The inverse problem of calculus of variations for autoparallels

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Motivation

Typically, in metric-affine gravity, autoparallels are not variational (!)

Finsler metrizability $\iff$ parametrization-invariant Lagrangian for autoparallels.

General Relativity	**Metric-affine gravity**	**Finsler gravity**
$ds^2 = a(dx, dx)$,	$ds^2 = a(dx, dx)$	$ds^2 = L(x, dx)$ - more general
∇ – Levi-Civita	∇ – arbitrary, linear	∇ – canonical (usually nonlinear)

The inverse problem of calculus of variations

Statement of the problem

Given a system of second order ordinary differential equations

$$\ddot{x}^i(t) = f^i\left(x^j(t), \dot{x}^j(t)\right), \quad \text{with} \quad 1 \le i, j \le n$$

that holds at times $0 \le t \le T$, does there exist a Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that the system (3) coincides with the Euler-Lagrange equations of L?

Varational functions

A system of functions

$$\varepsilon = \{\varepsilon_i\}_{i=1}^n, \qquad \varepsilon_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

is called **variational** if there exists a sufficiently smooth function

$$L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

such that

$$\varepsilon_i(x, \dot{x}, \ddot{x}) = \frac{\partial L}{\partial x^i}(x, \dot{x}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}(x, \dot{x})\right), \qquad i = 1, \ldots, n.$$

Characterization of variational systems

Helmholtz conditions

For a system of functions

$$\varepsilon = \{\varepsilon_i\}_{i=1}^n\,, \quad \varepsilon_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

the following conditions are equivalent:

(*i*) The system $\varepsilon = \{\varepsilon_i\}_{i=1}^n$ is variational.

(*ii*) The functions $\{\varepsilon_i\}_{i=1}^n$ satisfy

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} - \frac{\partial \varepsilon_l}{\partial \ddot{x}^i} = 0,$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^l} + \frac{\partial \varepsilon_l}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^l} + \frac{\partial \varepsilon_l}{\partial \ddot{x}^i}\right) = 0,$$

$$\frac{\partial \varepsilon_i}{\partial x^l} - \frac{\partial \varepsilon_l}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^l} - \frac{\partial \varepsilon_l}{\partial \dot{x}^i}\right) = 0.$$

Variational multipliers and the inverse problem

Variational multipliers

Let us assume that there exists a collection of functions $g_{ik} = g_{ik}\left(x^j(t), \dot{x}^j(t)\right)$, such that $\det g_{ik} \neq 0$, so that we can write

$$g_{ik}\left(x^j(t), \dot{x}^j(t)\right)\left(\ddot{x}^i(t) - f^i\left(x^j(t), \dot{x}^j(t)\right)\right) = 0.$$

Introducing the **variational multipliers**

$$\varepsilon_i\left(x^j(t), \dot{x}^j(t), \ddot{x}^j(t)\right) := g_{ik}\left(x^j(t), \dot{x}^j(t)\right)\left(\ddot{x}^i(t) - f^i\left(x^j(t), \dot{x}^j(t)\right)\right),$$

the inverse problem of calculus of variations is equivalent to the following question

Is the system ε_i variational?

The case of geodesics

If f^i is given by $f^i := \overset{\circ}{\Gamma}{}^i_{jk}\dot{x}^i\dot{x}^k$, then the equation in question is the geodesic equation

$$\ddot{\gamma}^i - \overset{\circ}{\Gamma}{}^i_{jk}\dot{\gamma}^i\dot{\gamma}^k = 0,$$

which is variational, and can be obtained from the metric Lagrangian $L = \int_0^1 \sqrt{a\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)}dt$.

