3+1 decomposition and Hamiltonian analyses in symmetric teleparallel theories

Dario Sauro

Theoretical Physics Institute, Friedrich Schiller University of Jena

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Motivation

The physical content of teleparallel theories

The coincident gauge is used in four-dimensional descriptions to get the purely dynamical part of the field equations

Is this gauge choice really necessary when we employ a $3+1$ decomposition?

If the answer is no, then we need to be able to covariantly "kill" some extrinsic and intrinsic parts of non-metricity, so that the theory automatically belongs to the teleparallel submanifold

How does the Hamiltonian depend on extrinsic tensors?

$3+1$ decomposition of a non-metric theory

A $3+1$ decomposition is identified by an orthonormal vector n_μ, and by introducing coordinates y^a on the hypersurface Σ we have that the basis vectors $e^\mu{}_a \equiv \frac{\partial x^\mu}{\partial y^a}$ are such that

$$
\begin{aligned}
& g_{\mu\nu} = \epsilon\, n_\mu n_\nu + e_\mu{}^a e_\nu{}^b h_{ab}\,, \qquad n^\mu n_\mu = \epsilon \quad \epsilon = \pm 1\,, \\
& e^\mu{}_a n_\mu = 0\,,
\end{aligned}
$$

where h_{ab} is the intrinsic metric

$$
ds^2_\Sigma = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} dy^a dy^b \equiv h_{ab} dy^a dy^b\,.
$$

Then, if A_μ is orthogonal to n^μ we can single out the three-connection as

$$
\begin{aligned}
D_a A_b &\equiv e^\alpha{}_a e^\beta{}_b \nabla_\beta A_\alpha \\
&= \nabla_\beta \left(e^\alpha{}_a A_\alpha \right) e^\beta{}_b - e^\beta{}_b A_\alpha \nabla_\beta e^\alpha{}_a \\
&= \partial_b A_a - \Gamma^c{}_{ab} A_c\,.
\end{aligned}
$$

Intrinsic and extrinsic connection

The three-connection can be decomposed in the usual manner

$$\Gamma^c{}_{ab} = \mathring{\Gamma}^c{}_{ab} + L^{(3)c}{}_{ab} = e^\beta{}_b e^{\rho c} \mathring{\nabla}_\beta e_{\rho a} + L^\rho{}_{\nu\mu} e^\nu{}_a e^\mu{}_b e_\rho{}^c \,.$$

The intrinsic non-metricity and extrinsic curvature are defined as

$$\begin{aligned} Q^{(3)}{}_{cab} \equiv& e^\rho{}_c e^\mu{}_a e^\nu{}_b Q_{\rho\mu\nu} = D_c h_{ab} \,, \\ K_{ab} \equiv& e^\mu{}_a e^\nu{}_b \nabla_\nu n_\mu = \mathring{K}_{ab} - L^\lambda{}_{\mu\nu} n_\lambda e^\mu{}_a e^\nu{}_b \,. \end{aligned}$$

Then, the Gauss-Weingarten relation

$$e^\nu{}_a \nabla_\nu e^\mu{}_b = e^\mu{}_c \Gamma^c{}_{ba} - \epsilon\, K_{ba} n^\mu$$

can be used to derive

$$K_{ab} = \frac{1}{2} \left(\mathcal{L}_n g_{\mu\nu} \right) e^\mu{}_a e^\nu{}_b - L^\lambda{}_{\mu\nu} n_\lambda e^\mu{}_a e^\nu{}_b \,.$$

The new extrinsic tensors

We can define the extrinsic tensors by enlisting all non-trivial three-dimensional expressions involving ${e_\mu}^a$, n_μ, h_{ab} and their first derivatives:

$$\text{Rank}-2: \quad \Psi^{ab} \equiv e^{\mu a} n^\rho \nabla_\rho {e_\mu}^b \,, \qquad \Xi_{ab} \equiv n^\rho {h_a}^c {h_b}^d \partial_\rho h_{cd}$$

$$\Phi^{ab} \equiv n^\mu e^{\rho a} \nabla_\rho {e_\mu}^b \,, \quad \Lambda_{ab} \equiv {e^\mu}_a n^\rho \nabla_\rho e_{\mu b} \,, \quad \Xi_{ab} = \Lambda_{(ab)} - \Psi_{(ab)} \,.$$

