial^\beta\mathcal{H} - \partial^\alpha\mathcal{H}_{\alpha\beta}\partial^\beta\mathcal{H} - \tfrac{1}{2}\partial_\gamma\mathcal{H}^{\alpha\beta}\partial^\gamma\mathcal{H}_{\alpha\beta} + \partial_\beta\mathcal{H}^{\alpha\beta}\partial^\gamma\mathcal{H}_{\alpha\gamma}\Big) - \theta_2\Big(\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} - \mathcal{H}^2\Big)\Big]$$

- Unitarity conditions:

$$\theta_1 < 0, \quad \theta_2 > 0$$

- Hypercube compactification $\Theta \equiv \tan^{-1}\theta$



<barker@fzu.cz>

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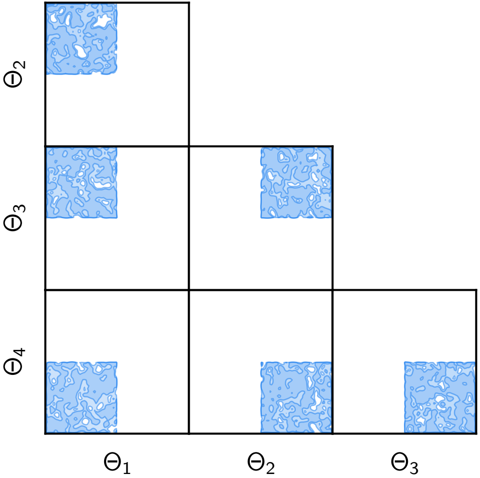
- Simple example of **Proca**:

$$\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[-\theta_3 \partial_{[\alpha}\mathcal{A}_{\beta]}\partial^{[\alpha}\mathcal{A}^{\beta]} - \tfrac{\theta_4}{2}\mathcal{A}_\alpha \mathcal{A}^\alpha \Big]$$

- Unitarity conditions:

$$\theta_1 < 0, \quad \theta_2 > 0, \quad \theta_3 > 0, \quad \theta_4 < 0$$



Numerical polology: new physics

- Shiny **new** theory of physics:

$$
\begin{aligned}
\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[& \tfrac{\theta_1}{2}\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} + \theta_5\mathcal{H}^2 + \theta_6\mathcal{H}\partial_\alpha\mathcal{A}^\alpha \\
& - \theta_6\mathcal{H}\partial_\alpha\partial^\alpha\phi - \theta_1\partial_\alpha\mathcal{A}^\alpha\partial_\beta\mathcal{A}^\beta + \theta_1\partial_\beta\mathcal{A}_\alpha\partial^\beta\mathcal{A}^\alpha \\
& + 2\theta_1\mathcal{H}^{\alpha\beta}\partial_\beta\partial_\alpha\phi - 2\theta_1\mathcal{H}_{\alpha\beta}\partial^\beta\mathcal{A}^\alpha + \theta_2\partial_\beta\mathcal{H}\partial^\beta\mathcal{H} \\
& + \theta_3\partial_\alpha\mathcal{H}^{\alpha\beta}\partial_\gamma\mathcal{H}_\beta{}^\gamma + \theta_4\partial^\beta\mathcal{H}\partial_\gamma\mathcal{H}_\beta{}^\gamma \\
& - \tfrac{\theta_3}{2}\partial_\gamma\mathcal{H}_{\alpha\beta}\partial^\gamma\mathcal{H}^{\alpha\beta}\Big]
\end{aligned}
$$

- Unitarity conditions:

$$
\theta_1 < 0, \quad \theta_3 < 0, \quad \theta_5 < \frac{-\theta_1{}^2 + \theta_1\theta_6 - \theta_6{}^2}{6\theta_1},
$$

$$
\theta_2 > \frac{2\theta_1{}^2\theta_3 - 2\theta_1\theta_3\theta_6 - 6\theta_1\theta_4\theta_6 - \theta_3\theta_6{}^2}{12\theta_1{}^2}
$$

	$\phi^{\#1}_{0^+}$	$h^{\#1}_{0^+}$	$h^{\#2}_{0^+}$	$\mathcal{A}^{\#1}_{0^+}$
$\phi^{\#1}_{0^+}\dagger$	0	$\frac{1}{2}\sqrt{3}\,k^2\,\theta_6$	$-\frac{Y_1 k^2}{2}$	0
$h^{\#1}_{0^+}\dagger$	$\frac{1}{2}\sqrt{3}\,k^2\,\theta_6$	$\frac{1}{2}(Y_2 + Y_3 k^2)$	$\frac{1}{2}(\sqrt{3}\,Y_4 k^2 + 2\sqrt{3}\,\theta_5)$	$\frac{1}{2}\,i\,\sqrt{3}\,k\,\theta_6$
$h^{\#2}_{0^+}\dagger$	$-\frac{Y_1 k^2}{2}$	$\frac{1}{2}(\sqrt{3}\,Y_4 k^2 + 2\sqrt{3}\,\theta_5)$	$\frac{1}{2}(Y_5 + Y_6 k^2)$	$-\frac{1}{2}\,i\,Y_1\,k$
$\mathcal{A}^{\#1}_{0^+}\dagger$	0	$-\frac{1}{2}\,i\,\sqrt{3}\,k\,\theta_6$	$\frac{i\,Y_1\,k}{2}$	0

	$h^{\#1}_{1^-}{}_\alpha$	$\mathcal{A}^{\#1}_{1^-}{}_\alpha$
$h^{\#1}_{1^-}\dagger^\alpha$	$\frac{\theta_1}{2}$	$-\frac{i\,\theta_1\,k}{\sqrt{2}}$
$\mathcal{A}^{\#1}_{1^-}\dagger^\alpha$	$\frac{i\,\theta_1\,k}{\sqrt{2}}$	$\theta_1\,k^2$

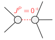
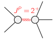
	$h^{\#1}_{2^+}{}_{\alpha\beta}$
$h^{\#1}_{2^+}\dagger^{\alpha\beta}$	$\frac{1}{2}(\theta_1 - k^2\,\theta_3)$