Riemannian and non-Riemannian geometry

Levi-Civita connection

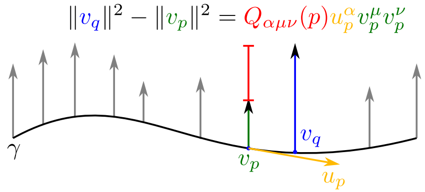
On a Lorentzian manifold (M, a) there exists a unique connection $\mathring{\nabla}$, which satisfies

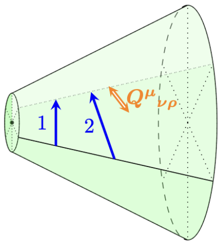
$$\mathring{\nabla} a = 0, \quad \text{and} \quad \mathring{\nabla}_X Y - \mathring{\nabla}_Y X - [X, Y] = 0 \quad \text{for all} \quad X, Y \in \Gamma(TM).$$

Nonmetricity

There do, however, exist affine linear connections ∇, for which

$$(\nabla_X a)(Y, Z) := -Q(X, Y, Z) \neq 0, \quad \forall X, Y, Z \in \Gamma(TM).$$





Vectorial nonmetricity: local description

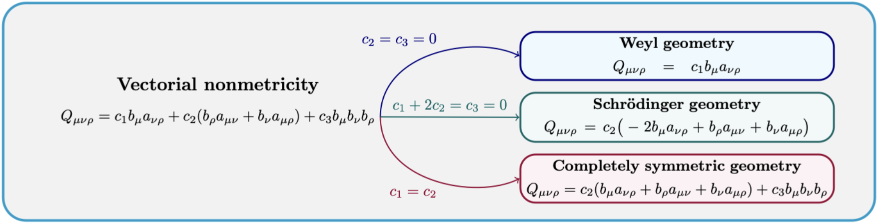
Nonmetricity tensor

For a connection with vectorial nonmetricity, $Q_{\mu\nu\rho}$ takes the form

$$Q_{\mu\nu\rho} = c_1 b_\mu a_{\nu\rho} + c_2 \left(b_\rho a_{\mu\nu} + b_\nu a_{\rho\mu} \right) + c_3 b_\mu b_\nu b_\rho .$$

Why this form of nonmetricity?

- In cosmology, the most general nonmetricity (Iosifidis 2003.07384) is obtained if we allow c_1, c_2, c_3 to depend on time.
- 'Natural' extension (all allowed terms without contractions $b_\mu b^\mu$).
- Phenomenological reasons: the new c_3 term plays a key role in describing dark energy in cosmology.



Autoparallel equation

Levi-Civita autoparallels

$$\overset{\circ}{\nabla}_X X = 0 \iff \ddot{\gamma}^\mu(t) + \overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho}\dot{\gamma}^\nu(t)\dot{\gamma}^\rho(t) = 0 \iff \text{extremals of } L = \int_0^1 \sqrt{|a\left(\dot{\gamma}(t), \dot{\gamma}(t)|\right)}dt.$$

1. Levi-Civita autoparallels are the extremals of the *Riemannian* length functional.

Nonmetric autoparallels

$$\ddot{\gamma}^\mu(t) + \left(\overset{\circ}{\Gamma}{}^\mu{}_{\nu\rho} + b^\mu a_{\nu\rho}\left(\frac{2c_2 - c_1}{2}\right) + \frac{c_1}{2}\delta^\mu_\nu b_\rho + \frac{c_1}{2}\delta^\mu_\rho b_\nu + \frac{c_3}{2}b^\mu b_\nu b_\rho\right)\dot{\gamma}^\nu(t)\dot{\gamma}^\rho(t) = 0. \quad (1)$$

Main question

Does there exist a reparametrization-invariant Lagrangian, whose extremals coincide with (1)?

Finsler metrics

Conic subbundle

Let M be a smooth manifold, and denote with (TM, π, M) its tangent bundle. We have naturally induced coordinates $(x, \dot{x}) \equiv (x^a, \dot{x}^a)$. A **conic subbundle** of TM is an open subset $\mathcal{A} \subset TM \setminus \{0\}$, such that:

1. $(x, \dot{x}) \in \mathcal{A} \implies (x, \alpha\dot{x}) \in \mathcal{A}, \ \ \forall \alpha > 0$ (conic property);
2. All fibers $\mathcal{A}_x := \mathcal{A} \cap T_x M$ are non-empty.

Pseudo-Finsler space

A **pseudo-Finsler space** is a pair (M, L), where M is a smooth manifold, and L is a smooth map

$$L : \mathcal{A} \to \mathbb{R}, \ \ (x, \dot{x}) \mapsto L(x, \dot{x}),$$

which satisfies:

1. $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x}), \ \ \forall \alpha > 0$ (positive 2-homogeneity);
2. $g_{\mu\nu}(x, \dot{x}) := \dfrac{1}{2} \dfrac{\partial^2 L}{\partial \dot{x}^\mu \dot{x}^\nu}(x, \dot{x})$ is nondegenerate $\forall (x, \dot{x}) \in \mathcal{A}$.