$$\text{Rank}-1: \quad \kappa_a \equiv {e^\mu}_a n^\rho \nabla_\rho n_\mu \,, \qquad \lambda^a \equiv n^\rho n^\mu \nabla_\rho {e_\mu}^a \,, \qquad \theta_a \equiv 2 n^\mu {e^\rho}_a \nabla_\rho n_\mu \,.$$

$$\text{Rank}-0: \quad \alpha \equiv n^\nu n^\nu \nabla_\nu n_\mu \,.$$

This yields the completeness relation of non-metricity

$$Q^{\rho\mu\nu} = 2\epsilon\, \alpha n^\rho n^\mu n^\nu + 2 n^\rho n^{(\mu} e^{\nu)a} \left(\kappa_a + \lambda_a\right) + n^\mu n^\nu e^{\rho a} \theta_a$$
$$+ \epsilon \left[2 e^{\rho a} e^{(\nu b} n^{\mu)} \left(K_{ab} + \Phi_{ab}\right) + n^\rho e^{\mu a} e^{\nu b} \left(\Xi_{ab} + 2\Psi_{(ab)}\right)\right] + e^{\rho c} e^{\mu a} e^{\nu b} {Q^{(3)}}_{cab} \,.$$

Gauss-Codazzi 1

By contracting the Riemann tensor with n^μ and $h^\mu{}_\alpha$ and by repeatedly using integration by parts we derive the Gauss-Codazzi relations.

Generalized Ricci relations:

$$\begin{aligned} R^\mu{}_{\lambda\rho\nu} n^\lambda n_\mu h^\rho{}_\alpha h^\nu{}_\beta =& 2K_{\mu[\alpha}\Phi_{\beta]}{}^\mu - D_{[\alpha}\theta_{\beta]}\,, \\ R^\mu{}_{\alpha\beta\nu} h^\alpha{}_\lambda n_\mu n^\beta h^\nu{}_\rho =& D_\rho\kappa_\lambda - \epsilon\,\kappa_\rho\kappa_\lambda + \epsilon\,\alpha K_{\lambda\rho} - K_{\rho\mu}\Phi_\lambda{}^\mu - \mathcal{L}_n K_{\lambda\rho}\,, \\ R^\mu{}_{\alpha\nu\beta} n^\alpha n^\beta h^\sigma{}_\mu h^\nu{}_\rho =& - D_\rho\lambda^\sigma + \epsilon\,\lambda^\sigma\left(\kappa_\rho - \theta_\rho\right) + \epsilon\,\alpha\Phi_\rho{}^\sigma \\ & - \Phi_\mu{}^\sigma\Phi_\rho{}^\mu + h^\sigma{}_\mu\mathcal{L}_n\Phi_\rho{}^\mu\,. \end{aligned}$$

Rank-one relation

$$R^\mu{}_{\lambda\rho\nu} n^\lambda n_\mu n^\rho h^\nu{}_\alpha = D_\alpha\alpha + \epsilon\,\alpha\left(\tfrac{1}{2}\theta_\alpha - \kappa_\alpha\right) + \kappa^\mu\Phi_{\alpha\mu} - \lambda^\mu K_{\alpha\mu} - \tfrac{1}{2}\mathcal{L}_n\theta_\alpha\,.$$

Gauss-Codazzi 2

First and second generalized Codazzi relations:

$$R^{\mu}{}_{\lambda\rho\nu} n_{\mu} h^{\lambda}{}_{\alpha} h^{\rho}{}_{\beta} h^{\nu}{}_{\sigma} = 2 D_{[\sigma} K_{\beta]\alpha} + \epsilon\, \theta_{[\beta} K_{\sigma]\alpha} \,,$$

$$R^{\mu}{}_{\lambda\rho\nu} n^{\lambda} h_{\mu\sigma} h^{\rho}{}_{\alpha} h^{\nu}{}_{\beta} = -2 D_{[\alpha} \Phi_{\beta]}{}^{\sigma} + \epsilon\, \theta_{[\beta} \Phi_{\alpha]}{}^{\sigma} \,.$$