Abbreviations used in matrices

$Y_1 == 2\,\theta_1 - \theta_6$ && $Y_2 == \theta_1 + 6\,\theta_5$ && $Y_3 == 6\,\theta_2 - \theta_3$ &&
$Y_4 == 2\,\theta_2 + \theta_4$ && $Y_5 == \theta_1 + 2\,\theta_5$ && $Y_6 == 2\,\theta_2 + \theta_3 + 2\,\theta_4$ &&
$\mathrm{Det}(0^+) == -\frac{1}{4}k^2(1+k^2)(2\,\theta_1{}^3 + 2\,\theta_1{}^2(k^2(6\,\theta_2 - \theta_3) + 6\,\theta_5 - \theta_6) + k^2\,\theta_3\,\theta_6{}^2 + 2\,\theta_1\,\theta_6(k^2(\theta_3 + 3\,\theta_4) + \theta_6))$ &&
$\mathrm{Det}(2^+) == \frac{1}{2}(\theta_1 - k^2\,\theta_3)$

Lagrangian

$$
\frac{1}{2}\theta_1\,h_{\alpha\beta}\,h^{\alpha\beta} + \theta_5\,h^\alpha{}_\alpha\,h^\beta{}_\beta + \theta_6\,h^\beta{}_\beta\,\partial_\alpha\mathcal{A}^\alpha - \theta_6\,h^\beta{}_\beta\,\partial_\alpha\partial^\alpha\phi - \theta_1\,\partial_\alpha\mathcal{A}^\alpha\,\partial_\beta\mathcal{A}^\beta + 2\,\theta_1\,h^{\alpha\beta}\,\partial_\beta\partial_\alpha\phi - 2\,\theta_1\,h_{\alpha\beta}\,\partial^\beta\mathcal{A}^\alpha + \theta_1\,\partial_\beta\mathcal{A}_\alpha\,\partial^\beta\mathcal{A}^\alpha + \theta_2\,\partial_\beta h^\chi{}_\chi\,\partial^\beta h^\alpha{}_\alpha + \theta_3\,\partial_\alpha h^{\alpha\beta}\,\partial_\chi h_\beta{}^\chi + \theta_4\,\partial^\beta h^\alpha{}_\alpha\,\partial_\chi h_\beta{}^\chi - \frac{1}{2}\theta_3\,\partial_\chi h_{\alpha\beta}\,\partial^\chi h^{\alpha\beta}
$$

Added source term(s):	$\phi\,\rho + \mathcal{A}^\alpha\,\mathcal{J}_\alpha + h^{\alpha\beta}\,\mathcal{T}_{\alpha\beta}$		
Source constraint(s)	# constraint(s)	Covariant form	
$i\,\rho^{\#1}_{0^+} + k\,\mathcal{J}^{\#1}_{0^+} == 0$	1	$\rho == \partial_\alpha\mathcal{J}^\alpha$	
$\mathcal{J}^{\#1\,\alpha}_{1^-} - 2\,i\,k\,\mathcal{T}^{\#1\,\alpha}_{1^-} == 0$	3	$\partial_\beta\partial^\alpha\mathcal{J}^\beta + 2\,\partial_\chi\partial^\chi\partial_\beta\mathcal{T}^{\alpha\beta} == \partial_\beta\partial^\beta\mathcal{J}^\alpha + 2\,\partial_\chi\partial_\beta\partial^\alpha\mathcal{T}^{\beta\chi}$	
Total # constraint(s):	4		
Resolved pole(s)	# polarization(s)	Square mass	Residue
	1	$-\frac{2\,\theta_1(\theta_1{}^2 + 6\,\theta_1\,\theta_5 - \theta_1\,\theta_6 + \theta_6{}^2)}{2\,\theta_1{}^2(6\,\theta_2 - \theta_3) + 2\,\theta_1(\theta_3 + 3\,\theta_4)\,\theta_6 + \theta_3\,\theta_6{}^2}$	(Hidden for brevity)
	5	$\frac{\theta_1}{\theta_3}$	$-\frac{2}{\theta_3}$
Resolved unitarity condition(s):	$\theta_1 < 0$ && $\theta_3 < 0$ && $\theta_5 < \frac{-\theta_1{}^2 + \theta_1\,\theta_6 - \theta_6{}^2}{6\,\theta_1}$ && $\theta_2 > \frac{2\,\theta_1{}^2\,\theta_3 - 2\,\theta_1\,\theta_3\,\theta_6 - 6\,\theta_1\,\theta_4\,\theta_6 - \theta_3\,\theta_6{}^2}{12\,\theta_1{}^2}$		

Numerical polology: new physics

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$$\begin{aligned}\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[&\tfrac{\theta_1}{2}\mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} + \theta_5\mathcal{H}^2 + \theta_6\mathcal{H}\partial_\alpha\mathcal{A}^\alpha \\ &- \theta_6\mathcal{H}\partial_\alpha\partial^\alpha\phi - \theta_1\partial_\alpha\mathcal{A}^\alpha\partial_\beta\mathcal{A}^\beta + \theta_1\partial_\beta\mathcal{A}_\alpha\partial^\beta\mathcal{A}^\alpha \\ &+ 2\theta_1\mathcal{H}^{\alpha\beta}\partial_\beta\partial_\alpha\phi - 2\theta_1\mathcal{H}_{\alpha\beta}\partial^\beta\mathcal{A}^\alpha + \theta_2\partial_\beta\mathcal{H}\partial^\beta\mathcal{H} \\ &+ \theta_3\partial_\alpha\mathcal{H}^{\alpha\beta}\partial_\gamma\mathcal{H}_\beta{}^\gamma + \theta_4\partial^\beta\mathcal{H}\partial_\gamma\mathcal{H}_\beta{}^\gamma \\ &- \tfrac{\theta_3}{2}\partial_\gamma\mathcal{H}_{\alpha\beta}\partial^\gamma\mathcal{H}^{\alpha\beta}\Big]\end{aligned}$$

- Unitarity conditions:

