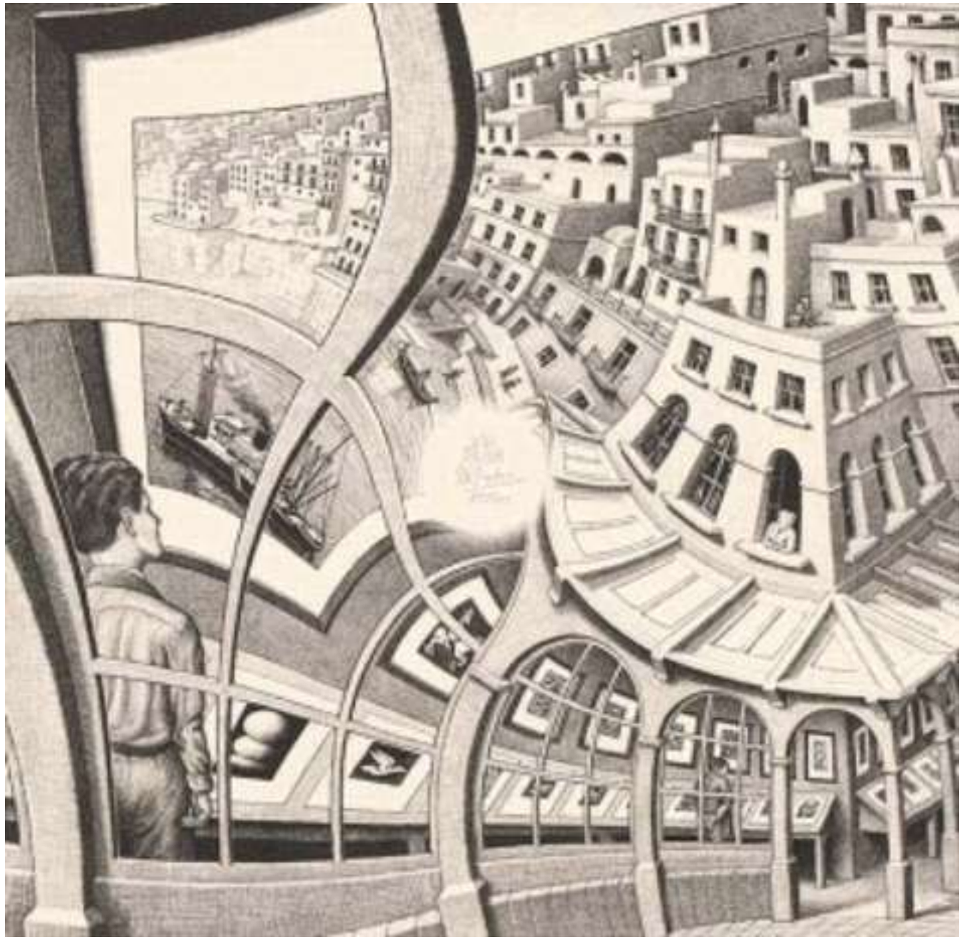
Gravitational Instantons and Chiral Reduction in Spin(4) Gauge theory of space-time with a Cartan-Khronon field.



UNIVERSITY OF TARTU

Papadolpoulos Stylianos (Stelios)

Overview

1. The shape of spacetime and gravity

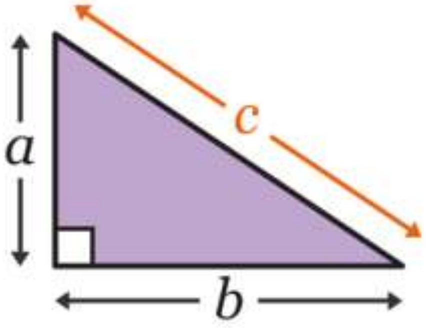
2. Gravitational instantons
   - What are they and where do they come from
   - Where they are being used
   - Topological Invariants
   - Why study them in a theory

3. Spin(4)-Cartan-Khronon theory of spacetime

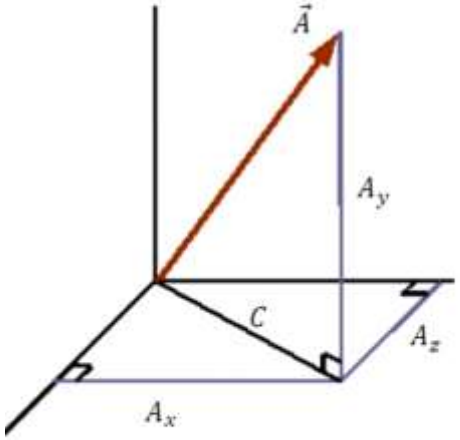
   Tomi Koivisto, Lucy Zheng, and Tom Zlosnik. A *spin*(4) gauge theory of space, time, gravitation, matter and dark matter. *arXiv preprint arXiv:2507.00968v2*, 2025. Submitted on 1 Jul 2025, last revised 30 Mar 2026.

4. Gravitational instanton solutions in the theory

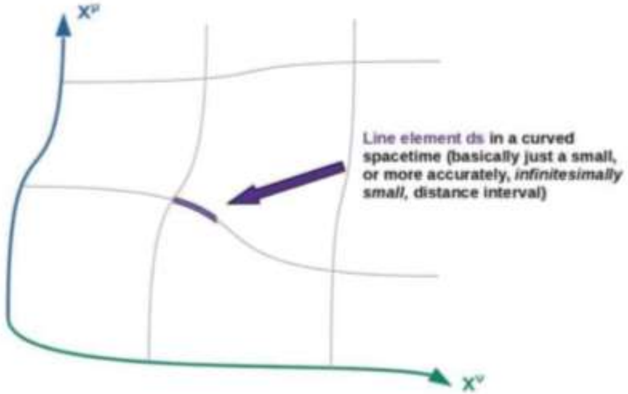
- The shape of spacetime and gravity
  - The Pythagorean Theorem



