

Vacuum Decay in Geometric Trinity of Gravity

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GeomGravX

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Talk based on:

V. Branchina, S. Capozziello, FC, C. Ferrara, arXiv:2606.28956

E. Bentivegna, V. Branchina, FC, D. Zappalà, JHEP 12 (2017) 100

E. Bentivegna, V. Branchina, FC, D. Zappalà, PRD 99 (2019) 9, 096029

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Metric-affine geometry

In a general metric-affine framework, the fundamental variables are

$$\{g_{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}\},$$

where the metric and the affine connection are, in principle, independent.

From the affine connection we can define three geometrical tensors:

$$R^{\alpha}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} - \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\rho}_{\beta\nu}\Gamma^{\alpha}_{\rho\mu} - \Gamma^{\rho}_{\beta\mu}\Gamma^{\alpha}_{\rho\nu},$$

$$T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\mu\nu}, \quad Q_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu}.$$

- **Curvature**: rotation of vectors under parallel transport.
- **Torsion**: non-closure of infinitesimal parallelograms.
- **Non-metricity**: change of lengths and angles under parallel transport.

Curvature, torsion and non-metricity are different geometrical ways of describing the affine structure of spacetime.

Affine connection and distortions

General Relativity describes gravity through the metric $g_{\mu\nu}$ and the curvature of the Levi-Civita connection:

$$\dot{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta} (\partial_{\mu}g_{\nu\beta} + \partial_{\nu}g_{\mu\beta} - \partial_{\beta}g_{\mu\nu}).$$

In a general metric-affine geometry, the affine connection can be decomposed as

$$\Gamma^{\alpha}_{\mu\nu} = \dot{\Gamma}^{\alpha}_{\mu\nu} + N^{\alpha}_{\mu\nu}, \quad N^{\alpha}_{\mu\nu} = K^{\alpha}_{\mu\nu} + L^{\alpha}_{\mu\nu}.$$

The distortion $N^{\alpha}_{\mu\nu}$ measures the departure from Riemannian geometry.

The contortion tensor is determined by torsion:

$$K^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta} (-T_{\mu\beta\nu} - T_{\nu\beta\mu} + T_{\beta\mu\nu}).$$

The disformation tensor is determined by non-metricity:

$$L^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta} (-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}).$$

GR: curvature

TEGR: torsion

STEGR: non-metricity

Symmetric Teleparallel Equivalent of GR

In STEGR, gravity is described by non-metricity only:

$$R^\alpha{}_{\beta\mu\nu} = 0, \quad T^\alpha{}_{\mu\nu} = 0, \quad Q_{\alpha\mu\nu} \neq 0.$$

The non-metricity scalar is

$$Q = \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} - \frac{1}{4} Q_\alpha Q^\alpha + \frac{1}{2} \tilde{Q}_\alpha Q^\alpha,$$

with

$$Q_\alpha = Q_{\alpha\mu}{}^\mu, \quad \tilde{Q}_\alpha = Q^\mu{}_{\mu\alpha}.$$

The key identity is

$$\mathring{R} = Q - \mathring{\nabla}_\mu (Q^\mu - \tilde{Q}^\mu).$$

$$S_{\text{STEGR}} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} Q + \mathcal{L}_m \right]$$

Teleparallel Equivalent of GR

In TEGR, gravity is described by torsion only:

$$R^\alpha{}_{\beta\mu\nu} = 0, \quad Q_{\alpha\mu\nu} = 0, \quad T^\alpha{}_{\mu\nu} \neq 0.$$

The fundamental variables are the tetrad and the spin connection:

$$\{e^a{}_\mu, \omega^a{}_{b\mu}\}, \quad g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}.$$

The torsion scalar is

$$T = \frac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T_\mu T^\mu,$$

with

$$T_\mu \equiv T^\alpha{}_{\alpha\mu}.$$

The key identity is

$$\dot{R} = T + 2\dot{\nabla}_\mu T^\mu.$$

$$S_{\text{TEGR}} = -\frac{1}{2\kappa} \int d^4x e T + \int d^4x e \mathcal{L}_m$$

Classical equivalence of the Geometric Trinity

Although GR, TEGR and STEGR are based on different geometrical objects, their field equations are dynamically equivalent.

$$\text{TEGR} \implies \mathring{G}_{\mu\nu} = \kappa T_{\mu\nu} \longleftarrow \text{STEGR}$$

Moreover, the connection field equations carry no additional dynamics:

connection equations \implies identically satisfied by Bianchi identities.