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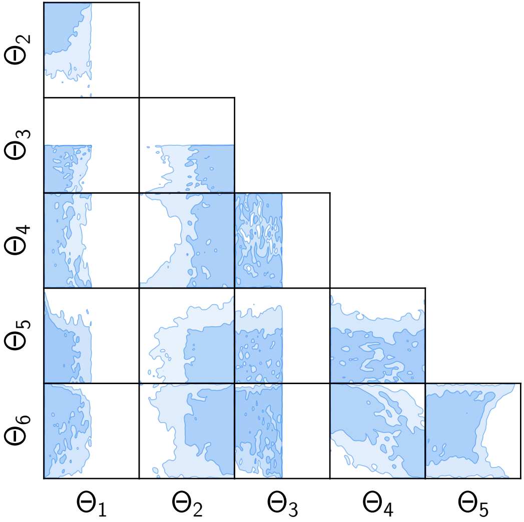
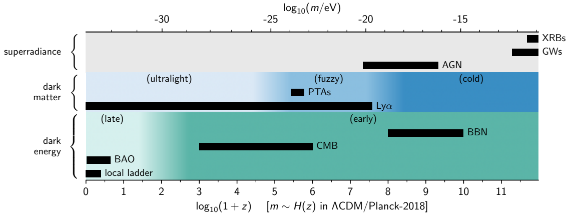
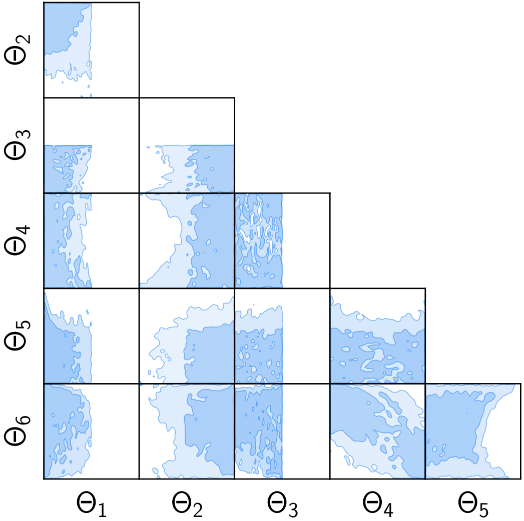
$$\theta_2 > \frac{2\theta_1{}^2\theta_3 - 2\theta_1\theta_3\theta_6 - 6\theta_1\theta_4\theta_6 - \theta_3\theta_6{}^2}{12\theta_1{}^2}$$



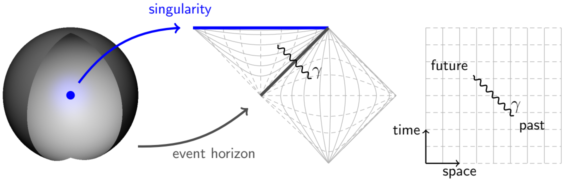
Numerical polology: astroparticle physics



- Theoretical self-consistency **necessary**, but not **sufficient**
- Theories must be subordinate to the observed phenomena
- Numerical framing allows observational constraints

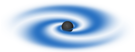


Numerical polology: black hole superradiance constraints

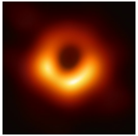


general BH idea   curved spacetime   flat spacetime

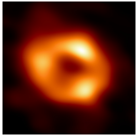
- BH **mass** M and **angular momentum** J
- Size of event horizon is $r_g \sim GM$



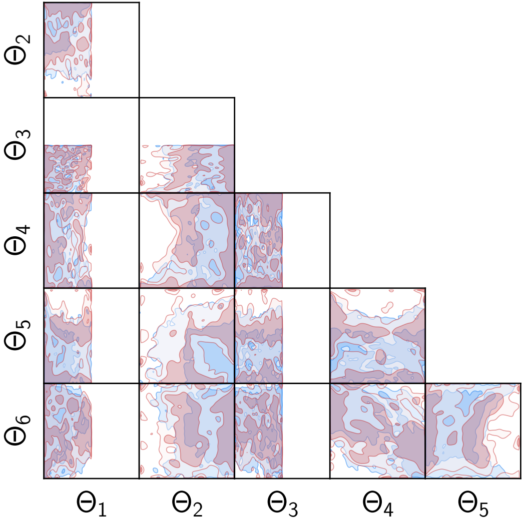
supermassive BHs in galaxy centres



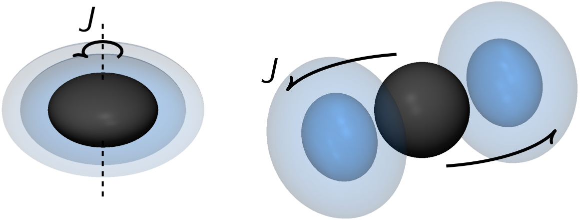
a long time ago in a galaxy far far away



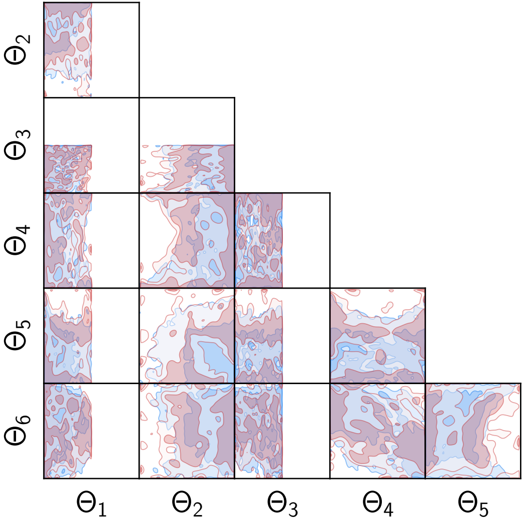
centre of our own galaxy



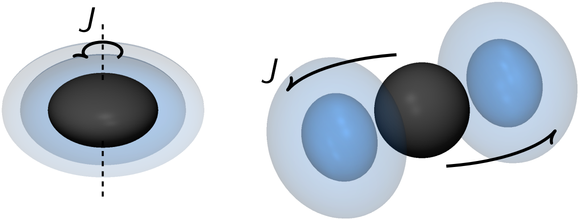
Numerical pology: black hole superradiance constraints



