Hamiltonian Formulation of Teleparallel Gravity with the Nieh–Yan Term

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Abstract

We present the Hamiltonian formulation of the Teleparallel Gravity in the language of differential forms. Using a (3+1) decomposition, we identify the dynamical variables and their canonically conjugate momenta. The Hamiltonian is shown to be a linear combination of the Hamiltonian and diffeomorphism constraints arising from the teleparallel field equations. Adding the Nieh-Yan boundary term naturally leads to a formulation closely related to Ashtekar variables.

1. Constrained Hamiltonian systems

Constrained Hamiltonian systems contain restrictions of the form $C = 0$, which must be satisfied on phase space to isolate the true degrees of freedom of the theory.
Constraints arise because the Hessian of the Lagrangian with respect to the velocities is singular, making the Legendre transformation

$$H(p,q) = p\dot{q} - L,$$

non-invertible. Usually, this occurs when the velocities $\dot{q}$ cannot be expressed in terms of the momenta p.

2. ADM formalism

The ADM formalism is the standard Hamiltonian formulation of General Relativity.
The time evolution vector t^μ is decomposed as

$$t^\mu = Nn^\mu + N^\mu,$$

where N and N^μ are the lapse and shift. The induced metric on Σ is

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu,$$

leading to the ADM decomposition

$$ds^2 = -N^2dt^2 + q_{ab}\left(dy^a + N^a dt\right)\left(dy^b + N^b dt\right).$$

Substituting this decomposition into the Einstein-Hilbert action and performing the Legendre transformation yields the Hamiltonian density

$$\mathcal{H} = N\mathcal{C} + N^a\mathcal{C}_a,$$

where $\mathcal{C}$ and $\mathcal{C}_a$ are the Hamiltonian and diffeomorphism constraints.
The non-polynomial form of $\mathcal{C}$ creates significant obstacles for canonical quantization.

3. Tetrad formalism in General Relativity

In General Relativity, we can work with the co-tetrad h^A instead of the metric, related by

$$g_{\mu\nu} = \eta_{AB}h^A{}_\mu h^B{}_\nu.$$

The Levi-Civita connection $\overset{\circ}{\omega}{}^A{}_B$ is a metric connection,

$$\overset{\circ}{\omega}_{AB} = -\overset{\circ}{\omega}_{BA},$$

with vanishing torsion

$$0 = \overset{\circ}{D}h^A = dh^A + \overset{\circ}{\omega}{}^A{}_B \wedge h^B,$$

and non-vanishing curvature

$$\overset{\circ}{\mathcal{R}}{}^A{}_B = d\overset{\circ}{\omega}{}^A{}_B + \overset{\circ}{\omega}{}^A{}_C \wedge \overset{\circ}{\omega}{}^C{}_B.$$

The Einstein-Hilbert action is

$$S_{\mathsf{EH}} = -\int_{\mathcal{M}} \overset{\circ}{R}_{AB} \wedge \star(h^A \wedge h^B).$$

4. Teleparallel formulation of General Relativity

In teleparallel geometry, we use a connection with vanishing curvature $R^A{}_B = 0$, but non-vanishing torsion

$$T^A = Dh^A = dh^A + h^B \wedge \omega^A{}_B.$$

The solution of these conditions is the **teleparallel connection**

$$\omega^A{}_B = \Lambda^A{}_C d(\Lambda^{-1})^C{}_B,$$

which is related to the Levi-Civita connection by

$$\omega^A{}_B = \overset{\circ}{\omega}{}^A{}_B + K^A{}_B,$$

where $K^A{}_B$ is the contortion tensor.
Using these relations, the Einstein-Hilbert Lagrangian can be rewritten as

$$-R_{AB} \wedge \star(h^A \wedge h^B) = T^A \wedge H_A - 2d(h^A \wedge \star T^A),$$

where H^A is the **excitation** 2-form defined by

$$T^A \wedge H_A = h\left(\frac{1}{4}T^A{}_{\mu\nu}T_A{}^{\mu\nu} + \frac{1}{2}T^A{}_{\mu\nu}T^{\nu\mu}{}_A - T^{\rho\mu}{}_\rho T^\sigma{}_{\mu\sigma}\right)d^4x.$$

The action of the **Teleparallel Equivalent of General Relativity** is then

$$S_{\mathsf{TG}} = \int_{\mathcal{M}} T^A \wedge H_A.$$

The field equations obtained from the teleparallel action are

$$DH^A + E^A = 0,$$

where E^A is the gravitational energy–momentum current.