GR, TEGR and STEGR are dynamically equivalent at the classical level.

They describe the same gravitational degrees of freedom through curvature, torsion and non-metricity, respectively.

Why this matters for vacuum decay

Gravity has an impact on the lifetime of a metastable state in QFT!

Classically, the three formulations give the same field equations:

$$\text{GR} \equiv \text{TEGR} \equiv \text{STEGR}.$$

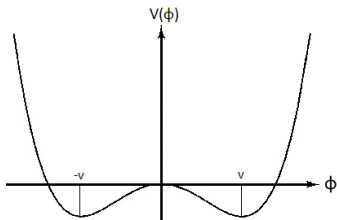
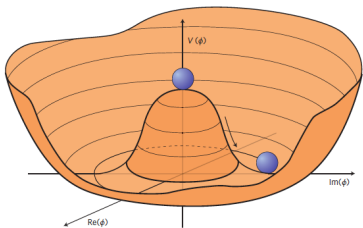
However, the decay rate of a metastable vacuum is not obtained only from the equations of motion, but from the Euclidean action evaluated on a non-trivial configuration, the **bounce solution**:

$$\Gamma \sim e^{-S_E[\text{bounce}]}$$

Does the classical equivalence of the Geometric Trinity survive at the level of the tunneling of a metastable vacuum?

Classical Higgs Potential

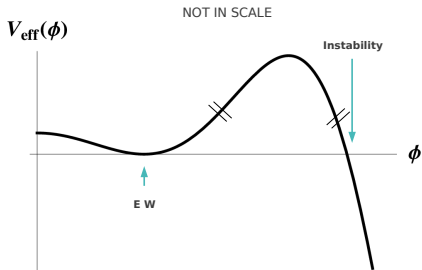
$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \quad \Rightarrow \quad v = \sqrt{\frac{-m^2}{2\lambda}}$$



- **Electroweak (EW) Scale** $v = \langle 0|\phi|0\rangle \sim 246 \text{ GeV}$.
- **Spontaneous Symmetry Breaking** \Rightarrow All particles acquire mass.

Effective Higgs Potential - Radiative corrections

$$\begin{aligned}
 V_{1\text{loop}}(\phi) = & -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{64\pi^2} \left[\left(m^2 + \frac{\lambda}{2}\phi^2\right)^2 \left(\ln\left(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2}\right) - \frac{3}{2}\right) + \right. \\
 & + 3 \left(m^2 + \frac{\lambda}{6}\phi^2\right)^2 \left(\ln\left(\frac{m^2 + \frac{\lambda}{6}\phi^2}{\mu^2}\right) - \frac{3}{2}\right) + \frac{6g^4}{16}\phi^4 \left(\ln\left(\frac{\frac{g^2}{4}\phi^2}{\mu^2}\right) - \frac{5}{6}\right) + \\
 & \left. + 3\frac{(g^2 + g'^2)^2}{16}\phi^4 \left(\ln\left(\frac{\frac{1}{4}(g^2 + g'^2)\phi^2}{\mu^2}\right) - \frac{5}{6}\right) - 12g_t^4\phi^4 \left(\ln\frac{g_t^2\phi^2}{\mu^2} - \frac{3}{2}\right) \right]
 \end{aligned}$$

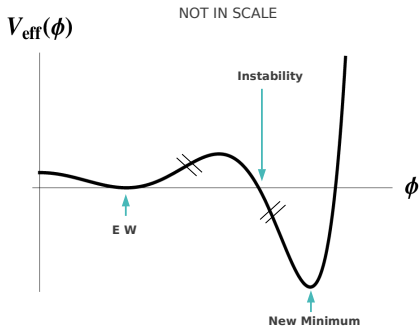


RG Improved Effective Potential

Solving RG equations for the SM couplings, we obtain the **RG improved effective potential**:

$$V_{\text{eff}}(\phi) \sim \frac{1}{4} \lambda_{\text{eff}}(\phi) \phi^4$$

where $\lambda_{\text{eff}}(\phi)$ is the running coupling $\lambda(\mu)$ with $\mu = \phi$.