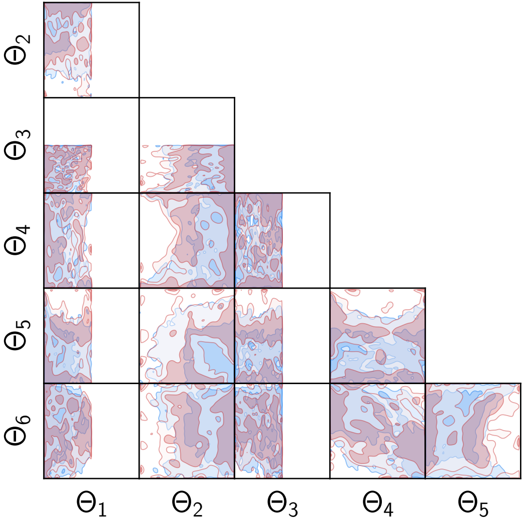
- Boson has Compton wavelength $\lambda \sim 1/m$
- BH event horizon $r_g \sim GM$
- Boson 'feels' BH when $\lambda \sim r_g$
- BH leaks J to boson at rate $\Gamma(M, m)$
- By astrophysics BH age is $\tau(M, J)$
- Observed J, M **rules out** m with $\tau\Gamma \gg 1$



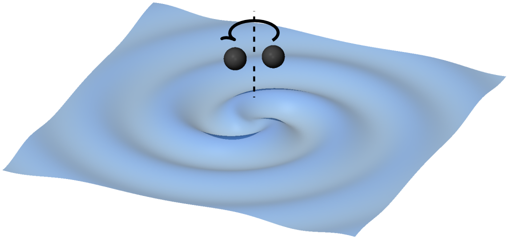
Numerical polology: black hole superradiance constraints



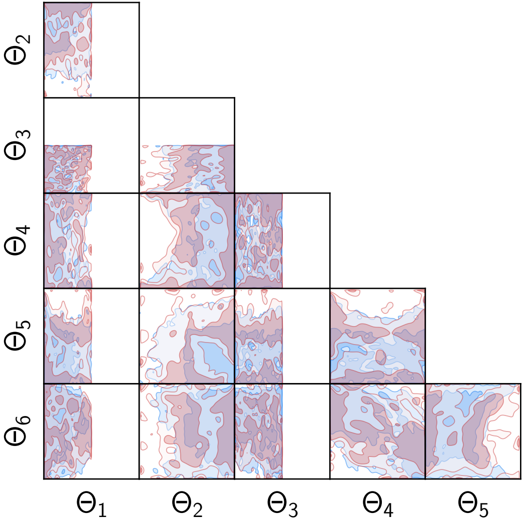
- Observed J, M **rules out** m with $\tau\Gamma \gg 1$
- By astrophysics, we expect J, M populations
- **Missing** J, M are a smoking gun for boson at m
- Two probes: **gravitational waves** and **astronomy**



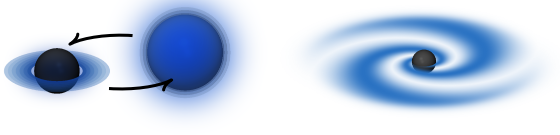
Numerical polology: black hole superradiance constraints

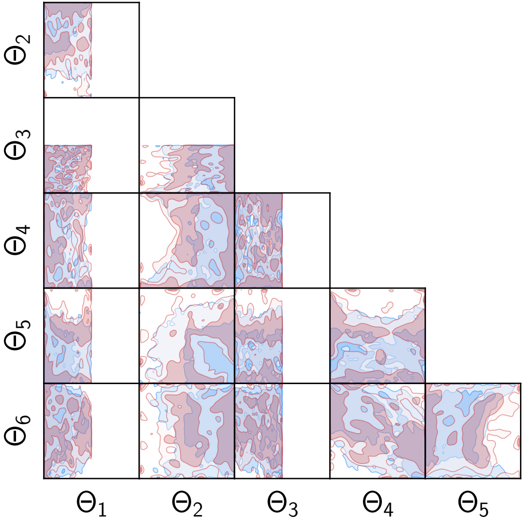


- Could use gravitational wave observatories
- Fairly small BHs merging
- J, M statistics inferred



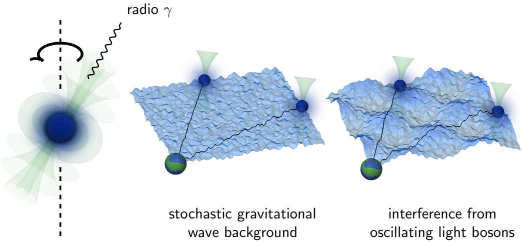
Numerical pology: black hole superradiance constraints



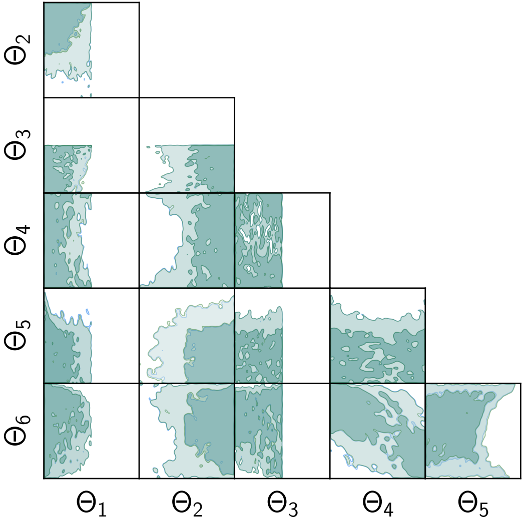


- Astronomy gives **crispest** cases
- Small BHs in X-ray binaries
- Supermassive BHs dominate galaxies
- Often high-J (which is good)
- BHSR codes from `sebhoof/bhsr` — see also `2406.10337`

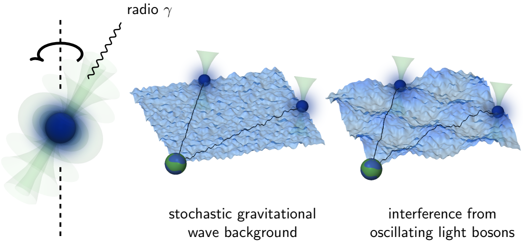
Numerical polology: pulsar timing array constraints



- Pulsars are rapidly rotating neutron stars
- Radio signals form precise clocks
- From astrophysics, expect GW background
- GWs delay/advance radio pulses
- New light bosons oscillate at $\omega \sim m$