5. 3 + 1 Decomposition of Differential Forms

Given a timelike observer vector field v satisfying $g(v,v) = -1$, we define $\tilde{v} := g(v,\cdot)$. Any differential form α can then be decomposed with respect to v as

$$\alpha = (\alpha)^\parallel + \tilde{v} \wedge (\alpha)_\perp,$$

where

$$(\alpha)^\parallel = i_v(\tilde{v} \wedge \alpha), \qquad (\alpha)_\perp = i_v\alpha.$$

6. Hamiltonian formalism in the language of differential forms

The dynamical variable in TEGR is the tetrad h^A. But ince derivatives of h^A enter the Lagrangian only through the torsion T^A, it can be shown that only the time derivative of the parallel part of the tetrad, $(\dot{h}^A)^\parallel$, appears in the Lagrangian, while $(\dot{h}^A)_\perp$ does not. The momenta are therefore given by

$$P_A = \frac{\partial(i_v L)}{\partial(\dot{h}^A)^\parallel}.$$

where $(\dot{h}^A)^\parallel = \mathcal{L}_v(h^A)^\parallel$.
The Hamiltonian 3-form is then

$$H = P_A \wedge (\dot{h}^A)^\parallel - i_v L,$$

which can be rewritten as

$$H = \left(\frac{\partial L}{\partial T^A} \wedge \dot{h}^A - i_v L\right)^\parallel.$$

This form of the Hamiltonian allows it to be related directly to the equations of motion,

$$H = -v^A\left(dH_A + E_A\right)^\parallel + \left(d\left(v^B H_B\right)\right)^\parallel.$$

The components v^A are straightforward generalizations of the lapse function and shift vector,

$$v^A \leftrightarrow \begin{pmatrix} N \\ N^I \end{pmatrix},$$

and act as Lagrange multipliers of the secondary constraints.
Up to a total derivative term, the Hamiltonian is given by the Hamiltonian constraint

$$\mathcal{C} = (DH_0 + E_0)^\parallel,$$

and the diffeomorphism constraint

$$\mathcal{C}_I = (DH_I + E_I)^\parallel.$$

7. Addition of Nieh-Yan term

The TEGR action can be supplemented by the boundary Nieh-Yan term

$$\mathcal{N} = T^A \wedge T_A = d\left(h_A \wedge T^A\right),$$

multiplied by an arbitrary parameter γ.
The momenta then become

$$P_A = H_A + \gamma T_A.$$

Using the identity

$$H^A \wedge H_A = -T^A \wedge T_A,$$

and choosing $\gamma = \pm i$, the equations of motion can be written in terms of a newly defined exterior covariant derivative acting on the momenta,

$$\tilde{D}P_A := DP_A + \frac{1}{2i}\left[\left(i_{h_A}\left(H^B - iT^B\right)\right) \wedge P_B\right] = 0.$$

The corresponding constraints are

$$\left(\tilde{D}P_A\right)^\parallel = 0.$$

Another interesting feature is that, although the canonical momenta are $(P_A)^\parallel$, the perpendicular part $(P_A)_\perp$ is analogue of the Ashtekar variables,

$$\frac{1}{2}(P_A)_\perp = A_{AI}h^I \equiv A_A.$$

Conclusions and Outlook

- A $3+1$ decomposition identifies the dynamical variables and their canonically conjugate momenta.
- The Hamiltonian is a linear combination of the Hamiltonian and diffeomorphism constraints arising from the teleparallel field equations.
- The inclusion of the Nieh-Yan boundary term leads to a formulation closely related to Ashtekar variables.

References

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