Bounce solution without gravity

S. Coleman, PRD 15 (1977) 2929

C. Callan, S. Coleman, PRD 16 (1977) 1762

Euclidean action for a single component real scalar field ϕ :

$$S[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right]$$

$V(\phi)$ potential with a **false vacuum** $\phi = \phi_{fv}$ and a **true vacuum** $\phi = \phi_{tv}$.

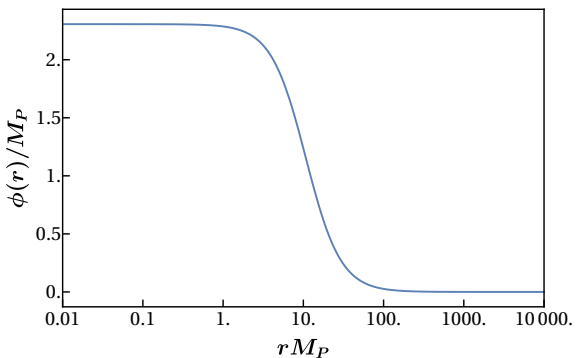
Bounce solution $\phi_b(r)$ is a particular solution to the Euclidean Euler-Lagrange equation with $O(4)$ symmetry, obtained within the appropriate boundary conditions

$$\begin{aligned} \ddot{\phi}(r) + \frac{3}{r} \dot{\phi}(r) &= \frac{dV}{d\phi} \\ \phi(\infty) &= \phi_{fv} \quad \dot{\phi}(0) = 0 \end{aligned}$$

where r is the radial coordinate.

Bounce solution in SM - flat spacetime

Numerical solution through shooting methods



Decay rate of the false vacuum

Decay time of the false vacuum:

$$\Gamma = \frac{1}{\tau} = T_U^3 \frac{B^2}{4\pi^2} \left| \frac{\det' [-\partial^2 + V''(\phi_b)]}{\det [-\partial^2 + V''(v)]} \right|^{-\frac{1}{2}} e^{-B} \equiv D e^{-B}$$

$B = S[\phi_b] - S[\phi_{fv}]$ **Tunneling Exponent**; e^{-B} gives the **tree-level contribution** to the decay rate; D is the **quantum fluctuation determinant**.

Good approximation to the prefactor $D \Rightarrow$ The EW vacuum tunneling time $\tau = \Gamma^{-1}$ is calculated as

$$\tau \simeq \left(\frac{\mathcal{R}^4}{T_U^3} \right) e^B \quad \Rightarrow \quad \tau_{\text{flat}} \sim 10^{639} T_U \gg \gg T_U$$

\mathcal{R} size of the bounce, defined as the value of r such that $\phi_b(\mathcal{R}) = \frac{1}{2}\phi_b(0)$.

Bounce solution in curved spacetime

S. Coleman, F. de Luccia, PRD 21 (1980) 3305

Euclidean action for a single component real scalar field ϕ :

$$S[\phi, g_{\mu\nu}] = \int d^4x \sqrt{g} \left[-\frac{R}{16\pi G} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right]$$

R is the Ricci scalar and G is the Newton constant.