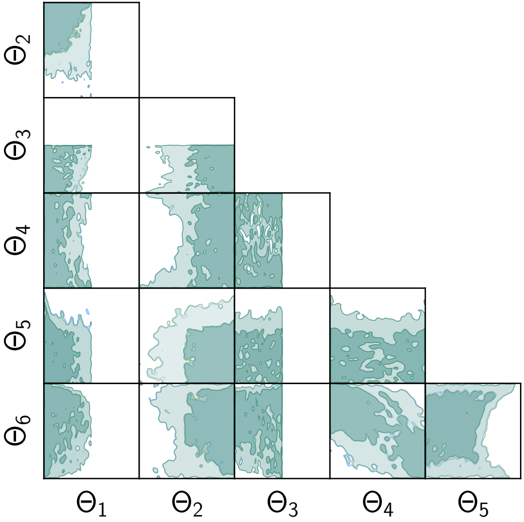
Numerical polology: pulsar timing array constraints



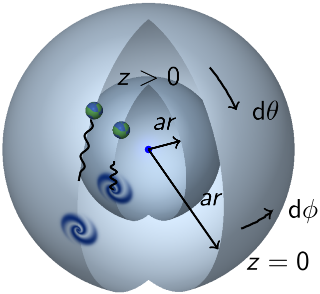
- Mass cut from 2312.12225

Arecibo (1963–2020)

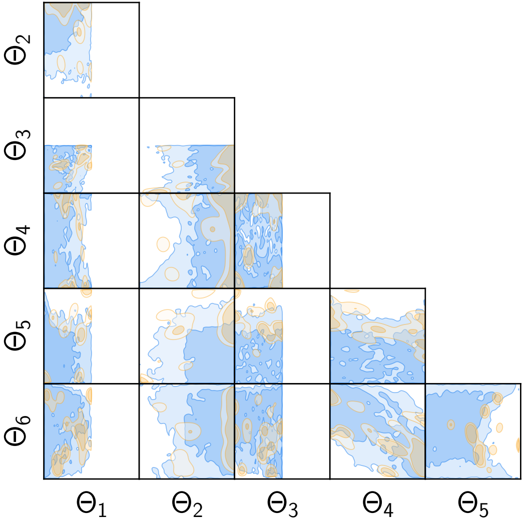
FAST (2016–)



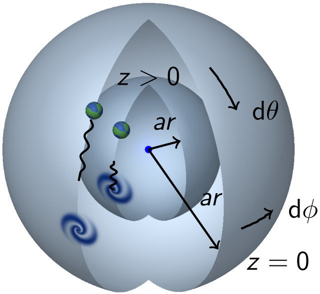
Numerical polology: dark energy constraints



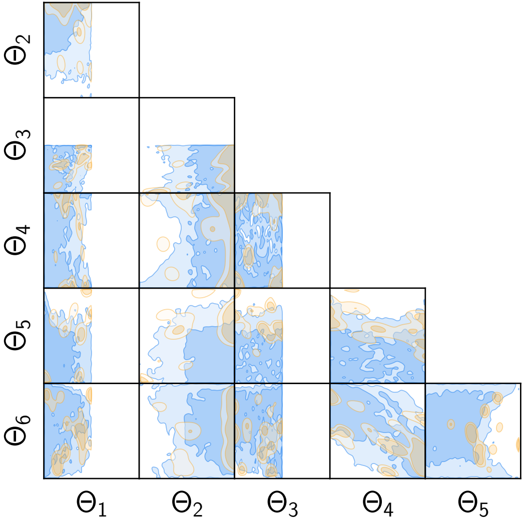
- Distance $a^2 r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$
- Expansion via **scale factor** $a(z)$
- Photons **stretch** with redshift z
- Take $a(0) = 1$ **today**, $a(z > 0) < 1$ **past**



Numerical polology: dark energy constraints



- Hubble **number** $H = \dot{a}/a$
- $H(z)$ encodes expansion history



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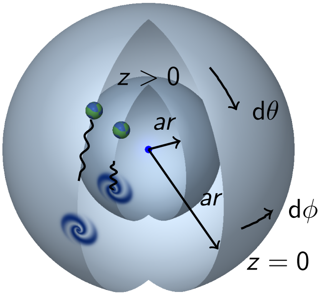
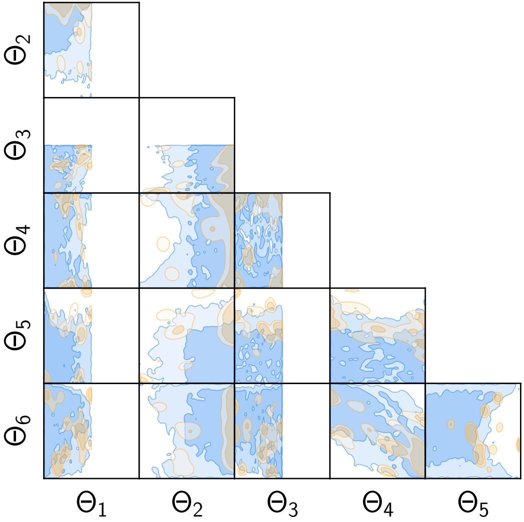


- Generally $H(z)^2$ equals **energy density**
- As matter dilutes away, $H(z)^2 \sim \Lambda$
- Bosons oscillate when $H(z) \lesssim m$
- These can **modify** expansion history