Finslerian arc length

Finslerian arc length

Let (M, L) be a pseudo-Finsler space. An **admissible curve** is a smooth mapping $\gamma : [0,1] \to M$, such that (the image of) its natural extension to TM

$$\gamma : [0,1] \to TM, \quad \tau \mapsto (\gamma(\tau), \dot{\gamma}(\tau))$$

lies in $\mathcal{A}$. For an admissible curve, its **Finslerian arc length** is defined as

$$l[\gamma] = \int_0^1 \sqrt{|L(\gamma(\tau), \dot{\gamma}(\tau))|} d\tau.$$

Finsler geometry — most general notion of parametrization-independent arc length.

Finsler geodesics and nonlinear connections

Finsler geodesics

The **geodesics** of a pseudo-Finsler space (M, L) are defined as critical curves of the Finslerian arc length $l[\gamma]$. In arc length parametrization they are given by

$$\ddot{x}^\mu(t) + 2G^\mu(x(t), \dot{x}(t)) = 0, \quad \text{where} \quad G^\mu(x, \dot{x}) = \frac{1}{4} g^{\mu\nu} \left(\frac{\partial^2 L}{\partial \dot{x}^\nu \partial x^\kappa} \dot{x}^\kappa - \frac{\partial L}{\partial x^\nu} \right)$$

are called the **spray coefficients**, which give rise to a canonical nonlinear connection.

Nonlinear connection

A **nonlinear connection** N on $\mathcal{A} \subseteq TM \setminus \{0\}$ is a smooth assignment

$$N : \mathcal{A} \to T\mathcal{A}, \quad (x, \dot{x}) \mapsto H_{(x,\dot{x})}\mathcal{A} \subseteq T_{(x,\dot{x})}\mathcal{A}\,,$$

where $H_{(x,\dot{x})}\mathcal{A}$ is an n-dimensional horizontal subspace complementary to the vertical subspace

$$V_{(x,\dot{x})}\mathcal{A} := \ker d\pi_{(x,\dot{x})} = Span\left\{ \frac{\partial}{\partial \dot{x}^\mu} \right\}.$$

Nonlinear connections

A nonlinear connection gives a *local adapted basis* of $T\mathcal{A}$

$$\left\{\delta_\mu := \partial_\mu - G^\nu{}_\mu(x,\dot{x})\frac{\partial}{\partial \dot{x}^\nu}\,;\,\frac{\partial}{\partial \dot{x}^\mu}\right\},\;\; H\mathcal{A} = Span\left\{\delta_\mu\right\},\;\; V\mathcal{A} = Span\left\{\frac{\partial}{\partial \dot{x}^\mu}\right\}.$$

The spray coefficients G^ν induce *canonically* a nonlinear connection via

$$G^\nu{}_\mu(x,\dot{x}) = \frac{\partial G^\nu(x,\dot{x})}{\partial \dot{x}^\mu}\,.$$

If (M, L) is pseudo-Riemannian, that is, $L = a_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$ is quadratic in $\dot{x}$, then

$$G^\nu{}_\mu(x,\dot{x}) = \mathring{\Gamma}^\nu{}_{\mu\rho}(x)\dot{x}^\rho$$

is linear in $\dot{x}$. Though, typically, $G^\nu{}_\mu(x,\dot{x})$ are *not linear*, but just 1-*homogeneous* in $\dot{x}$.
Key property of the nonlinear connection:

$$\boxed{\delta_\mu L = 0.}$$

Berwald spaces and metrizability

Berwald space

A pseudo-Finsler space (M, L) is called **Berwald** iff its spray coefficients are quadratic in $\dot{x}$

$$2G^{\mu}(x, \dot{x}) = \Gamma^{\mu}{}_{\nu\rho}(x)\dot{x}^{\nu}\dot{x}^{\rho}.$$

Metrizability

A symmetric affine connection $\nabla \in Conn(M)$ on M is

1. **Finsler metrizable** if there exists a Finsler function $L : \mathcal{A} \to \mathbb{R}$, such that the autoparallel curves of ∇ coincide with the Finsler geodesics of L, that is, locally,