$$c^2 = a^2 + b^2$$

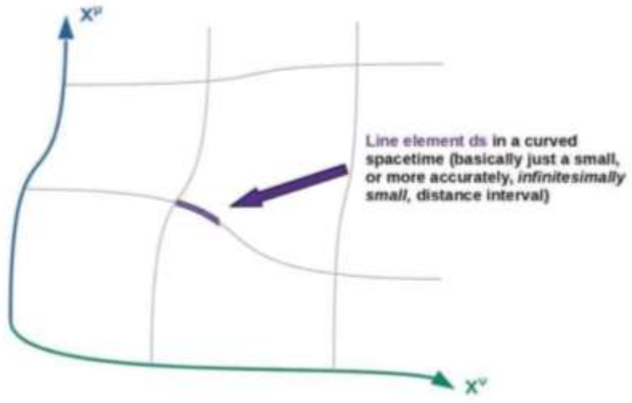


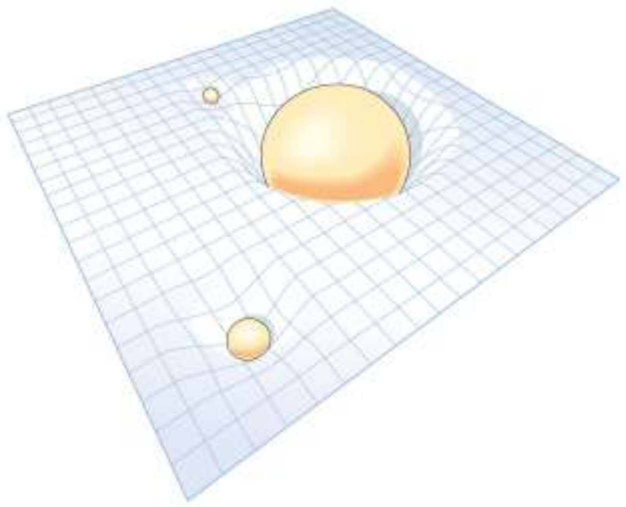
$$|\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2$$



$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

- The shape of spacetime and gravity





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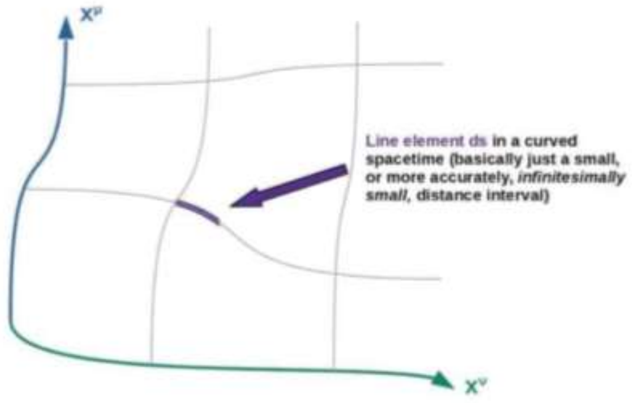
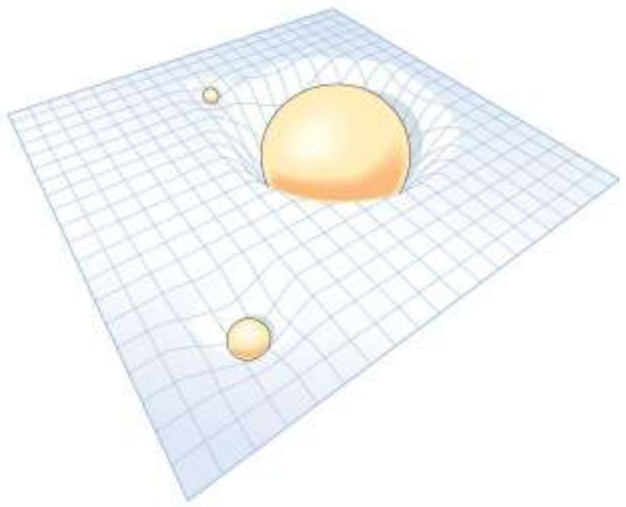
- The shape of spacetime and gravity





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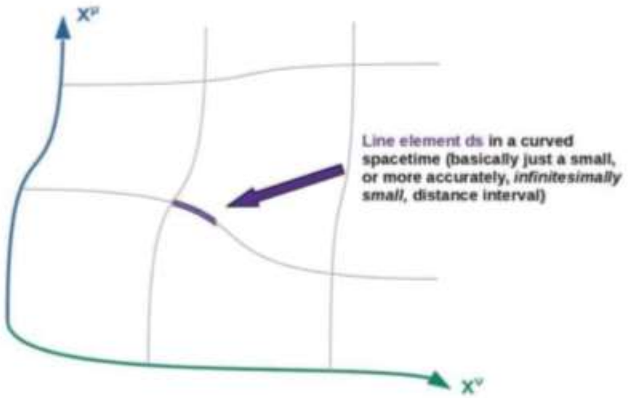