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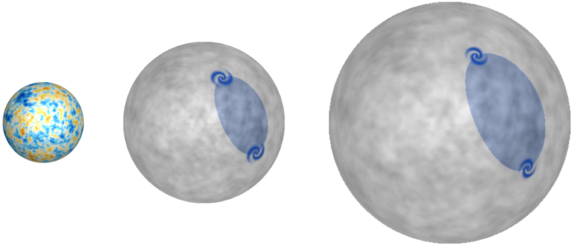
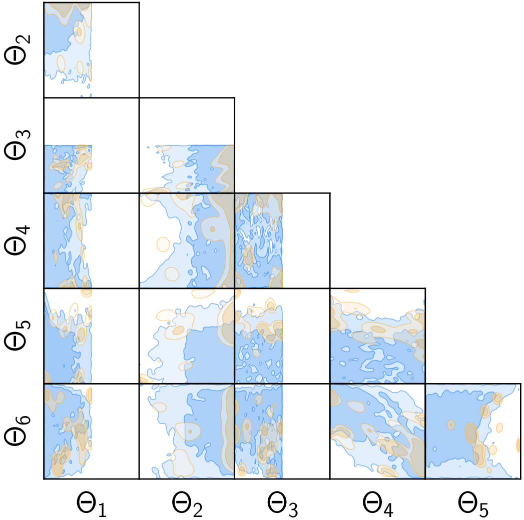
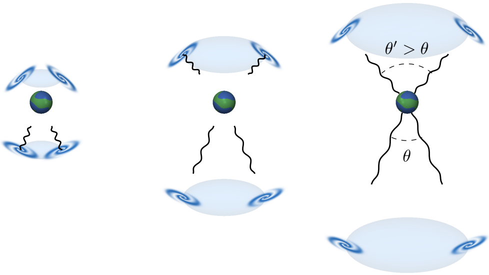


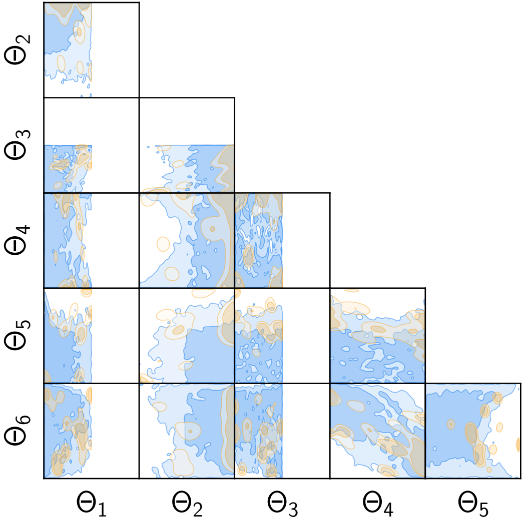


- Plasma and radiation at early times
- Correlated structure from acoustic waves
- Structures determine galaxy positions
- Produces instances of BAO scale

Numerical polology: dark energy constraints

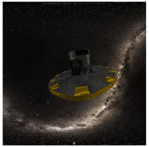
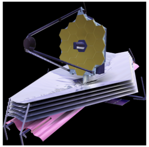
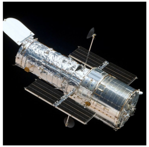
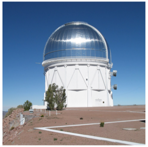


- Photons also stretch with **redshift** z
- **Angular size shrinks** with z
- We can measure **expansion history**

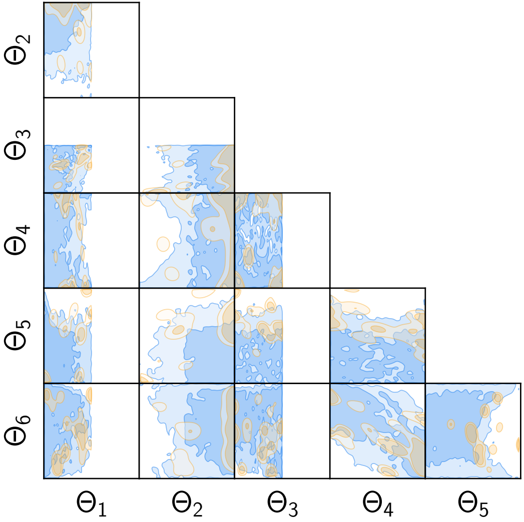


<barker@fzu.cz>

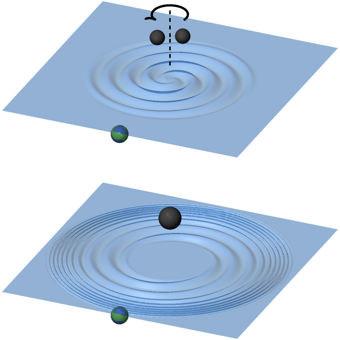
Numerical polology: dark energy constraints

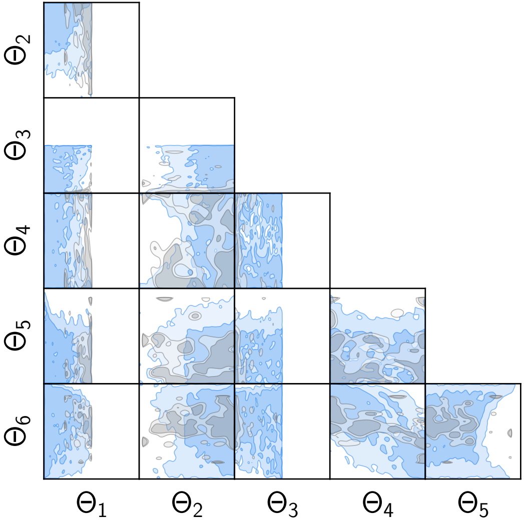






- Full picture is more complicated
- Calibrate supernovae in nearby galaxies
- Distant supernovae give $H(z \lesssim 2)$
- Distant galaxy correlations give $H(z \lesssim 3)$
- Techniques inspired by 2503.08658



Numerical pology: gravitational wave dispersion constraints

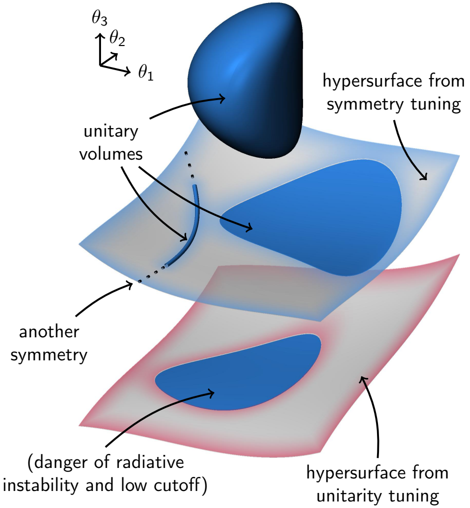




- This theory also has a **massive graviton**
- Gravitational waves **below** speed of light

Numerical polology: the 'untuned' algorithm

- Untuned $\mathcal{S}(\theta)$ has no unitary *volume*, only isolated unitary **hypersurfaces**