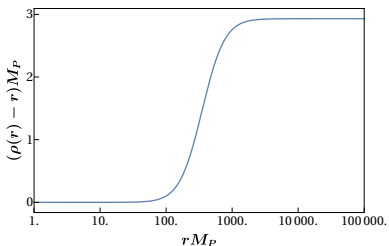
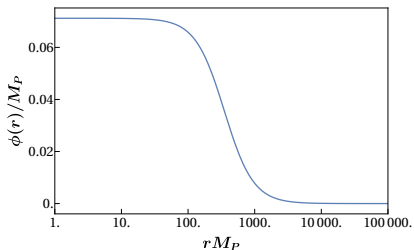
Requiring $O(4)$ symmetry, the (Euclidean) metric takes the form:

$$ds^2 = dr^2 + \rho^2(r) d\Omega_3^2$$

The **bounce solution** is now given by $\phi_b(r)$ and $\rho_b(r)$, solutions of the coupled equations:

$$\begin{aligned} \ddot{\phi} + 3 \frac{\dot{\rho}}{\rho} \dot{\phi} &= \frac{dV}{d\phi} & \dot{\rho}^2 &= 1 + \frac{\kappa}{3} \rho^2 \left(\frac{1}{2} \dot{\phi}^2 - V(\phi) \right) \\ \phi(\infty) &= \phi_{fv} & \dot{\phi}(0) &= 0 & \rho(0) &= 0 \end{aligned}$$

Bounce solution in SM - curved spacetime



$$\tau_{\text{flat}} \sim 10^{639} T_U$$

 \Rightarrow

$$\tau_{\text{grav}} \sim 10^{661} T_U$$

Gravity tends to **stabilize the EW vacuum** with respect to the flat spacetime background!

Asymptotic behavior of the bounce solution

Starting from the CDL equations

$$\ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} = \frac{dV}{d\phi}, \quad \dot{\rho}^2 = 1 + \frac{\kappa}{3}\rho^2 \left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right),$$

the boundary conditions are

$$\phi(\infty) = \phi_{\text{fv}} = 0, \quad \dot{\phi}(0) = 0, \quad \rho(0) = 0.$$

- **False-vacuum region, $r \rightarrow \infty$:**

$$\rho(r) = r + k, \quad \phi(r) = \frac{c_1}{r^2} + O(r^{-3}) \Rightarrow r^2\phi(r) \sim \text{const.}$$

- **Center of the bounce, $r \rightarrow 0$:**

$$\phi(r) = A_0 + \frac{1}{8}V'(A_0)r^2 + \dots, \quad \rho(r) = r - \frac{\kappa}{18}V(A_0)r^3 + \dots$$

These asymptotic behaviors will be crucial to evaluate the boundary contributions to the tunneling exponent in STEGR and TEGR.

The main question

Vacuum decay rate is controlled by the Euclidean action evaluated on a non-trivial configuration:

$$\Gamma \sim e^{-B}, \quad B = S_E[\text{bounce}] - S_E[\text{false vacuum}].$$

Since TEGR and STEGR differ from GR by boundary terms, the non-trivial question is:

Do these boundary terms contribute to B ?

Vacuum decay is a testing ground to see whether the Geometric Trinity equivalence survives also considering a quantum phenomenon.

Vacuum decay in STEGR

In STEGR, gravity is described by the non-metricity scalar Q . The Euclidean action is

$$S_E[\phi, g, \Gamma] = \int d^4x \sqrt{g} \left[-\frac{Q}{2\kappa} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right].$$

The STEGR connection is constrained to be

$$R^\alpha{}_{\beta\mu\nu} = 0, \quad T^\alpha{}_{\mu\nu} = 0, \quad Q_{\alpha\mu\nu} \neq 0.$$

Key identity

$$\mathring{R} = Q - \mathring{\nabla}_\mu (Q^\mu - \tilde{Q}^\mu).$$

STEGR action differs from the Einstein-Hilbert one only by a boundary term.

The bounce equations are still the CDL equations of GR.

$O(4)$ -invariant connection in STEGR

For the $O(4)$ -symmetric bounce geometry

$$ds^2 = dr^2 + \rho^2(r)d\Omega_3^2, \quad x^\mu = (r, \chi, \theta, \varphi),$$

we need an affine connection compatible with the same symmetry.

