SHADOW CONSTRAINTS OF NON-RIEMANNIAN BLACK HOLES

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Abstract

The increased precision of horizon-scale imaging achieved by interferometric instruments motivates a closer examination of the shadow of the black hole (BH) and its constraints on the geometry in the center of Sgr A*. While these have been applied to solutions in general relativity (GR) and in its metric modifications, we investigate the shadow sizes set by various non-Riemannian theories of gravity, namely in the framework of metric-affine, teleparallel, Ricci-based and Palatini theories, incorporating the effects of nonlinear electrodynamics (NLED). Our analysis sets constraints on theories capable of mimicking BHs in GR and allows for the exclusion of some solutions.

Theory

In **metric-affine theories**, the metric $g_{\mu\nu}$ and the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ are treated as independent variables. General connection is characterised by curvature, torsion and nonmetricity,

$$R^{\sigma}{}_{\rho\mu\nu} \equiv \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho}\,,$$

$$T^{\sigma}{}_{\mu\nu} \equiv \Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\nu\mu},$$

$$Q_{\rho\mu\nu} \equiv \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}{}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\rho\nu} g_{\mu\lambda}\,,$$

while a generic gravitational action can involve all possible quadratic combinations of these quantities.

Teleparallel gravity assumes a flat connection, i.e. vanishing curvature. If torsion is taken to be zero as well then we have **symmetric teleparallel**, and if nonmetricity is zero then **metric teleparallel** theory. In both cases one can rewrite the Riemannian Ricci scalar $\mathring{R}$ as a teleparallel scalar and a Riemannian divergence (boundary term), e.g. in the symmetric case

$$\mathring{R} = Q + \mathring{\nabla}_\mu(\hat{Q}^\mu - Q^\mu) = Q + B_Q\,,$$

where

$$Q \equiv -\frac{1}{4}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + \frac{1}{2}Q_{\lambda\mu\nu}Q^{\mu\nu\lambda} + \frac{1}{4}Q_\mu Q^\mu - \frac{1}{2}Q_\mu \hat{Q}^\mu$$

$$Q_\mu \equiv Q_{\mu\nu}{}^{\nu}\,, \quad \hat{Q}_\mu \equiv Q_{\nu\mu}{}^{\nu}\,.$$

Extended gravitational Lagrangians would then be of type $f(Q)$ or $f(T)$, or scalar-tensor like $f(\phi, Q)$, etc.

Ricci-based gravity (RBG) denotes a subclass of theories in the metric-affine/Palatini formalism with the action

$$\mathcal{S}(g,\Gamma,\psi) = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathcal{L}_G(g_{\mu\nu}, R_{\mu\nu}(\Gamma)) + \mathcal{S}_m(g_{\mu\nu},\psi_m).$$

Instead of the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, the gravitational Lagrangian is written in terms of a general function $M^{\mu}{}_{\nu} = g^{\mu\alpha}R_{\alpha\nu}$, where $R_{\mu\nu}(\Gamma)$ is the symmetric Ricci tensor of the affine connection. Some notable examples of RBGs are Palatini $f(R)$ and Eddington-inspired Born-Infeld (EiBI) gravity. To achieve a different behaviour from GR, these theories require a matter coupling, usually some form of electrodynamics.

Methodology

Given a spherically symmetric spacetime with the metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)\,d\Omega_2^2,$$

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2,$$

the photon trajectories follow null geodesics, with the equations of motion

$$\frac{B(r)}{C(r)}\left(\frac{dr}{d\phi}\right)^2 = \frac{C(r)}{A(r)}\left(\frac{1}{b^2} - \frac{V_{\text{eff}}}{L^2}\right),$$

where $b = L/E$ is the impact parameter and the effective potential is defined as

$$V_{\text{eff}}(r) = \frac{A(r)}{C(r)}L^2.$$

The impact parameter obtains a critical value when the photon reaches a turning point, marked by $dr/d\phi = 0$, yielding

$$b(r) = \sqrt{\frac{C(r)}{A(r)}}.$$

When this corresponds to a maximum of the effective potential $V_{\text{eff}}(r)$, the respective r marks the photon sphere radius r_{ph},