Numerical polology: the 'untuned' algorithm

- Untuned $\mathcal{S}(\theta)$ has no unitary *volume*, only isolated unitary **hypersurfaces**

- $\det \mathrm{O}_{J^P}(\theta; k)$ is a polynomial in $z \equiv k^2$ for Hermitian $\mathrm{O}_{J^P}(\theta; k)$:

$$\det \mathrm{O}_{J^P}(\theta; k) = \sum_{j=0}^{n_{J^P}} c_j^{J^P}(\theta)\, z^j$$

- Evaluate numerically at $n_{J^P}+1$ test points $z_i = i$, invert the Vandermonde system

$$d_i = \sum_{j=0}^{n_{J^P}} V_{ij}\ c_j^{J^P}(\theta), \quad V_{ij} \equiv i^j$$

- Note that V_{ij} inverse can be precomputed

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- Indices: $j \geq 1$ for non-target J^P; $j \geq 2$ for the target sector (the chosen pole is kept)
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$$\log \mathcal{L}(\mathrm{QFT}|\theta) = -\left|\mathrm{r}(\theta)\right|^2$$

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  can also be obtained numerically
- Drop samples with final $|\mathrm{r}(\theta)| > 10^{-6}$

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- For **residues** plug the survivors back into the closed-form no-ghost formula

$$Z_{s_{J^P}}(\theta) = 2P\left[\sum_n n\,|m_{s_{J^P}}(\theta)|^{n-2}\,\mathrm{w}_{s_{J^P}}^{m(\theta)\dagger}\cdot \mathrm{O}_{J^P}^{(n)}(\theta)\cdot \mathrm{w}_{s_{J^P}}^{m(\theta)}\right]^{-1}$$

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- Need to reweight to restore uniformity of original prior on θ-space

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Numerical polology: textbook examples

- General vector theory:

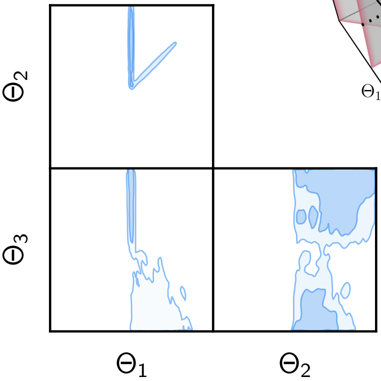
$$\mathcal{S}(\theta) = \int \mathrm{d}^4 x \left[\frac{\theta_2}{2}\partial_\alpha \mathcal{A}^\alpha \partial_\beta \mathcal{A}^\beta - \frac{\theta_1}{2}\partial_\alpha \mathcal{A}_\beta \partial^\alpha \mathcal{A}\right.$$
$$\left. - \frac{\theta_3}{2}\mathcal{A}_\alpha \mathcal{A}^\alpha\right]$$

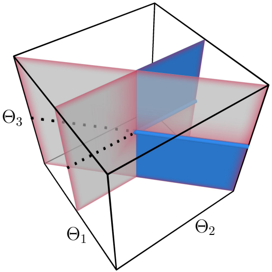
- Proca theory at $\theta_2 = \theta_1$:

$$\theta_1 > 0, \quad \theta_3 < 0$$

- Klein–Gordon-like at $\theta_1 = 0$:

$$\theta_2 > 0, \quad \theta_3 > 0$$





<barker@fzu.cz>

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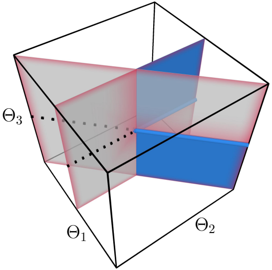

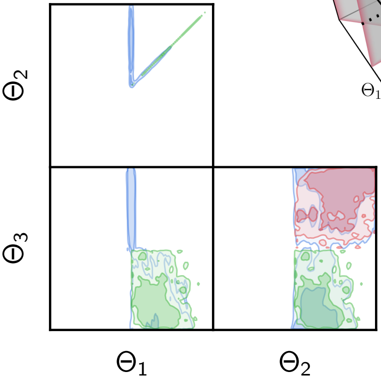

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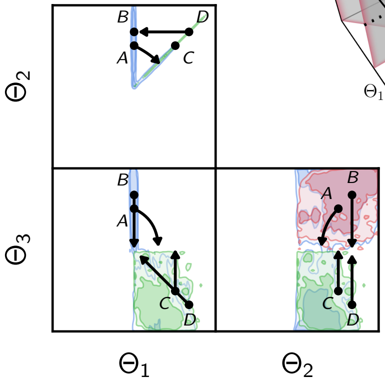
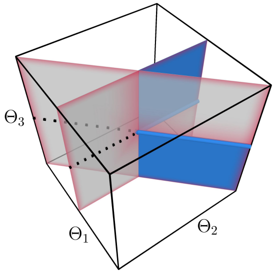
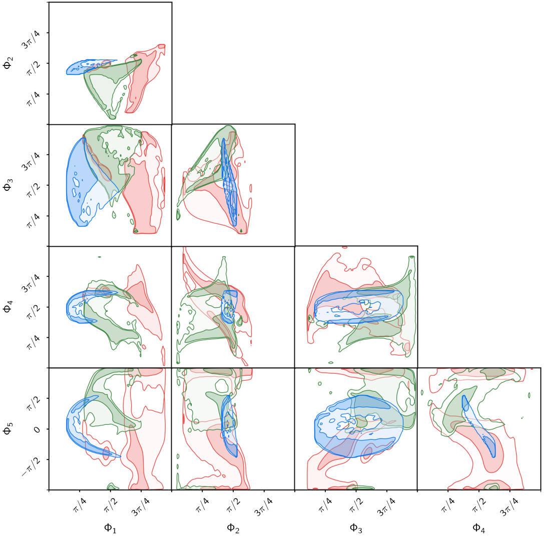