Third generalized Codazzi relation

$$R^{\rho}{}_{\sigma\mu\nu} h_{\rho\alpha} h^{\sigma}{}_{\beta} h^{\mu}{}_{\lambda} n^{\nu} = \Big[\frac{1}{2} \Big(D_{\alpha} + \epsilon \Big(\kappa_{\alpha} - \tfrac{1}{2}\theta_{\alpha} \Big) \Big) \Big(\Xi_{\beta\lambda} + 2\Psi_{(\beta\lambda)} - 2\Phi_{(\beta\lambda)} \Big) - (\alpha \leftrightarrow \beta) \Big] - \frac{1}{2} \Big(D_{\lambda} - \epsilon \Big(\kappa_{\lambda} - \tfrac{1}{2}\theta_{\lambda} \Big) \Big) \Big(\Xi_{\alpha\beta} + 2\Psi_{(\alpha\beta)} - 2\Phi_{[\alpha\beta]} \Big) + \epsilon \Big(\kappa_{\beta} \Phi_{\lambda\alpha} - \lambda_{\alpha} K_{\lambda\beta} \Big) + L^{(3)\rho}{}_{\beta\lambda} \Phi_{\rho\alpha} - L^{(3)}{}_{\alpha\lambda\rho} \Phi_{\beta}{}^{\rho} - h^{\rho}{}_{\alpha} h^{\nu}{}_{\beta} h^{\mu}{}_{\lambda} \mathcal{L}_{n} L^{(3)}{}_{\rho\nu\mu} \,.$$

The teleparallel limit - 1

In the teleparallel limit the right-hand sides of the Gauss-Codazzi relations must vanish identically

Distinction between D_α and $\mathcal{L}$:

- We solve for $\mathcal{L}F$
- We treat DF as an expression homogeneous in F.

Using these prescriptions we find that the teleparallel condition is met in the first and second Codazzi and Ricci relations provided that

$$K_{\mu\nu} = 0\,, \qquad \Phi_{\mu\nu} = 0\,, \qquad \kappa_\mu = 0\,, \qquad \lambda_\mu = 0\,,$$

Furthermore, the rank-one relation is satisfied if

$$\mathcal{L}_n \theta_\mu = D_\mu \alpha + \frac{\epsilon}{2} \alpha\, \theta_\mu\,.$$

The teleparallel limit - 2

Let us define

$$S_{ab} + A_{ab} \equiv n^{\nu} e^{\rho}{}_{a} e^{\mu}{}_{b} Q_{\rho\mu\nu} = K_{ab} + \Phi_{ab}\,, \quad \Sigma_{ab} \equiv n^{\rho} e^{\mu}{}_{a} e^{\nu}{}_{b} Q_{\rho\mu\nu} = \Xi_{ab} + 2\Psi_{(ab)}\,,$$

where $S_{[ab]} = 0$ and $A_{(ab)} = 0$. Then, the third Codazzi relations implies

$$h^{\rho}{}_{\alpha} h^{\nu}{}_{\beta} h^{\mu}{}_{\lambda} \mathcal{L}_n L^{(3)}{}_{\rho\nu\mu} = \frac{1}{2}\left(D_{\alpha} - \frac{\epsilon}{2}\theta_{\alpha}\right)\Sigma_{\beta\lambda} - \left(D_{(\beta} - \frac{\epsilon}{2}\theta_{(\beta}\right)\Sigma_{\lambda)\alpha}\,.$$

Thus, the teleparallel expression of the completeness relation of non-metricity turns out to be

$$Q_{\rho\mu\nu} = 2\epsilon\,\alpha n_{\mu} n_{\nu} n_{\rho} + n_{\mu} n_{\nu} \theta_{\rho} + \epsilon\, n_{\rho} \Sigma_{\mu\nu} + Q^{(3)}{}_{\rho\mu\nu}\,.$$

Variational problem in symmetric teleparallel theories

The Lagrange multiplier enters the Lagrangian in the following geometric expression, which does not require the existence of a metric structure

$$\mathcal{L}_{LM} = R^{\rho}{}_{\lambda} \wedge \kappa_{\rho}{}^{\lambda} = R^{\rho}{}_{\lambda[\mu\nu]}\kappa_{\rho}{}^{\lambda}{}_{[\alpha\beta]}\, dx^{\mu} \wedge dx^{\nu} \wedge dx^{\alpha} \wedge dx^{\beta} \,.$$

The Lagrange multiplier enjoys a gauge-like symmetry transformation

$$\delta_{LM}\kappa^{\rho}{}_{\alpha} \equiv \delta_{LM}\kappa^{\rho}{}_{\alpha\mu\nu} dx^{\mu} \wedge dx^{\nu} = \nabla\lambda^{\rho}{}_{\lambda} \,.$$