$$\frac{d}{dr}(b^2(r_{\text{ph}})) = 0.$$

Evaluating the critical impact parameter at the photon sphere radius yields the apparent **shadow radius** $r_{\text{sh}} = b(r)|_{r=r_{\text{ph}}}$. Assuming a Gaussian distribution, Vagnozzi et al. [1] found the following constraints for the apparent shadow radius of Sgr A*:

$$4.55 < r_{\text{sh}}/M < 5.22 \quad (1\sigma)$$

$$4.21 < r_{\text{sh}}/M < 5.56 \quad (2\sigma).$$

Metric-affine solutions

The Reissner-Nordström-type metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2}} + r^2 d\Omega_2^2$$

is reproduced in **metric-affine theory with dynamical torsion and nonmetricity**, where the effective post-Riemannian charge is built from independent spin, dilation, and shear charges, [2–5]

$$\mathcal{Q}^2 = d_1\kappa_s^2 - 4e_1\kappa_d^2 - 2f_1\kappa_{sh}^2,$$

carried by axial torsion,

$$T^{\theta}{}_{t\phi} = -T^{\theta}{}_{t\phi} = -\frac{\kappa_s}{r}\,, \quad \ldots\,,$$

trace of nonmetricity Weyl vector,

$$Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda\,, \quad W_t = \frac{\kappa_d}{r}\,, \quad \ldots\,,$$

and traceless part of nonmetricity

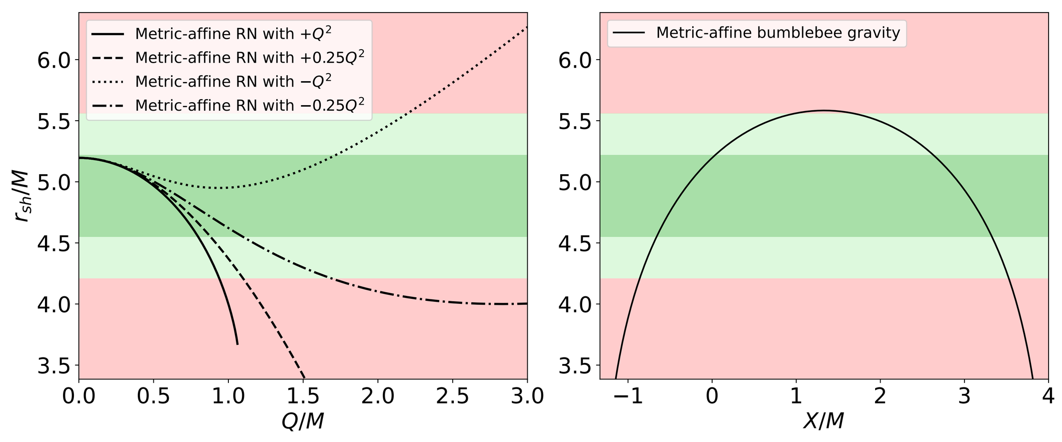
$$Q_{t\theta\theta} = r(\kappa_d + \kappa_{sh} + c_2 r)\,, \quad \ldots\,,$$

respectively, while c_2, d_1, e_1, f_1 are constants in the action.

Metric-affine bumblebee gravity describes a metric-affine extension to the gravitational sector via the Lorentz-symmetry-breaking coefficient $X = \xi b^2$, with the metric [6]

$$ds^2 = -\frac{\left(1 - \frac{2M}{r}\right)}{\sqrt{\left(1 + \frac{3X}{4}\right)\left(1 - \frac{X}{4}\right)}}dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)}\sqrt{\frac{\left(1 + \frac{3X}{4}\right)}{\left(1 - \frac{X}{4}\right)^3}} + r^2 d\Omega_2^2.$$

The calculated shadow radius can be seen in the following figure, for RN-equivalents on the left and the bumblebee gravity solution on the right.



Shadow radius constraints in metric-affine solutions.