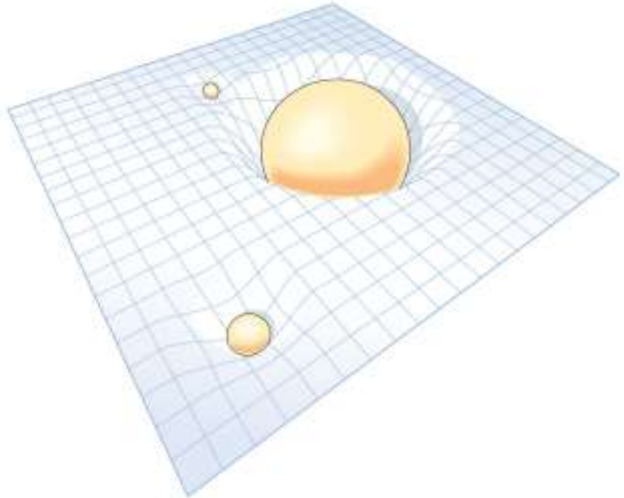

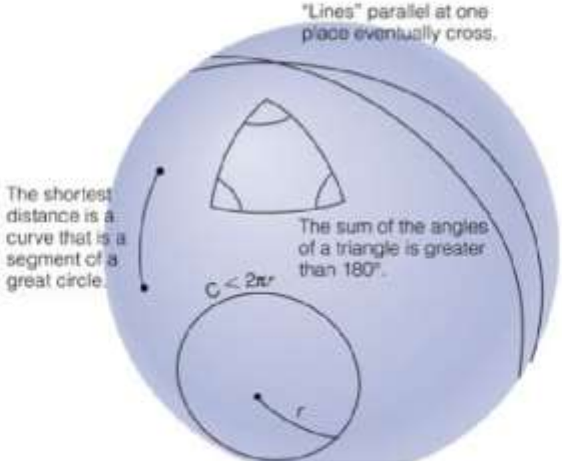

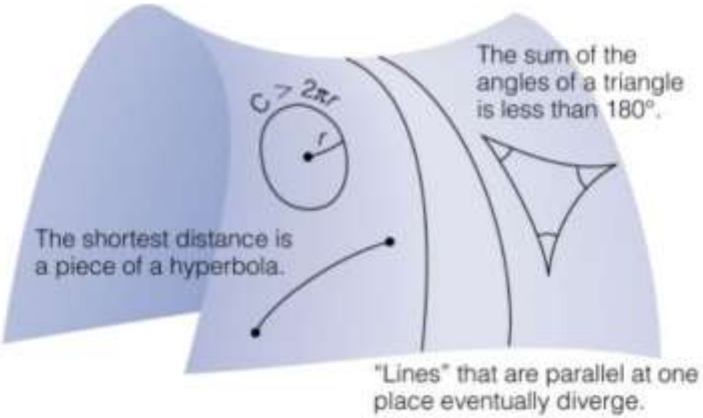

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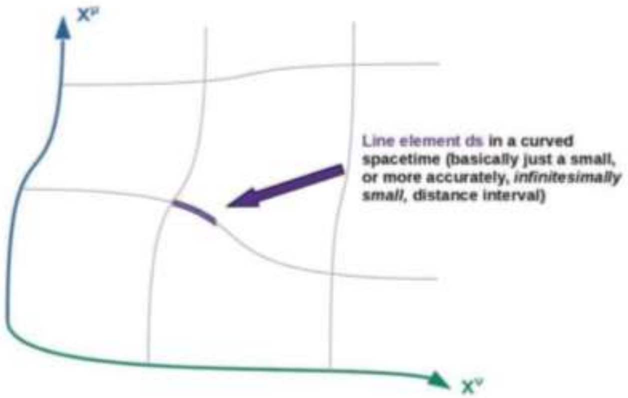

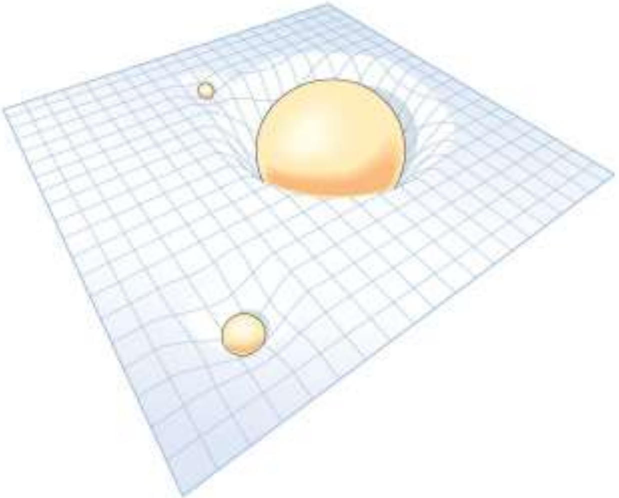

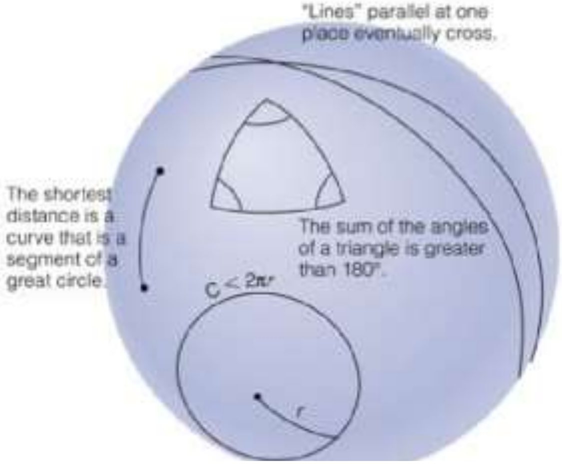

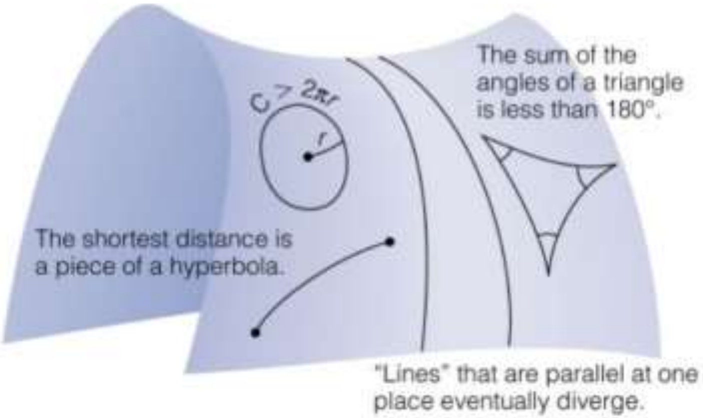

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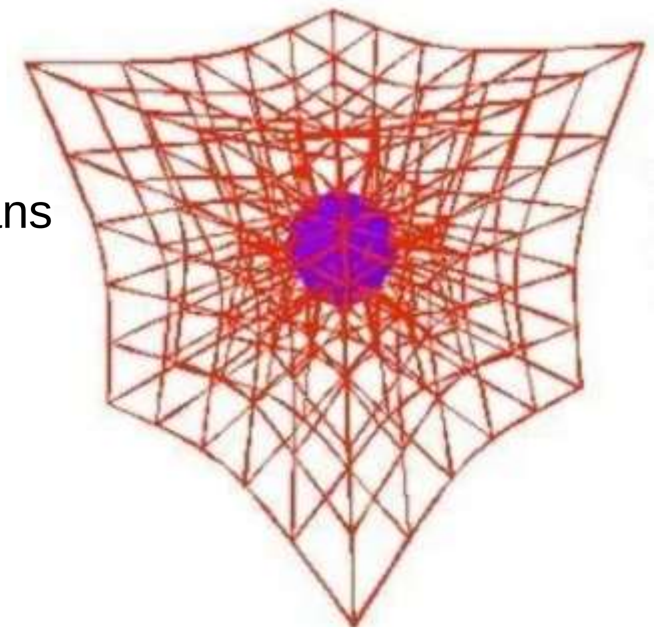


The need of source: Mass and Energy

Shape without source?