The connection is required to be invariant under the $O(4)$ Killing vectors ξ :

$$\mathcal{L}_\xi \Gamma^\alpha_{\mu\nu} = 0.$$

Explicitly, for an affine connection,

$$\mathcal{L}_\xi \Gamma^\alpha_{\mu\nu} = \xi^\beta \partial_\beta \Gamma^\alpha_{\mu\nu} - \Gamma^\beta_{\mu\nu} \partial_\beta \xi^\alpha + \Gamma^\alpha_{\beta\nu} \partial_\mu \xi^\beta + \Gamma^\alpha_{\mu\beta} \partial_\nu \xi^\beta + \partial_\mu \partial_\nu \xi^\alpha.$$

In STEGR we also impose the geometrical constraints

$$R^\alpha_{\beta\mu\nu} = 0, \quad T^\alpha_{\mu\nu} = 0.$$

These conditions fix the connection up to a single arbitrary function $c(r)$.

STEGR connection for the CDL geometry

The $O(4)$ -invariant, flat and torsionless connection is:

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} -\frac{1 + \dot{c}(r)}{c(r)} & 0 & 0 & 0 \\ 0 & c(r) & 0 & 0 \\ 0 & 0 & c(r) \sin^2 \chi & 0 \\ 0 & 0 & 0 & c(r) \sin^2 \chi \sin^2 \theta \end{pmatrix},$$

$$\Gamma^x_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c(r)} & 0 & 0 \\ -\frac{1}{c(r)} & 0 & 0 & 0 \\ 0 & 0 & -\sin \chi \cos \chi & 0 \\ 0 & 0 & 0 & -\sin \chi \cos \chi \sin^2 \theta \end{pmatrix},$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{1}{c(r)} & 0 \\ 0 & 0 & \cot \chi & 0 \\ -\frac{1}{c(r)} & \cot \chi & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{pmatrix}, \quad \Gamma^\varphi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{c(r)} \\ 0 & 0 & 0 & \cot \chi \\ 0 & 0 & 0 & \cot \theta \\ -\frac{1}{c(r)} & \cot \chi & \cot \theta & 0 \end{pmatrix}.$$

This connection allows us to compute explicitly the STEGR boundary term on the bounce and false-vacuum configurations.

Non-metricity scalar and traces

Using the $O(4)$ -invariant STEGR connection, the non-metricity scalar is

$$Q = \frac{3(2\dot{\rho}(r)^2 + \dot{c}(r) + 2)}{\rho(r)^2} + \frac{3c(r)\dot{\rho}(r)}{\rho(r)^3} + \frac{9\dot{\rho}(r)}{\rho(r)c(r)} - \frac{3\dot{c}(r)}{c(r)^2}.$$

The two independent non-metricity traces have only one non-vanishing component:

$$Q_r = \frac{6\dot{\rho}(r)}{\rho(r)} + \frac{2(\dot{c}(r) + 4)}{c(r)}, \quad \tilde{Q}_r = \frac{5\rho(r)^2 + 2\rho(r)^2\dot{c}(r) - 3c(r)^2}{c(r)\rho(r)^2}.$$

The function $c(r)$ remains arbitrary in general: the connection equation is identically satisfied.

In the Euclidean false-vacuum background,

$$\rho_{\text{fv}}(r) = r + k,$$

we fix $c(r)$ by requiring trivial non-metricity,

$$Q = 0 \quad \implies \quad c(r) = -r - k.$$

This choice makes the STEGR connection reduce to the Levi-Civita connection of Euclidean Minkowski space in spherical coordinates.

STEGR tunneling exponent

$$S^{\text{STEGR}}[\phi, g, \Gamma] = S^{\text{GR}}[\phi, g] + \frac{1}{2\kappa} \int d^4x \sqrt{g} \dot{\nabla}_\mu (Q^\mu - \tilde{Q}^\mu).$$

For the $O(4)$ -symmetric connection, the boundary contribution reduces to

$$\Delta S_Q[\rho, c] = \frac{1}{2\kappa} \int d^4x \sqrt{g} \dot{\nabla}_\mu (Q^\mu - \tilde{Q}^\mu) = \frac{3\pi^2}{\kappa} \left[\rho \left(2\rho\dot{\rho} + \frac{\rho^2}{c} + c \right) \right]_0^\infty.$$