Teleparallel solutions

The **Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) nonmetricity BH** is a static solution found in nonmetricity scalar-tensor teleparallel gravity [7]:

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2,$$

$$\Gamma^{r}{}_{\theta\theta} = M - r, \qquad \Phi(r) = \Phi_0\left(1 - \frac{M}{r}\right)^{-1/2}.$$

The **Bahamonde-Järv-Lember-Valcarcel (BJLV) BH** described by the Lambert function $W(z)$ defined as $W(z)e^{W(z)} = z$ [7]:

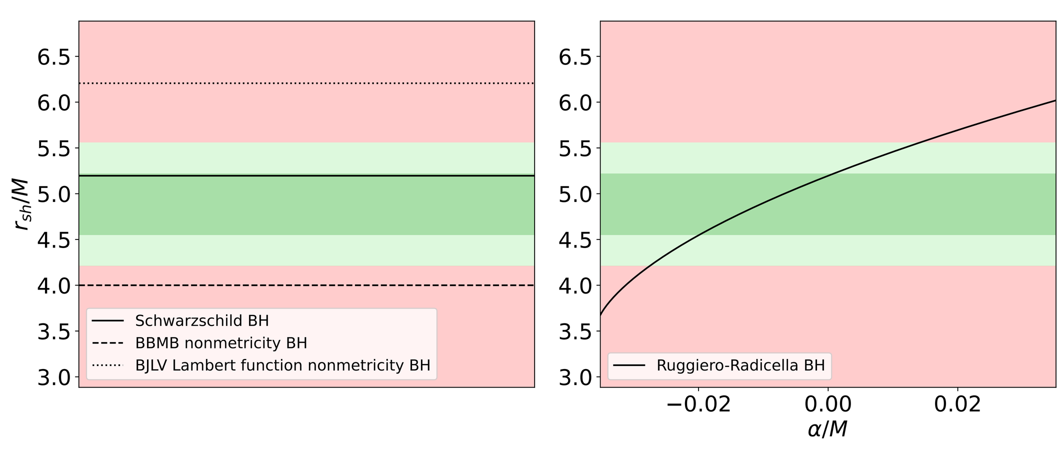
$$ds^2 = -\left(1 + W\left(\frac{-M}{r}\right)\right)^2 dt^2 + \left(1 + W\left(\frac{-M}{r}\right)\right)^{-2} dr^2 + r^2 d\Omega_2^2,$$

$$\Gamma^{r}{}_{\theta\theta} = -r\left[1 + W\left(-\frac{M}{r}\right)\right], \quad \Phi(r) = \Phi_0\left(-\frac{M}{rW\left(-\frac{M}{r}\right)}\right)^{1/2},$$

The **Ruggiero–Radicella BH** in $f(T) = T + \alpha T^2$ gravity [8]:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{32\alpha}{r^2}\right)dt^2 + \left(1 + \frac{2M}{r} + \frac{96\alpha}{r^2}\right)dr^2 + r^2 d\Omega_2^2.$$

The BBMB, BJLV produce a constant shadow radius, but do not reproduce Schwarzschild (left). The Ruggiero-Radicella shadow radius (right) increases with the scalar hair parameter.



Shadow radius constraints in teleparallel solutions.

Ricci-based gravity example

The solution [9] is of Reissner-Nordström type and corresponds to a coupling between EiBI gravity and Born-Infeld electrodynamics:

$$ds^2 = -\frac{A(x)}{\Omega_+}dt^2 + \frac{dr^2}{A(x)\Omega_+} + r^2 d\Omega_2^2,$$

$$\text{with} \quad A(x) = 1 - \frac{2M_0}{x(r)} + \frac{Q^2}{x^2(r)},$$

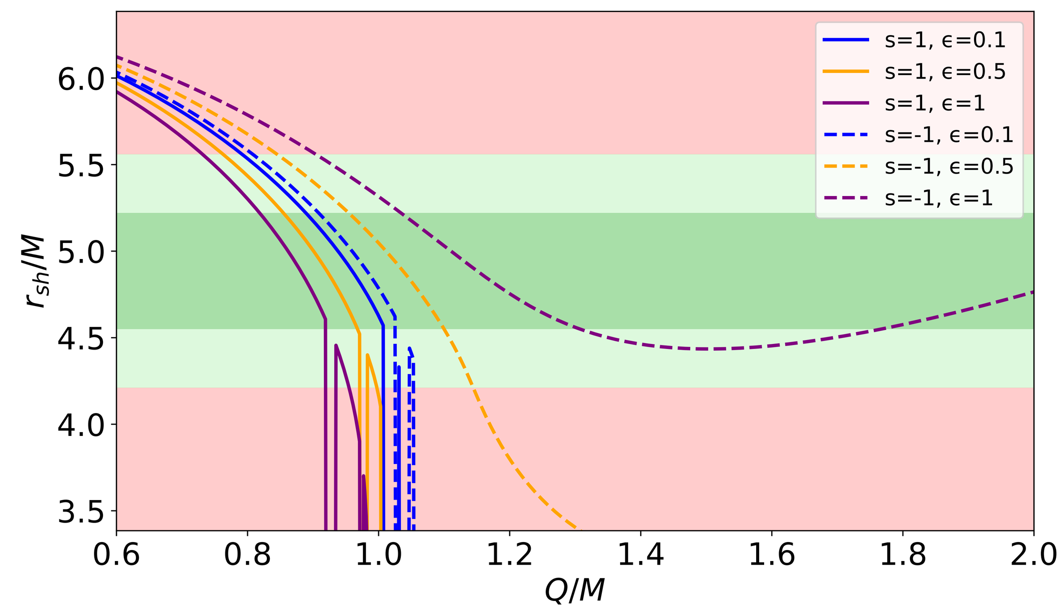
$$x^2 = \frac{1}{2}\left(r^2 + \sqrt{r^2 + 4s\,|\epsilon|Q^2}\right), \quad \Omega_+ = \frac{1}{1 + \frac{s|\epsilon|Q^2}{x^4}}.$$