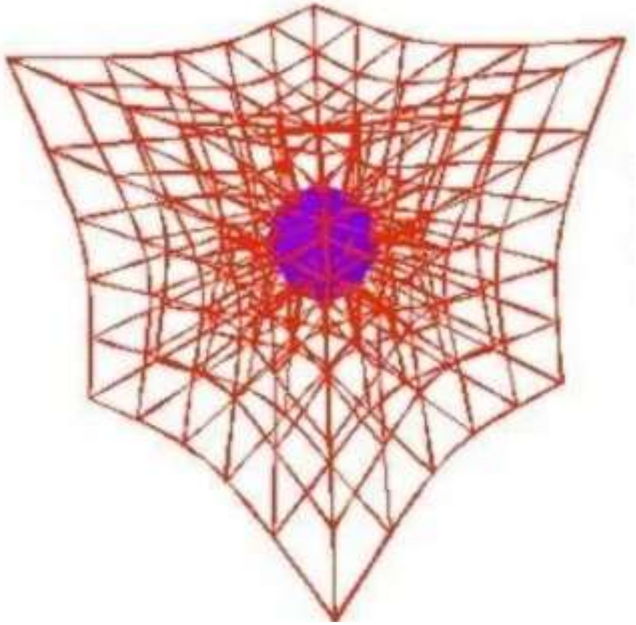
Gravity as a purely geometric phenomenon means curvature is just “shape”.

Must distinguish between contracting curvature (Ricci, tied to matter) and tidal curvature (Weyl, tied to shape).

$$[\nabla_\mu, \nabla_\nu] \neq 0$$

(Extension to Torsion and Non-Metricity)

- The shape of spacetime and gravity



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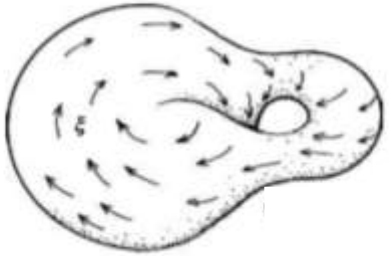
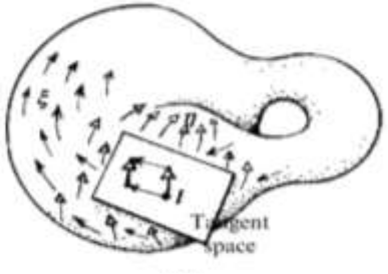
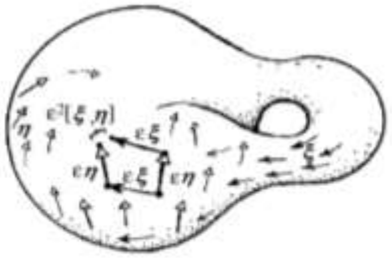




Parallel Transport

Gravity is the nontrivial geometry of parallel transport:
Curvature, Torsion and Non-metricity.

Shape and Bundles



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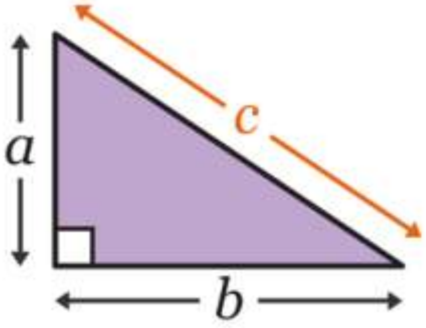
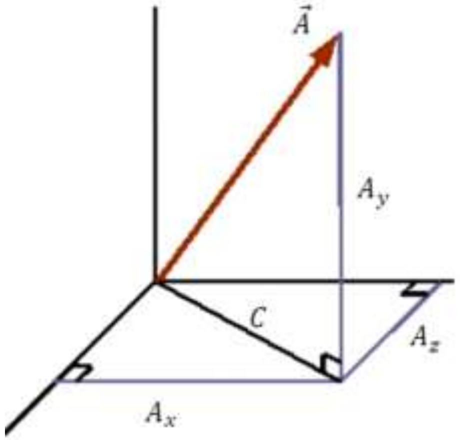
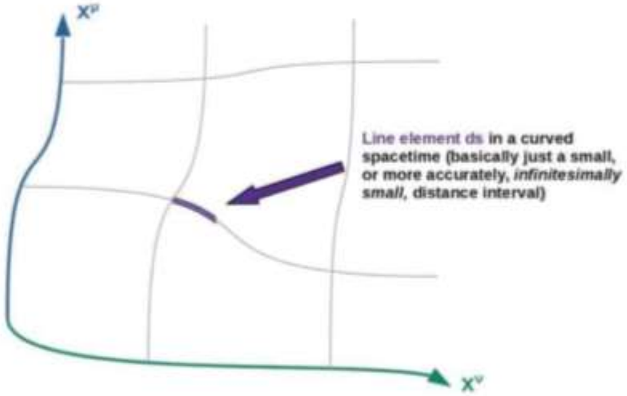
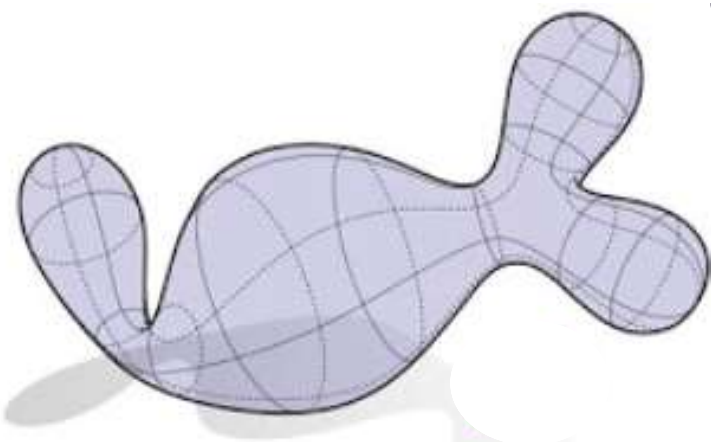
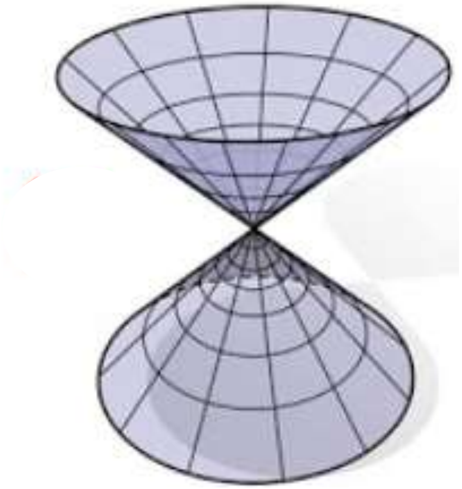