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\nu\rho}(x)\dot{x}^{\nu}\dot{x}^{\rho} = 0 \quad \Longleftrightarrow \quad \text{Finsler geodesics of } L\,.$$

2. **Pseudo-Riemann-metrizable** if there exists a pseudo-Riemannian metric a such that ∇ coincides with the Levi-Civita connection of a, that is, locally,

$$\Gamma^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\nu\rho}[a]\,.$$

Finsler metrizability

Bucataru-Dahl theorem

The Helmholtz conditions for an affine connection $\iff$ for each local chart domain, $\exists g_{ab} = g_{ab}(x, \dot{x})$ symmetric and non-degenerate such that

$$\nabla g_{ab} = 0, \quad g_{ac} R^c_{\ d} = g_{dc} R^c_{\ a}.$$

Local Finsler metrizability

A symmetric affine connection ∇ on a manifold M is locally Finsler metrizable if and only if, corresponding to each local chart, there exists a symmetric matrix $g_{ab} = g_{ab}(x, \dot{x})$ with positively 0-homogeneous in $\dot{x}$ components such that

$$\nabla g_{ab} = 0, \quad g_{ac} R^c_{\ d} = g_{dc} R^c_{\ a}.$$

Main conclusion

Let ∇ be a torsion-free affine connection on M. Then, the following are equivalent:

1. The autoparallel equations are variational, with positively 0-homogeneosu in $\dot{x}$ variational multipliers.
2. ∇ is Finsler metrizable.

Finsler metrizability of vectorial nonmetricity

Input data

A connection with vectorial nonmetricity, specified by

1. A pseudo-Riemannian metric a, locally given by its components $a_{\mu\nu}$.
2. A one-form b, locally given by its components b_μ .
3. Three real coefficients c_1, c_2, c_3.

The input data induces a connection on TM with horizontal basis vectors

$$\delta_\mu = \overset{\circ}{\delta}_\mu - \frac{2c_2 - c_1}{2} \dot{x}_\mu b^\nu \frac{\partial}{\partial \dot{x}^\nu} - \frac{c_1}{2} \left(b_\mu \dot{x}^\mu \delta^\nu_\mu + b_\mu \dot{x}^\nu \right) \frac{\partial}{\partial \dot{x}^\nu} - \frac{c_3}{2} b_\mu b_\rho \dot{x}^\rho b^\nu \frac{\partial}{\partial \dot{x}^\nu}.$$

Main question

Find a Finsler Lagrangian L, and conditions on the input data (b, c_1, c_2, c_3), such that

$$\delta_\mu L = 0$$

can be solved.

Finsler metrizability of vectorial nonmetricity

(α, β)-metric

Let $b = b_\mu(x)dx^\mu$ be a nonvanishing one-form with contraction

$$B := b_\mu \dot{x}^\mu.$$

An **(α, β)-metric** is a Finsler structure on M whose fundamental function is given by

$$L(x, \dot{x}) = A\Phi(s), \quad s := \frac{B^2}{A}, \quad \text{with} \quad A = a_{\mu\nu}(x)dx^\mu dx^\nu,$$

where Φ is a nontrivial smooth function.

Generalized (α, β)-metric

For the most general *algebraic* dependence of the Finsler Lagrangian L on the metric $a_{\mu\nu}$ and the one-form b_μ, we consider $L = A\Phi\left(|b|, p\right)$, where

$$|b| = \sqrt{|\langle b, b\rangle|}, \quad b_\mu = |b|u_\mu, U = u_\mu \dot{x}^\mu, p = \frac{U^2}{A}, \quad \epsilon = u_\mu u^\mu.$$

Finsler metrizability of vectorial nonmetricity

Disformation tensor

For a symmetric affine connection $\nabla \in Conn(M)$, the disformation tensor is locally given by

$$D^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(Q_{\nu\rho}{}^{\mu} + Q_{\rho}{}^{\mu}{}_{\nu} - Q^{\mu}{}_{\nu\rho}\right).$$