Therefore, being $S_E^{\text{STEGR}}[\phi_{\text{fv}}, \rho_{\text{fv}}] = 0$

$$B_{\text{STEGR}} = S_E^{\text{STEGR}}[\phi_b, \rho_b] = B_{\text{GR}} + \Delta S_Q[\rho_b, c_b].$$

From the asymptotic behavior

$$\rho_b(r) \sim r + k, \quad c_b(r) \sim -r - k,$$

with k finite for $r \rightarrow \infty$ and $k = 0$ for $r \rightarrow 0$, we get

$$\Delta S_Q[\rho_b, c_b] = 0 \quad \Rightarrow \quad B_{\text{STEGR}} = B_{\text{GR}}$$

Vacuum decay in TEGR

In TEGR, gravity is described by the torsion scalar T . The Euclidean action is

$$S_E[\phi, e, \omega] = \int d^4x e \left[\frac{T}{2\kappa} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right].$$

The TEGR geometry is constrained to be

$$R^\alpha{}_{\beta\mu\nu} = 0, \quad Q_{\alpha\mu\nu} = 0, \quad T^\alpha{}_{\mu\nu} \neq 0.$$

Key identity

$$\dot{R} = T + 2\dot{\nabla}_\mu T^\mu.$$

TEGR action differs from the Einstein-Hilbert one only by a boundary term.

The bounce equations are still the CDL equations of GR.

$O(4)$ -invariant tetrads in TEGR

For the $O(4)$ -symmetric bounce geometry

$$ds^2 = dr^2 + \rho^2(r)d\Omega_3^2, \quad x^\mu = (r, \chi, \theta, \varphi),$$

we need tetrads reproducing the CDL metric:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}.$$

A convenient diagonal tetrad is

$$e^a{}_\mu = \text{diag}(1, \rho(r), \rho(r) \sin \chi, \rho(r) \sin \chi \sin \theta).$$

Equivalently, one can use a non-diagonal tetrad $\tilde{e}^a{}_\mu$ related to $e^a{}_\mu$ by a local $SO(4)$ transformation:

$$\tilde{e}^a{}_\mu = \Lambda^a{}_b(\chi, \theta, \varphi) e^b{}_\mu.$$

In the Weitzenböck gauge,

$$\{\tilde{e}^a{}_\mu, 0\},$$

while the equivalent diagonal description is

$$\{e^a{}_\mu, \omega^a{}_{b\mu}\}.$$

TEGR spin connection

The inertial spin connection associated with the diagonal tetrad is obtained from the $SO(4)$ rotation matrix $\Lambda^a_b(\chi, \theta, \varphi)$:

$$\omega^a_{b\mu} = \Lambda^a_c \partial_\mu \Lambda^c_b.$$

The non-vanishing components are:

$$\omega^a_{b\chi} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega^a_{b\theta} = \begin{pmatrix} 0 & 0 & -\sin \chi & 0 \\ 0 & 0 & -\cos \chi & 0 \\ \sin \chi & \cos \chi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega^a_{b\varphi} = \begin{pmatrix} 0 & 0 & 0 & -\sin \chi \sin \theta \\ 0 & 0 & 0 & -\cos \chi \sin \theta \\ 0 & 0 & 0 & -\cos \theta \\ \sin \chi \sin \theta & \cos \chi \sin \theta & \cos \theta & 0 \end{pmatrix}, \quad \omega^a_{br} = 0.$$

The affine connection is obtained from the tetrad postulate:

$$\Gamma^\lambda_{\nu\mu} = e^\lambda_a \left(\partial_\mu e^a_\nu + \omega^a_{b\mu} e^b_\nu \right).$$

The couples $\{\tilde{e}^a_\mu, 0\}$ and $\{e^a_\mu, \omega^a_{b\mu}\}$ are related by a local Lorentz transformation and define the same affine connection and torsion tensor.

