# A Variational Framework for Linearised Theories

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# Contents

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1. In order to prove that $|h_{\mu\nu}| \ll 1$, one should compute it pointwise and the result would depend on the coordinate system;
2. Metrics do not have a vector or affine structure, thus linear combinations are not well defined.

Moreover, since GR is not a linear theory, solutions of linearised equations are not exact solutions of Einstein equations.

Chiaffredo (thesis, 2020); Chiaffredo, Fatibene et al. (2024)

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The second equations are called linearised equations.

They are equations for vertical vector fields $X$.

Solutions of linearised equations are called Jacobi fields.

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Let us see the explicit computation in the simple case of a first-order Lagrangian $\boldsymbol{L} = L(x^\mu, y^i, y^i_\mu)$. The first variation $\delta\boldsymbol{L}$ is

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Linearised equations can be computed to be

$$\begin{aligned}&(\partial_i p_k - d_\mu(\partial_i^\mu p_k))\, X^k + \\ &+ (\partial_i p_k^\nu - \partial_i^\nu p_k - d_\mu(\partial_i^\mu p_k^\nu))\, d_\nu X^k + \\ &- \partial_i^\mu p_k^\nu d_{\mu\nu} X^k = 0\end{aligned}$$

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# Linearised Equations

They can be recast as

$$X^k \partial_k (p_i - d_\alpha p_i^\alpha) + d_\nu X^k \partial_k^\nu (p_i - d_\alpha p_i^\alpha) + d_{\mu\nu} X^k \partial_k^{\mu\nu} (p_i - d_\alpha p_i^\alpha) = 0,$$

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which in turn can be written more compactly as $j^2 X(\mathbb{E}_i(L)) = 0$. In particular, the second-order jet prolongation of Jacobi fields is tangent to the submanifold representing field equations of $\boldsymbol{L}$.

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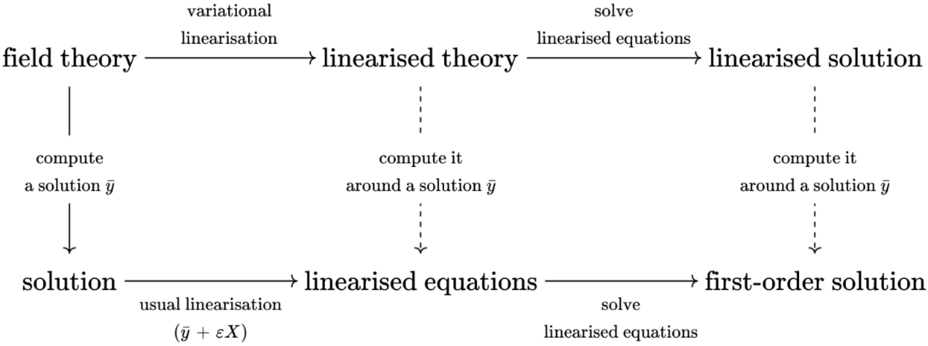
5. Eventually, one can recover the usual results by fixing a specific background.

## Summarising diagram



# Why standard GR

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- It is the best known and most studied gravitational theory;
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We will also include cosmological constant, for the sake of generality.

Let us stress that this can be applied to any gravitational theory, as long as it comes from a variational principle.

Chiaffredo (thesis, 2020); Chiaffredo, Fatibene et al. (2024); Isaia (thesis, 2024)

# Linearised Einstein Equations

Let us consider Hilbert-Einstein Lagrangian with cosmological constant:

$$\boldsymbol{L} = \sqrt{g}\,(R - 2\Lambda)\, d\sigma.$$

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The Jacobi Lagrangian is

$$L_J = \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) X^{\mu\nu} + \sqrt{g}\,(-\nabla_\varepsilon \nabla_\mu X^{\varepsilon\mu} + g_{\mu\nu} \Box X^{\mu\nu})\,.$$

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Linearised Einstein Equations (LEE) are

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\begin{aligned}
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These equations are rather complicated, but they can be significantly simplified.

First, one can cancel the terms corresponding to Einstein Equations.

Chiaffredo (thesis, 2020); Isaia (thesis, 2024)

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Then, one obtains

$$\frac{1}{2} \Box \bar{X}_{\mu\nu} - R_{\mu\alpha\lambda\nu} \bar{X}^{\lambda\alpha} + \frac{2\Lambda}{m-2} \left( \bar{X}_{\mu\nu} - g_{\varepsilon(\mu} g_{\nu)\alpha} \bar{X}^{\alpha\varepsilon} \right) = 0.$$

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In particular, if $\Lambda = 0$,

$$\frac{1}{2} \Box \bar{X}_{\mu\nu} - R_{\mu\alpha\lambda\nu} \bar{X}^{\lambda\alpha} = 0,$$

which is analogous to the GW equations around a general background.

Wald (1984)

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- Through Utiyama-like theorems, solutions were found, which happened to be globally well-defined and non-trivial vector fields reducing to the null field on-shell;
- It was proven that, if $X^{\mu\nu}$ is a solution, then so is $X^{\mu\nu} - 2\nabla_\alpha \xi^{(\nu} g^{\mu)\alpha}$, which is a gauge transformation.

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We are doing this using techniques from canonical analysis.