- General symmetric rank-two field theory:

$$\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[\theta_1 \mathcal{H}_{\alpha\beta}\mathcal{H}^{\alpha\beta} + \theta_2 \mathcal{H}^2 + \theta_3 \partial_\beta \mathcal{H}\partial^\beta \mathcal{H} + \theta_4 \partial_\alpha \mathcal{H}^{\alpha\beta}\partial_\gamma \mathcal{H}_\beta{}^\gamma + \theta_5 \partial^\beta \mathcal{H}\partial_\gamma \mathcal{H}_\beta{}^\gamma + \theta_6 \partial_\gamma \mathcal{H}_{\alpha\beta}\partial^\gamma \mathcal{H}^{\alpha\beta}\Big]$$

- Fierz–Pauli branch
- Klein–Gordon-like branch
- Proca-like branch (can say this is **numerically discovered**)
- Sample on hyperspherical ‘sky’



Numerical pology: torsion and non-metricity

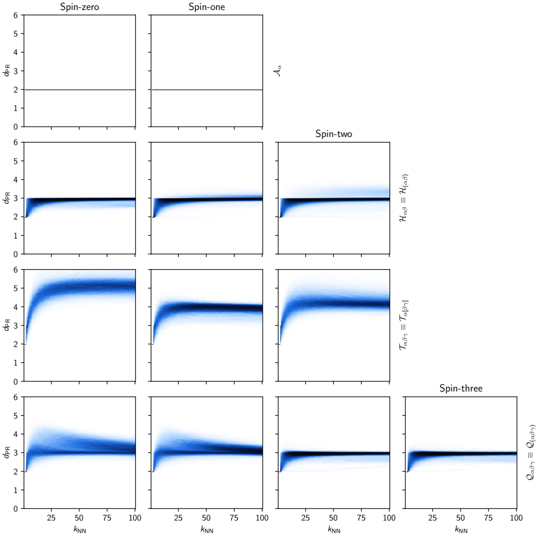
- Turn back to these models:

$$\begin{aligned}\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[&\theta_1 \mathcal{T}_{\alpha\beta\chi}\mathcal{T}^{\alpha\beta\chi} + \theta_2 \mathcal{T}_{\alpha\beta\chi}\mathcal{T}^{\beta\alpha\chi} + \theta_3 \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\mathcal{T}^{\chi}{}_{\beta\chi} \\ &+ \theta_4 \partial_\beta \mathcal{T}^{\delta}{}_{\chi\delta}\partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_5 \partial_\chi \mathcal{T}^{\delta}{}_{\beta\delta}\partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_6 \partial_\beta \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\alpha\chi}{}^{\delta} \\ &+ \theta_7 \partial_\alpha \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\beta\chi}{}^{\delta} + \theta_8 \partial_\beta \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}_{\chi\alpha}{}^{\delta} + \theta_9 \partial^\chi \mathcal{T}^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta \mathcal{T}_{\chi\beta}{}^{\delta} \\ &+ \theta_{10} \partial_\alpha \mathcal{T}^{\alpha\beta\chi}\partial_\delta \mathcal{T}^{\delta}{}_{\beta\chi} + \theta_{11} \partial_\delta \mathcal{T}_{\alpha\beta\chi}\partial^\delta \mathcal{T}^{\alpha\beta\chi} + \theta_{12} \partial_\delta \mathcal{T}_{\beta\alpha\chi}\partial^\delta \mathcal{T}^{\alpha\beta\chi}\Big]\end{aligned}$$

$$\begin{aligned}\mathcal{S}(\theta) = \int \mathrm{d}^4x \Big[&\theta_1 \mathcal{Q}_{\alpha\beta\chi}\mathcal{Q}^{\alpha\beta\chi} + \theta_2 \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta}\mathcal{Q}_{\beta}{}^{\chi}{}_{\chi} \\ &+ \theta_3 \partial_\beta \mathcal{Q}_{\chi}{}^{\delta}{}_{\delta}\partial^\chi \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta} + \theta_4 \partial_\chi \mathcal{Q}_{\beta}{}^{\delta}{}_{\delta}\partial^\chi \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta} \\ &+ \theta_5 \partial_\alpha \mathcal{Q}^{\alpha\beta\chi}\partial_\delta \mathcal{Q}_{\beta\chi}{}^{\delta} + \theta_6 \partial^\chi \mathcal{Q}^{\alpha}{}_{\alpha}{}^{\beta}\partial_\delta \mathcal{Q}_{\beta\chi}{}^{\delta} \\ &+ \theta_7 \partial_\delta \mathcal{Q}_{\alpha\beta\chi}\partial^\delta \mathcal{Q}^{\alpha\beta\chi}\Big]\end{aligned}$$

- Model **dimensionality**:

Field	0	1	2	3
$\mathcal{A}_\alpha$	2	2	–	–
$\mathcal{H}_{\alpha\beta} \equiv \mathcal{H}_{(\alpha\beta)}$	3	3	3	–
$\mathcal{T}_{\alpha\beta\gamma} \equiv \mathcal{T}_{\alpha[\beta\gamma]}$	5	4	4	–
$\mathcal{Q}_{\alpha\beta\gamma} \equiv \mathcal{Q}_{(\alpha\beta\gamma)}$	3	3	3	3



Numerical polology: the full pipeline

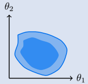
Basic HEP stack

- Unitarity
- Gauge symmetries
- Mass spectrum
- Spurion analysis

Basic pheno stack

- Non-interacting superradiance
- Pulsar timing arrays
- Background quintessence
- Graviton dispersion

Modern sampling



- θ-posterior — **always shrinking**
- Derived params — **always growing**

Advanced HEP stack

- Interaction ansatz
- Automated noether
- Dimensional regularisation with $\overline{\text{MS}}$
- Cutoff and renormalisation group

Advanced pheno stack

- Interacting superradiance
- Boltzmann integration
- Small-scale hydrodynamics
- Mode-mixing

Thanks for your attention!

- Quite a strong analytic claim:

  "***There can be no spin-zero or spin-two torsion or non-metricity particles in a perturbative, parity-preserving EFT.****"

  **With caveats.*

- Numerical claim:

  "***Prior chains are available for free single-particle torsion and non-metricity models: these have between three and five effective model parameters.***"

- QFT and data should be all you need
- Symbolic techniques do not scale...
- ...but numerical techniques do!
- Sampling native to precision cosmology
- Allows model discovery and model testing in one Bayesian framework

<barker@fzu.cz>