Incorporating NLED, the propagation of photons in the geometric optics limit no longer follows the null geodesics of the background spacetime metric, but an effective metric defined as

$$ds^2_{\text{eff}} = \delta_1\left[\left(1 + \frac{4s\,|\epsilon|F}{(1 - 4Fs\,|\epsilon|)}\right)(-A(x)\,dt^2 + B(x)\,dr^2) + r^2 d\Omega_2^2\right],$$

$$\text{where} \quad F = \frac{Q^2}{r^4 - 4s|\epsilon|Q^2}$$

and the component $B(r)$ and the prefactor δ_1 do not contribute to the r_{sh}. Fixing $s = \pm 1$, the shadow radius depends on two parameters Q and ϵ.



Shadow radius constraints in a RBG solution.

Summary

Model	1σ constraints	2σ constraints
RN with Q^2	$\lvert Q/M\rvert \lesssim 0.80$	$\lvert Q/M\rvert \lesssim 0.94$
RN with $0.25Q^2$	$\lvert Q/M\rvert \lesssim 0.88$	$\lvert Q/M\rvert \lesssim 1.10$
RN with $-Q^2$	$\lvert Q/M\rvert \lesssim 1.71$	$\lvert Q/M\rvert \lesssim 2.20$
RN with $-0.25Q^2$	$\lvert Q/M\rvert \lesssim 1.10$	$\lvert Q/M\rvert \lesssim 1.68$
Bumblebee gravity	$-0.66 \lesssim X/M \lesssim 0.03$ $2.63 \lesssim X/M \lesssim 3.32$	$-0.86 \lesssim X/M \lesssim 0.98$ $1.68 \lesssim X/M \lesssim 3.52$
Schwarzschild	allowed	allowed
BBMB	excluded	excluded
BJLV	excluded	excluded
Ruggiero-Radicella	$-0.02 \lesssim \alpha/M \lesssim 0.001$	$-0.03 \lesssim \alpha/M \lesssim 0.01$
RBG $s = 1,\ \epsilon = 0.1$	$0.89 \lesssim Q/M \lesssim 1.01$	$0.79 \lesssim Q/M \lesssim 1.03$
RBG $s = 1,\ \epsilon = 0.5$	$0.85 \lesssim Q/M \lesssim 0.97$	$0.76 \lesssim Q/M \lesssim 1.00$
RBG $s = 1,\ \epsilon = 1.0$	$0.82 \lesssim Q/M \lesssim 0.92$	$0.73 \lesssim Q/M \lesssim 0.95$
RBG $s = -1,\ \epsilon = 0.1$	$0.91 \lesssim Q/M \lesssim 1.03$	$0.81 \lesssim Q/M \lesssim 1.05$
RBG $s = -1,\ \epsilon = 0.5$	$0.95 \lesssim Q/M \lesssim 1.10$	$0.85 \lesssim Q/M \lesssim 1.14$
RBG $s = -1,\ \epsilon = 1.0$	$1.04 \lesssim Q/M \lesssim 1.31$ $1.77 \lesssim Q/M \lesssim 2.37$	$0.90 \lesssim Q/M \lesssim 2.63$

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