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- A theory is linear if its bundle is a vector or an affine bundle, and the space of solutions is a vector space.
  It has been proven that, at least in a few crucial cases, if a theory is linear then its linearised equations are two copies of field equations.
- A first step towards a theory of conservation laws in this context has been made. In a natural theory, a vector field on $B$ induces naturally a vector field $\hat{\Xi}$ on $V(B)$. It has been proven that, if $\Xi$ is a Lagrangian symmetry for $\boldsymbol{L}$, then $\hat{\Xi}$ is a Lagrangian symmetry for $\boldsymbol{L}_J$.

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- The main advantage is globality: in terms of GW, this means studying GW around any possible background at the same time.
- We are currently studying methods to solve LEE or, at least, find some properties of solutions.
- The framework is more general: it could be applied also to extended theories of gravitation or to cosmological perturbation theory.

# Main references


[1] F. Chiaffredo, *A Geometric Framework for Perturbation Theory in Field Theories*, Master Thesis, Università degli Studi di Torino, 2019/2020.

[2] F. Chiaffredo, L. Fatibene, M. Ferraris, E. Ricossa, D. Usseglio, *A variational framework for higher order perturbations*, International Journal of Geometric Methods in Modern Physics, 21.10 (2024), 2440007.

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[4] L. Fatibene, *Relativistic Theories, Gravitational Theories and General Relativity*, (unpublished); version 2.0.1, http://www.fatibene.org/book.html.

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[6] L. Fatibene, P. Isaia, *work in progress*.

# A similar situation from Differential Geometry

When approximating a curve $\gamma$ on a manifold $M$,
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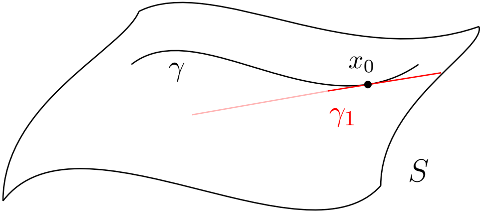

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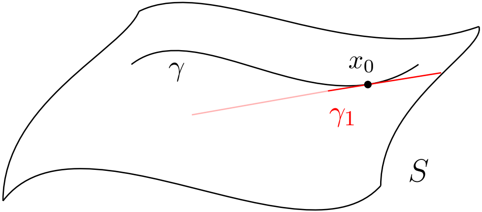


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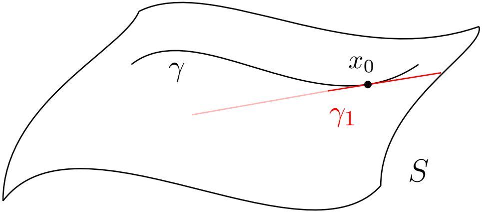


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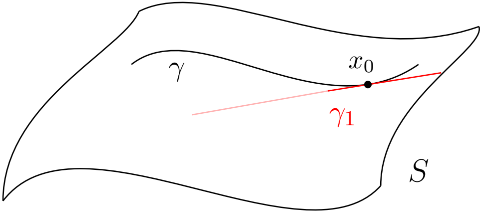


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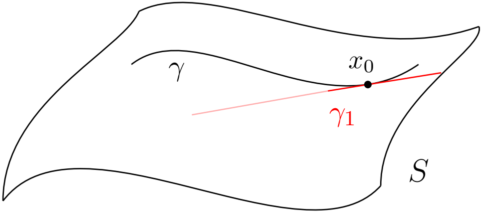


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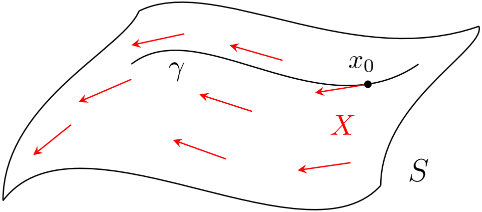

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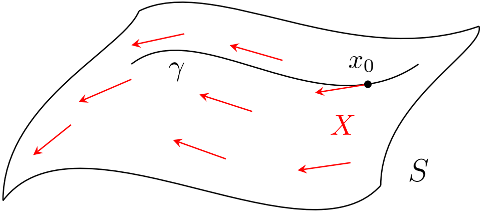


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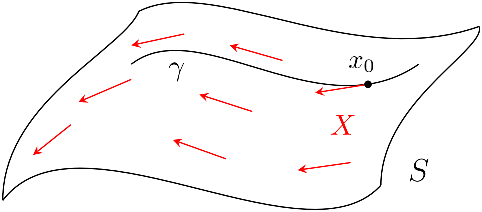


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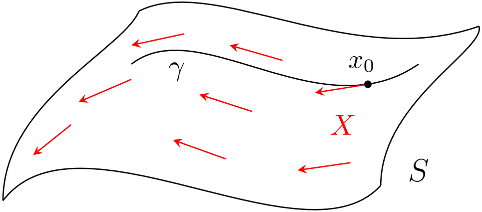


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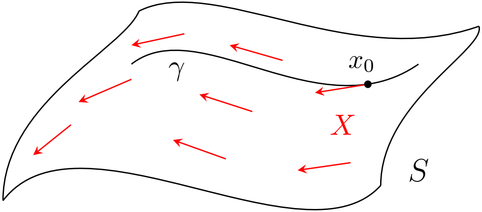


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**Cons**: non-locality, harder computations