TEGR connection for the CDL geometry

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\rho(r) & 0 & 0 \\ 0 & 0 & -\rho(r) \sin^2 \chi & 0 \\ 0 & 0 & 0 & -\rho(r) \sin^2 \chi \sin^2 \theta \end{pmatrix},$$

$$\Gamma^x_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{\rho(r)} & 0 & 0 \\ \frac{\dot{\rho}(r)}{\rho(r)} & 0 & 0 & 0 \\ 0 & 0 & -\sin \chi \cos \chi & 0 \\ 0 & 0 & 0 & -\sin \chi \cos \chi \sin^2 \theta \end{pmatrix},$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{1}{\rho(r)} & 0 \\ 0 & 0 & \cot \chi & 0 \\ \frac{\dot{\rho}(r)}{\rho(r)} & \cot \chi & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{pmatrix}, \quad \Gamma^\varphi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\rho(r)} \\ 0 & 0 & 0 & \cot \chi \\ 0 & 0 & 0 & \cot \theta \\ \frac{\dot{\rho}(r)}{\rho(r)} & \cot \chi & \cot \theta & 0 \end{pmatrix}.$$

This is the $O(4)$ -invariant, flat and metric-compatible connection used to compute the TEGR boundary term.

Torsion scalar and torsion vector

Using the $O(4)$ -invariant TEGR connection, the torsion scalar is

$$T = -\frac{6}{\rho(r)^2} (\dot{\rho}(r) - 1)^2.$$

The torsion vector has only one non-vanishing component:

$$T^r = -\frac{3}{\rho(r)} (\dot{\rho}(r) - 1).$$

Therefore, the TEGR boundary term is entirely controlled by the radial behavior of the CDL metric function $\rho(r)$:

$$2\overset{\circ}{\nabla}_{\mu} T^{\mu} = \frac{2}{e} \partial_{\mu} (e T^{\mu}) = \frac{2}{e} \partial_r (e T^r).$$

In contrast with STEGR, there is no arbitrary function: the relevant quantity is fixed directly by $\rho(r)$.

TEGR tunneling exponent

$$S^{\text{TEGR}}[\phi, e, \omega] = S^{\text{GR}}[\phi, g] - \frac{1}{2\kappa} \int d^4x e 2\dot{\nabla}_\mu T^\mu.$$

For the $O(4)$ -symmetric connection, the boundary contribution reduces to

$$\Delta S_T[\rho_b] = -\frac{1}{2\kappa} \int d^4x e \dot{\nabla}_\mu T^\mu = \frac{3\pi^2}{\kappa} [\rho^2 (\dot{\rho} - 1)]_0^\infty.$$

Therefore, being $S_E^{\text{TEGR}}[\phi_{\text{fv}}, \rho_{\text{fv}}] = 0$

$$B_{\text{TEGR}} = S_E^{\text{TEGR}}[\phi_b, \rho_b] = B_{\text{GR}} + \Delta S_T[\rho_b].$$

From the asymptotic behavior

$$\rho_b(r) \sim r + k,$$

with k finite for $r \rightarrow \infty$ and $k = 0$ for $r \rightarrow 0$, we get

$$\Delta S_T[\rho_b] = 0 \quad \Rightarrow \quad B_{\text{TEGR}} = B_{\text{GR}}$$

Main result

The bounce solution is the same in the three equivalent formulations:

$$(\phi_b, \rho_b)_{\text{GR}} = (\phi_b, \rho_b)_{\text{STEGR}} = (\phi_b, \rho_b)_{\text{TEGR}}.$$

The boundary terms relating the three Euclidean actions do not contribute to the difference between bounce and false-vacuum actions.

$$B_{\text{GR}} = B_{\text{STEGR}} = B_{\text{TEGR}}$$

Therefore,

$$\Gamma_{\text{GR}} = \Gamma_{\text{STEGR}} = \Gamma_{\text{TEGR}}$$

at the semiclassical level, for the $O(4)$ -symmetric Minkowski-to-AdS vacuum decay.

The classical equivalence of the Geometric Trinity extends to the tunneling exponent.

Thank you for the attention!