The teleparallel action is

$$S[g,\Gamma,\kappa] = \int_{\mathcal{V}} \left[d^4x\sqrt{-g}\, \left(\overline{\mathrm{Q}} + \mathring{\nabla}_{\mu}\left(Q^{\mu} - \tilde{Q}^{\mu}\right)\right) + R^{\rho}{}_{\lambda} \wedge \kappa_{\rho}{}^{\lambda} \right] \,.$$

By varying the first term we find

$$\begin{aligned}\delta\sqrt{-g}\,\overline{\mathrm{Q}} \equiv& \sqrt{-g}\left[\bar{q}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{\mathrm{Q}}\right]\delta g^{\mu\nu} + \sqrt{-g}Y^{\rho}{}_{\mu\nu}\nabla_{\rho}\delta g^{\mu\nu} + \sqrt{-g}X_{\rho}{}^{\mu\nu}\delta\Gamma^{\rho}{}_{(\mu\nu)} \\ =& \sqrt{-g}\left[\bar{q}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\overline{\mathrm{Q}} - \nabla_{\rho}Y^{\rho}{}_{\mu\nu} - \frac{1}{2}Q_{\rho}Y^{\rho}{}_{\mu\nu}\right]\delta g^{\mu\nu} + \sqrt{-g}X_{\rho}{}^{\mu\nu}\delta\Gamma^{\rho}{}_{(\mu\nu)} \,,\end{aligned}$$

Field equations

The variation of the boundary term yields

$$\delta \int_{\mathcal{V}} \sqrt{-g}\mathring{\nabla}_\mu \left(Q^\mu - \tilde{Q}^\mu\right) = \int_{\partial\mathcal{V}} \epsilon \sqrt{|h|} n^\mu h^{\nu\lambda} \partial_\mu \delta g_{\nu\lambda} \, ,$$

which requires introducing the counter-term

$$S_B = -\frac{1}{16\,\pi} \oint_{\partial\mathcal{V}} \sqrt{|h|}\, \epsilon\, n^\rho h^{\mu\nu} \left(\nabla_\mu g_{\rho\nu} - \nabla_\rho g_{\mu\nu}\right) \, .$$

Finally the resulting field equations are

$$\bar{q}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overline{\mathrm{Q}} - \nabla_\rho Y^\rho{}_{\mu\nu} - \frac{1}{2} Q_\rho Y^\rho{}_{\mu\nu} = 0 \, ,$$
$$X_\rho{}^{(\lambda\mu)} \sqrt{-g}\, d^4x + dx^{(\mu} \wedge \nabla \kappa_\rho{}^{\lambda)} = 0 \, ,$$
$$R^\rho{}_{\lambda[\mu\nu]} dx^\mu dx^\nu = 0 \, .$$

The metric field equations are insensitive to the presence of the Lagrange multiplier.

Metric velocity: Palatini vs teleparallel

In the presence of non-metricity the Lie derivative of the metric along $t^\mu = Nn^\mu + N^\mu$ reads

$$\begin{aligned}\mathcal{L}_t g_{\mu\nu} &= t^\alpha \nabla_\alpha g_{\mu\nu} + g_{\mu\alpha} \nabla_\nu t^\alpha + g_{\nu\alpha} \nabla_\mu t^\alpha \\ &= N\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) + \nabla_\mu N_\nu + \nabla_\nu N_\mu + n_\mu \nabla_\nu N + n_\nu \nabla_\mu N \\ &\quad + Nn^\alpha \left(Q_{\alpha\mu\nu} - 2Q_{(\mu\nu)\alpha}\right) + N^\alpha \left(Q_{\alpha\mu\nu} - 2Q_{(\mu\nu)\alpha}\right) .\end{aligned}$$

By projecting onto the three-dimensional we find

$$\dot{h}_{ab} = N\left(\Xi_{ab} + 2\Psi_{(ab)} - 2\Phi_{(ab)}\right) + 2D_{(a}N_{b)} + N^c \left({Q^{(3)}}_{cab} - 2{Q^{(3)}}_{(ab)c}\right) .$$

Using the new variables in the Palatini case we obtain

$$\begin{aligned}\dot{h}_{ab} &= 2\left[N\mathring{K}_{ab} + D_{(a}N_{b)} + N_c {L^{(3)c}}_{ab}\right] \\ &= 2\left(N\mathring{K}_{ab} + \mathring{D}_{(a}N_{b)}\right) .\end{aligned}$$