Smooth shapes with gravity and no source – Gravitational Instantons

- The shape of spacetime and gravity
  - The comparison

The "shape" of spacetime and gravity is fundamentally signature-dependent.

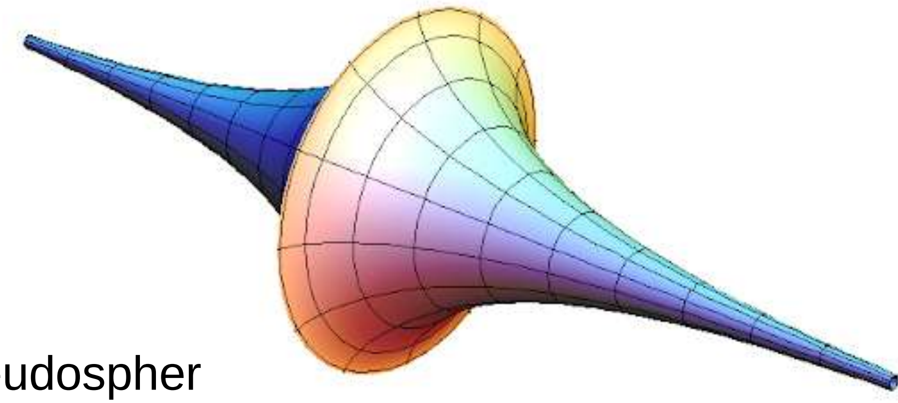
Euclidean signature allows source-free curvature as a topological phenomenon.

Lorentzian signature forbids it because it would violate causality or lead to singularities.

Gravitational instantons are precisely the Euclidean manifolds that exploit this allowed freedom, and they have no direct Lorentzian counterparts.

- The shape of spacetime and gravity
  - Pseudo-Riemannian Geometry

In Lorentzian signature:
the Einstein equations Rμv=0 admit non-flat solutions (gravitational waves, Schwarzschild, Kerr),



Pseudosphere

but these all appear with some property:

- singularities
- non-compact
- asymptotically flat.

In Lorentzian signature, topology cannot support cu
without introducing singularities or non-compactnes

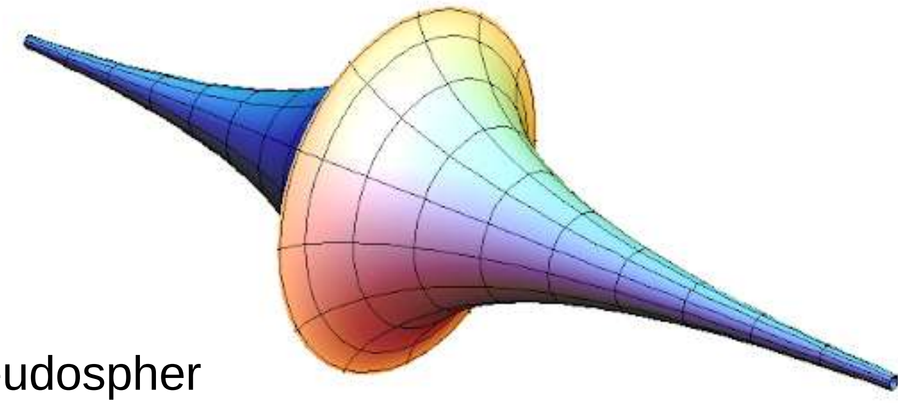
The same equations force flatness if compact, or singularities if non-compact. The reason is causal structure + energy conditions.

- The shape of spacetime and gravity
  - Pseudo-Riemannian Geometry

Singularity Theorems (Hawking-Penrose):

Even if you allow non-compact Lorentzian Ricci-flat manifolds, they generically develop singularities unless they are exactly flat or have special symmetries.

This is fundamentally tied to the convergence condition and the existence of trapping surfaces.



Pseudospher
e

"The classical singularity theorems of R. Penrose and S. Hawking from the 1960s show that, given a pointwise energy condition (and some causality as well as initial assumptions), spacetimes cannot be geodesically complete“

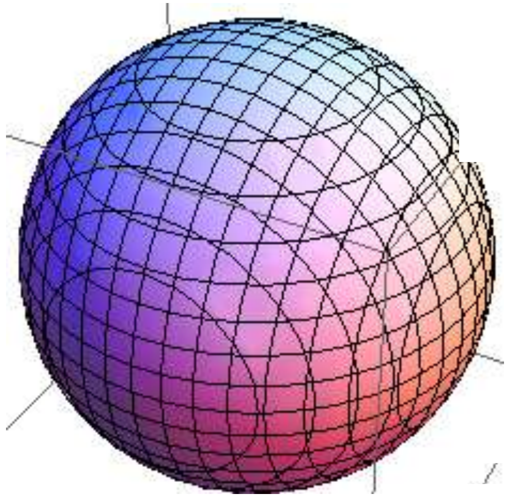
Graf, M., Kontou, EA., Ohanyan, A. et al. Hawking-Type Singularity Theorems for Worldvolume Energy Inequalities. Ann. Henri Poincaré 26, 3871–3906 (2025).doi.org/10.1007/s00023-024-01502-6

- The shape of spacetime and gravity
  - Riemannian Geometry

In Euclidean signature, the Einstein equations $R_{\mu\nu}=0$ (vacuum) permit compact, non-singular, Ricci-flat manifolds with non-zero Riemann curvature.