Characterization of Finsler metrizability by (α, β)-metrics

The Finsler (α, β)-metric $L = A\Phi$ metrizes the symmetric affine connection $\Gamma = \overset{\circ}{\Gamma} + D \in Conn(M)$ if and only if Φ and B solve the system

$$\Phi' B\left(A\overset{\circ}{\delta}_{\mu} B - AD^{\nu}{}_{\mu} b_{\nu} + BD^{\nu}{}_{\mu}\dot{x}_{\nu}\right) = \Phi AD^{\nu}{}_{\mu}\dot{x}_{\nu}, \quad \forall \mu = 0, .., 3.$$

Finsler metrizability of vectorial nonmetricity

Main theorem for (α,β) -metrics

A connection with vectorial nonmetricity

$$\Gamma^{\mu}{}_{\nu\rho} = \mathring{\Gamma}^{\mu}{}_{\nu\rho} + b^{\mu}a_{\nu\rho}\left(\frac{2c_2 - c_1}{2}\right) + \frac{c_1}{2}\delta^{\mu}_{\nu}b_{\rho} + \frac{c_1}{2}\delta^{\mu}_{\rho}b_{\nu} + \frac{c_3}{2}b^{\mu}b_{\nu}b_{\rho}$$

with c_1, c_2, c_3 not all zero is Finsler metrizable by a Berwald-type (α,β)-metric $L = A\Phi(s)$ if and only if one of the following happens:

1. $c_2 = c_3 = 0$ and there exists a constant $\lambda \neq 0$, such that

$$\mathring{\nabla}_{\mu}b_{\nu} = \frac{c_1}{2}\left(-\langle b,b\rangle a_{\mu\nu} + \left(\frac{1}{\lambda} + 1\right)b_{\mu}b_{\nu}\right).$$

2. $c_2 = 0, c_3 \neq 0$ and there exists $\tau \in \mathbb{R}$ such that

$$\mathring{\nabla}_{\mu}b_{\nu} = \frac{c_3}{2}\left(-\frac{c_1}{c_3}\langle b,b\rangle a_{\mu\nu} + \left(\frac{c_1}{c_3} + \tau + \langle b,b\rangle\right)b_{\mu}b_{\nu}\right).$$

Finsler metrizability of vectorial nonmetricity

The (α, β)-metrics

Under these conditions, the function L belongs to one of the following four families:

1. Power law for $c_2 = c_3 = 0$
$$L = \kappa A s^{\lambda}, \quad \kappa \in \mathbb{R}^* .$$
2. Generalized m-kropina for $c_1 \neq 0, \tau \neq 0$
$$L = \kappa A s^{\frac{c_1}{\tau c_3}} (s + \tau)^{1 - \frac{c_1}{\tau c_3}}, \quad \kappa \in \mathbb{R}^* .$$
3. Riemannian for $c_1 = 0, \tau \neq 0$
$$L = \kappa \left(\tau A + B^2 \right), \quad \kappa \in \mathbb{R}^* .$$
4. Exponential type for $c_1 \neq 0, \tau = 0$
$$L = \kappa B^2 e^{-\frac{c_1}{c_3 s}} .$$

Finsler metrizability of vectorial nonmetricity

Excluding the (α, β)-cases

If either $\langle b, b \rangle$ is constant or $\Phi'_{|b|}$ vanishes identically, the generalized (α, β) metric reduces to the standard (α, β) case. To avoid this degeneracy, we shall assume throughout that

$$\langle b, b \rangle \neq \text{const},$$

and, apart from possible isolated points,

$$\Phi'_{|b|} \neq 0.$$

Characterization of generalized (α, β)-metrizability

A symmetric connection ∇ on M is metrizable by a generalized (α, β)-metric if and only if its distortion D satisfies, in any local chart, the condition

$$\frac{1}{2} A^2 \Phi'_{|b|} \partial_\mu |b| + \Phi'_p U \left(A \overset{\circ}{\delta}_\mu U - A \left(D^\nu{}_\mu u_\nu \right) + U \left(D^\nu{}_\mu \dot{x}_\nu \right) \right) = A \left(D^\nu{}_\mu \dot{x}_\nu \right) \Phi \quad \forall \mu = 0, \ldots, 3 \,.$$

Finsler metrizability of vectorial nonmetricity

Lemma

The contraction of the Levi-Civita covariant derivative of U along u^μ vanishes

$$u^\mu \overset{\circ}{\delta}_\mu U = 0\,.$$

Moreover, $u^\mu \partial_\mu |b|$ only depends on $|b|$ and not on the individual components of b_μ.