However, taking the teleparallel limit we have

$$\dot{h}_{ab} = N\Sigma_{ab} + 2\mathring{D}_{(a}N_{b)} .$$

STEGR Hamiltonian

Using the teleparallel completeness relation we obtain the $3+1$-decomposed STEGR action

$$S_{STEGR} = \frac{1}{16\pi}\int_{t_1}^{t_2} dt \left\{ \int_{\Sigma_t} N\sqrt{h}\, d^3y \left[-\mathbb{Q}^{(3)} + \frac{1}{4}\Sigma_{ab}\Sigma^{ab} - \frac{1}{4}\Sigma^2 + \left(\tilde{L}^{(3)}{}_a - L^{(3)}{}_a \right)\theta^a \right] \right\} .$$

The momentum reads

$$p^{ab} = \frac{\sqrt{h}}{32\pi}\left(\Sigma^{ab} - h^{ab}\Sigma \right) ,$$

yielding the following expression of the Hamiltonian density

$$\mathcal{H}_{ST} = \frac{16\pi N}{\sqrt{h}}\left(p^{ab}p_{ab} - \frac{1}{d-2}P^2 \right) + 2p^{ab}\mathring{D}_{(a}N_{b)} + \frac{\sqrt{h}N}{16\pi}\left[\mathbb{Q}^{(3)} + \left(L_a - \tilde{L}_a \right)\theta^a \right] .$$

Constraints and d.o.f. count - 1

The variation of the Hamiltonian yields

$$\delta H_P = \int_{\Sigma_t} d^3y \left(\mathcal{P}^{ab}\delta h_{ab} + \mathcal{H}_{ab}\delta p^{ab} + \mathcal{C}\delta N - 2\mathcal{C}_a\delta N^a + \Theta^a\delta\theta_a + \Lambda_c{}^{ab}\delta L^c{}_{ab} \right) .$$

The deviation of the first four tensor densities from the Riemannian result is parametrized by tensors quadratic in $L^c{}_{ab}$ and θ_a

$$\mathcal{C} = \mathring{\mathcal{C}} + \frac{\sqrt{h}}{16\pi} c(L,\theta)\,, \qquad \mathcal{C}_a = \mathring{\mathcal{C}}_a\,,$$
$$\mathcal{P}^{ab} = \mathring{\mathcal{P}}^{ab} + \frac{\sqrt{h}}{16\pi} \Upsilon^{ab}(L,\theta)\,, \qquad \mathcal{H}_{ab} = \mathring{\mathcal{H}}_{ab}\,,$$
$$H_{ST} = \mathring{H} + f(\Theta^a, \Lambda_c{}^{ab}, N, h)\,.$$

Since c, Υ^{ab} and f are of second degree in $\theta_a, L^c{}_{ab}$ any arbitrary function of the canonical variables Ω satisfies

$$\{H,\Omega\} \approx \{\mathring{H},\Omega\}\,, \quad \{\mathcal{P}^{ab},\Omega\} \approx \{\mathring{\mathcal{P}}^{ab},\Omega\}\,, \quad \{\mathcal{C},\Omega\} \approx \{\mathring{\mathcal{C}},\Omega\}\,.$$

Constraints and d.o.f. count - 2

The new primary constraints satisfy

$$\{\pi_\theta{}^a, H\} = -\frac{\partial H}{\partial \theta_a} \approx 0\,,$$
$$\{\pi_{Lc}{}^{ab}, H\} = -\frac{\partial H}{\partial L^c{}_{ab}} \approx 0\,.$$

This means that they are first-class, therefore, they do not give rise to further secondary constraints. Since the number of new canonical variables is twice as that of new first-class constraints and

$$\#(d.o.f.) = \frac{\#(c.v.)\ -\ \#(s.c.c.)}{2} - \#(f.c.c.)\,,$$

the number of propagating degrees of freedom equals the Riemannian one.

Conclusions and outlooks

- This formalism can be applied to any symmetric teleparallel theory, and it allows to carry out Hamiltonian analyses without ever relying on the coincident gauge
- The problematic absence of a boundary term (energy of a Black-Hole at infinity)
- New possible applications: teleparallel equivalent of Starobinsky and Stelle actions

The end

Thank you for your attention!

Questions are welcome