The positive-definite metric allows for closed geodesics

Sphere



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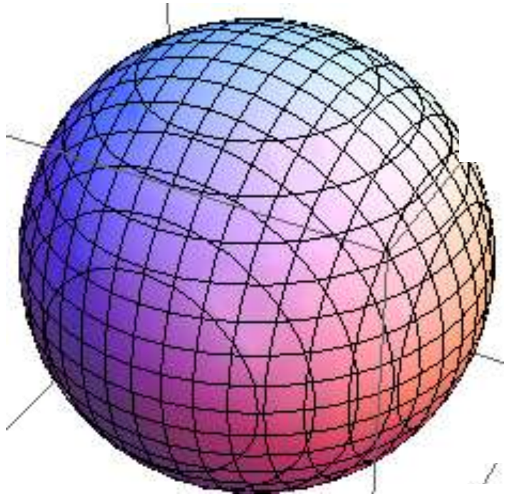


Sphere

The positive-definite metric allows for closed geodesics

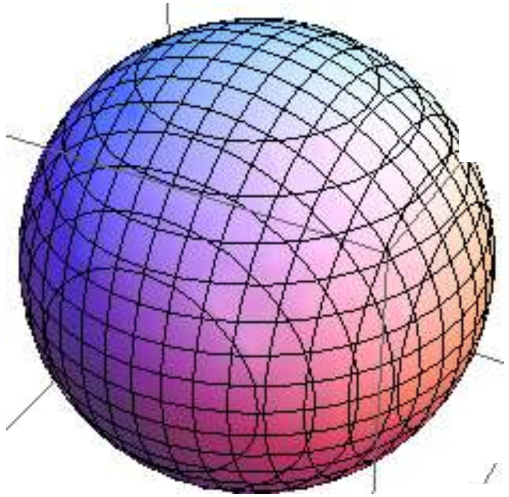
No distinction between timelike/spacelike directions. (compact Cauchy surfaces)

Curvature can be "trapped" in compact manifolds (or sub-) without violating any energy condition because there is no causal structure to break.

Existence Theorems: Yau's proof of the Calabi conjecture guarantees the existence of Ricci-flat Kähler metrics on compact complex manifolds with vanishing first Chern class.

- The shape of spacetime and gravity
  - Riemannian Geometry

Sphere

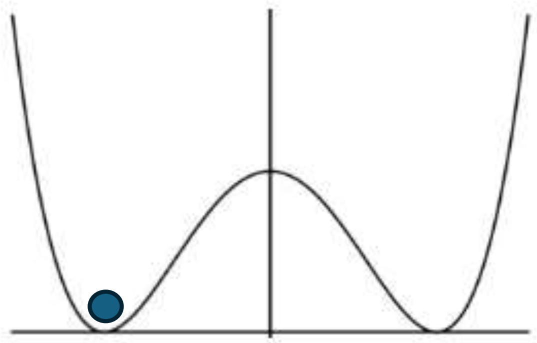


Ricci-flat, compact, non-flat manifolds exist (Calabi-Yau, hyper-Kähler).

Instantons offer an opportunity to study the mechanism of gravity based on the shape of the manifold, a gauge field theory mechanism.

- Gravitational instantons
  - Why the name “instantons” and where do they come from

- Gravitational instantons
  - What are they and where do they come from



In quantum field theory, **an instanton** is a solution localized in **Euclidean time**.

Finite-action solutions to the Euclidean field equations that mediate tunnelling between **topologically distinct vacuum states.**

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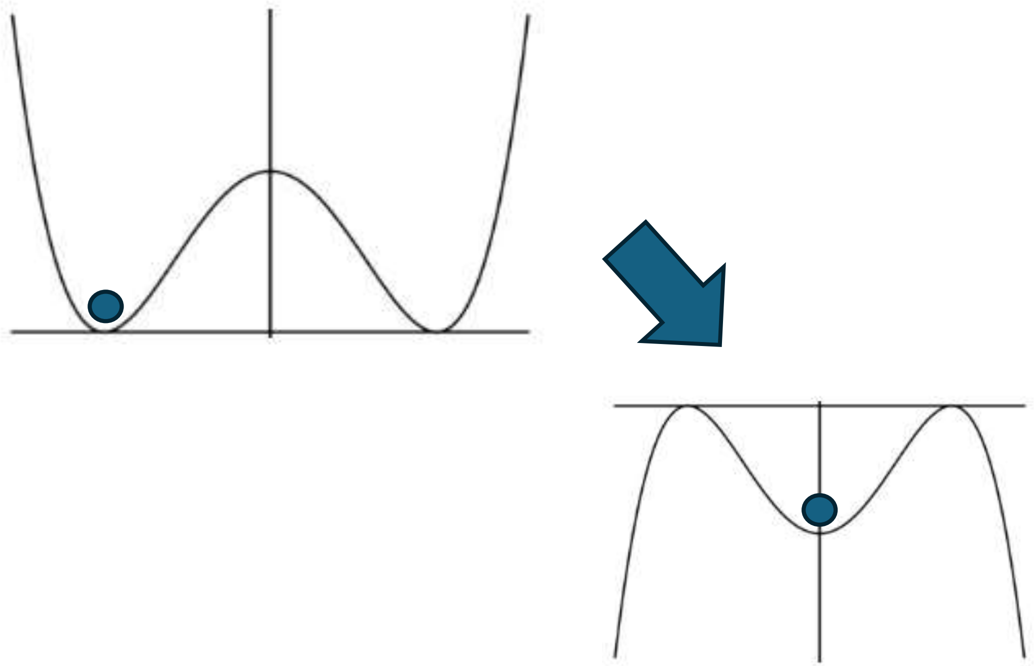