Lemma

If a connection with vectorial nonmetricity is metrizable by a generalized (α, β)-metric, then there exists a function $\lambda = \lambda(|b|)$, such that

$$\partial_\mu |b| = \lambda u_\mu .$$

Lemma

If a connection with vectorial nonmetricity is metrizable by a generalized (α, β)-metric, then the one-form u must satisfy

$$\overset{\circ}{\nabla}_\mu u_\nu = \tau \left(a_{\mu\nu} - \epsilon u_\mu u_\nu \right) .$$

Finsler metrizability of vectorial nonmetricity

Main theorem for generalized (α,β)-metrics

A connection $\nabla = \overset{\circ}{\nabla} + D$ with nonzero vectorial nonmetricity is pseudo-Finsler-metrizable by a generalized (α,β)-metric $L = A\Phi(|b|, p)$ if and only if the following conditions are simultaneously satisfied:

1. $c_3 = 0$, or $c_1 = c_2 = 0$.
2. There hold the equalities

$$d|b| = \lambda u, \quad \overset{\circ}{\nabla} u = \tau a - \epsilon u \otimes u,$$

   where $\lambda = \lambda(|b|)$ is an arbitrary, nowhere zero smooth function and $\tau = \tau(|b|)$ is defined in terms of λ as

$$\tau = \frac{c_1|b|}{c_1 C_1 e^{(c_1+2c_2)\rho(|b|)} - 2\epsilon}, \qquad \text{with} \qquad \rho(|b|) = \int \frac{|b|}{\lambda(|b|)} d|b|$$

   and $C_1 \in \mathbb{R}$ is an arbitrary constant satisfying $c_2(c_1 - 2c_2)C_1 = 0$.

Finsler metrizability of vectorial nonmetricity

The generalized (α, β)-metrics

1. If $c_3 \neq 0, c_1 = c_2 = 0$, then

$$\Phi(|b|, p) = \frac{p}{\epsilon} \exp\left(c_3 \epsilon \int \frac{|b|^3}{\lambda(|b|)} d|b|\right) F\left(\frac{e^{-c_3 \epsilon \int \frac{|b|^3}{\lambda(|b|)} d|b|}(\epsilon - p)}{p\epsilon}\right).$$

2. If $c_2 = c_3 = 0, c_1 \neq 0$, then

$$\Phi(|b|, p) = \exp\left(\left(\frac{\epsilon}{c_1} - \frac{C_1}{2} e^{c_1 \rho}\right)\left(c_1 \epsilon p + \frac{C_2 - 2c_1 \rho}{C_1 e^{c_1 \rho} - 2\frac{\epsilon}{c_1}}\right)\right).$$

3. If $c_3 = 0, c_2 \neq 0, c_1 - 2c_2 \neq 0$, then

$$\Phi(|b|, p) = e^{\frac{\epsilon}{c_1}\left(\epsilon c_1^2 \rho + p^2 c_2 + p\epsilon(c_1 - 2c_2) + C_3\right)}.$$

4. If $c_3 = 0, c_2 \neq 0, c_1 - 2c_2 = 0$, then

$$\Phi(|b|, p) = e^{\left(\frac{\epsilon}{2} - \frac{1}{2} C_1 c_2 e^{4\rho c_2}\right) p^2 + \left(2c_2 \rho - \frac{1}{2} C_4\right)}.$$

In cases $2-4$, C_1, C_2, C_3, C_4 are real constants, and ρ is defined as

$$\rho = \int \frac{|b|}{\lambda(|b|)} d|b|.$$

Summary, main achievements

Main results

1. *Physical:* Weyl, Schrödinger and completely symmetric autoparallels are Finsler geodesics "in disguise".
2. *Mathematical:* Classification of Berwald (α, β) and generalized (α, β) metrics.

Connection	(α, β)-metrizable	Generalized (α, β)-metrizable
Weyl	✓	✓
Schrödinger	✗	✓
Completely symmetric	✓	✓

Future directions

1. Classify all Berwald Finsler metrics, which are not pseudo-Riemann metrizable.
2. Check whether the found metrics solve the field equations of Finsler gravity in vacuum.