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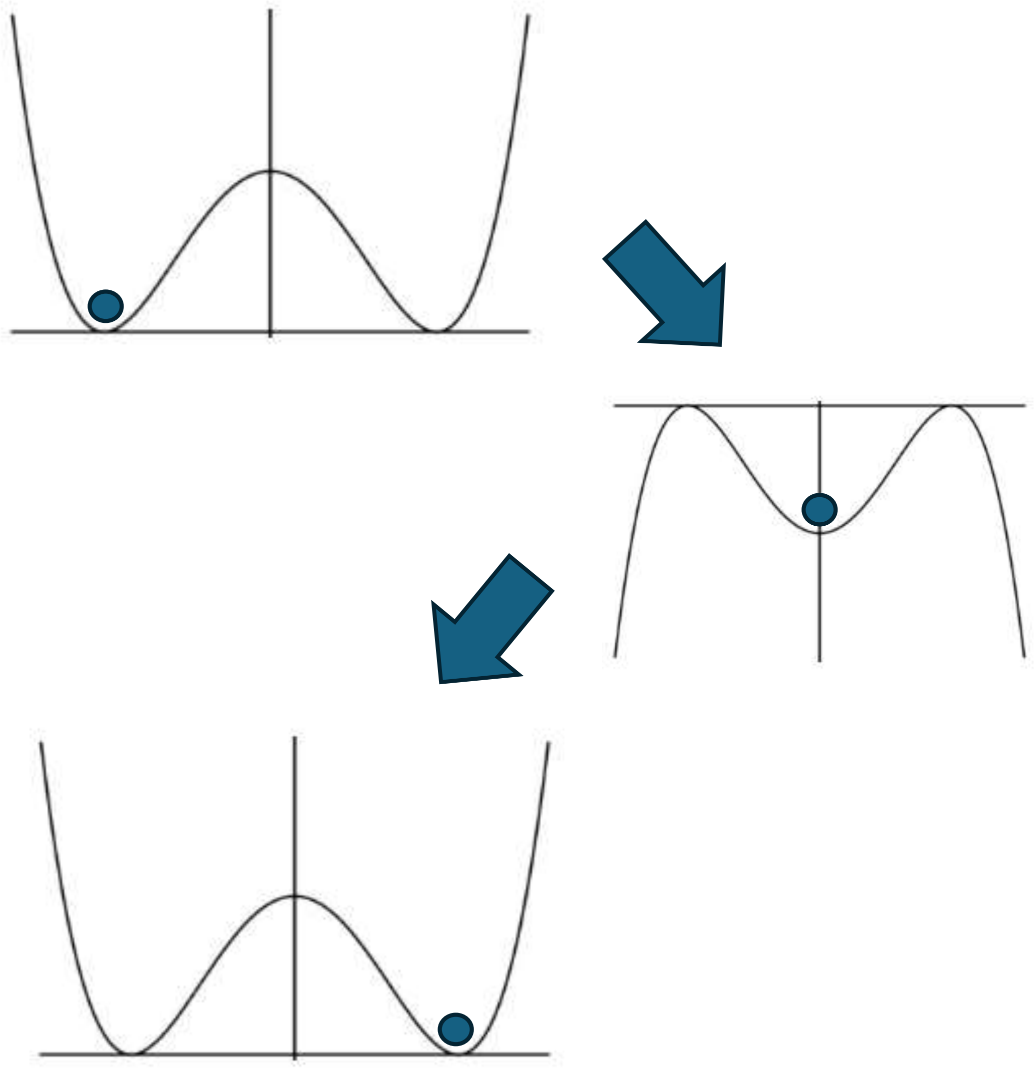

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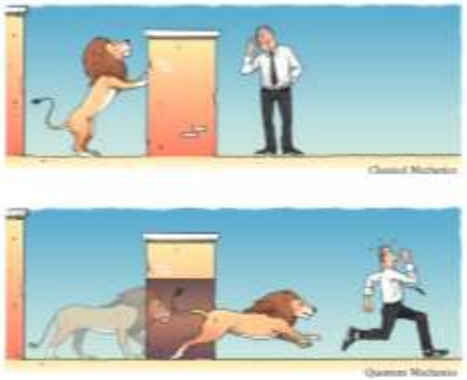
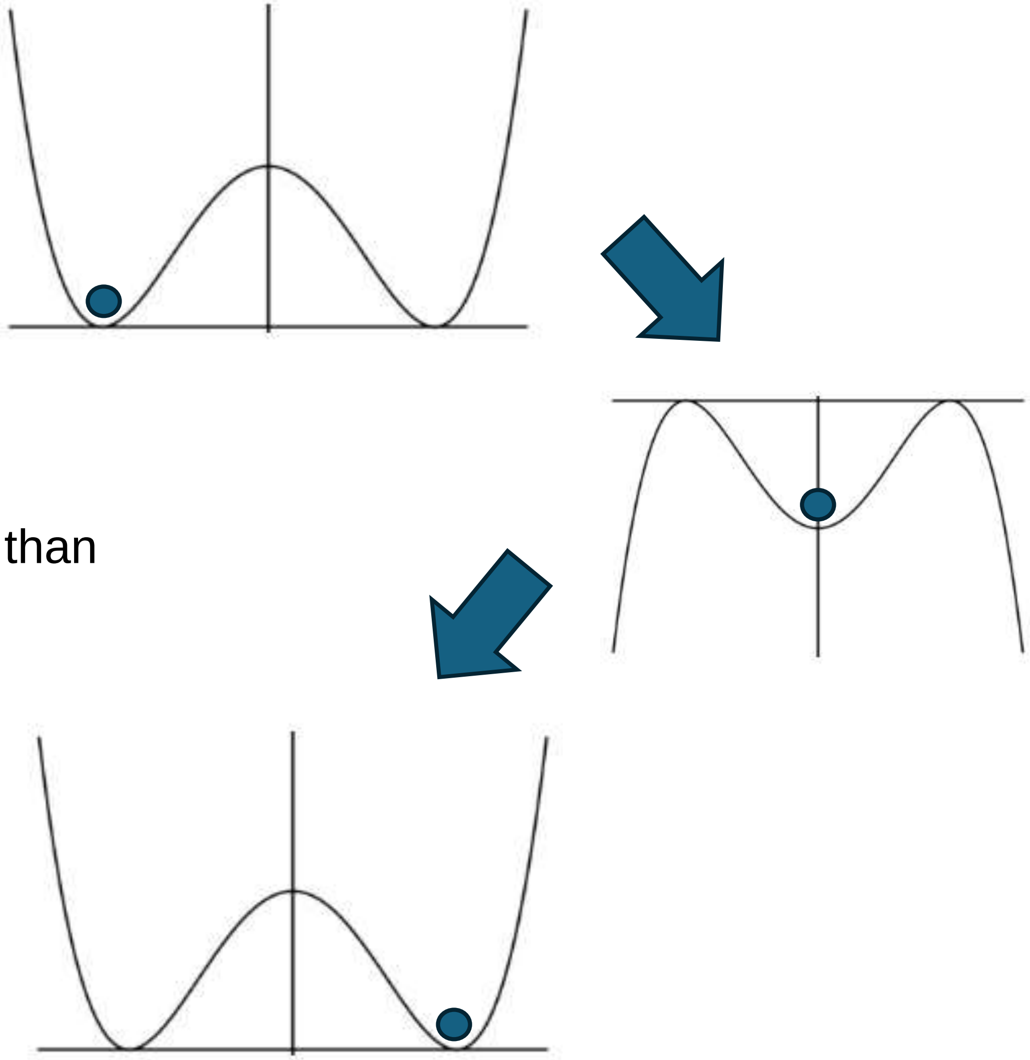
So it represents an event that happens at an **“instant”** rather than persisting as a particle-like object in Lorentzian time.

The term "pseudoparticles" was used for such localized “excitations” in Euclidean time.





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Non-trivial field configurations that do not occur “naturally” from the charges of the theory (magnetic monopoles).

Field equation solutions that can be used to offer insight in QFT effects (tunnelling).

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Many methods to derive them in Gauge field theories.

Duality properties

(Note this, we will need it…)

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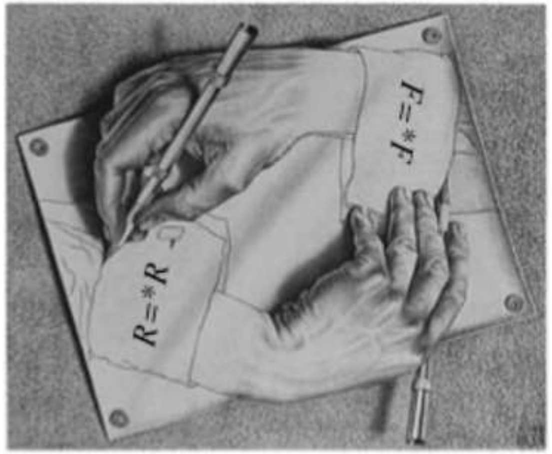
Tohru Eguchi and Andrew Hanson, 1978

Self-Duality property.

Geometrical spaces with specific characteristic “deformities”.



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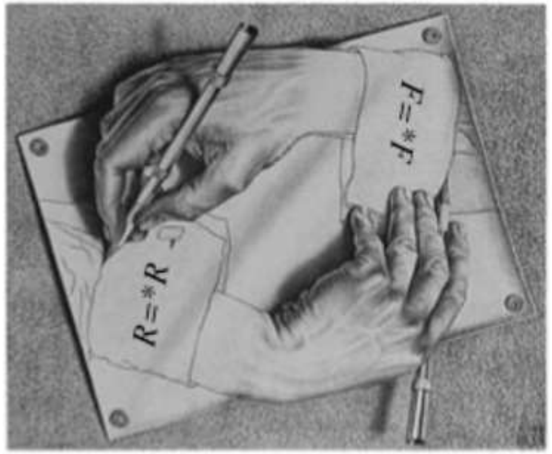

The Gibbons-Hawking classification from 1979 was fundamental.

They showed that the fixed point set of an isometry can be
either a **nut** (an isolated point)
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This classification is used to relate the topology of an instanton to its topological invariants like the Euler characteristic

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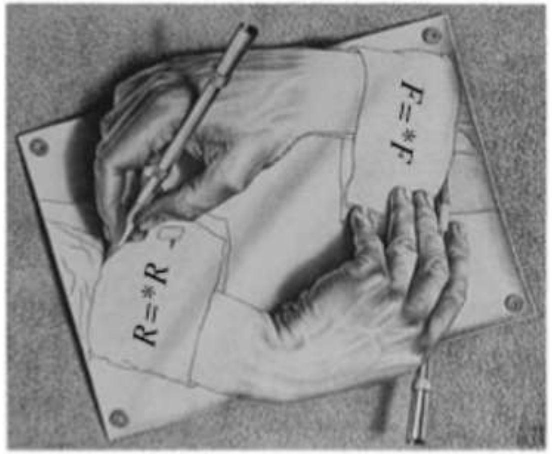

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Essentially, "shapes with gravity".

- Gravitational instantons
    - What are they and where do they come from

They serve as a powerful bridge between theoretical physics and pure mathematics, serving as "laboratories" for understanding four-dimensional geometry.

Studying them advances our understanding of fundamental geometric concepts like, **holonomy** and **topology**.

- Gravitational instantons
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They serve as a powerful bridge between theoretical physics and pure mathematics, serving as "laboratories" for understanding four-dimensional geometry.

They are non-compact, complete, Ricci-flat Riemannian 4-manifolds

Studying them advances our understanding of fundamental geometric concepts like, **holonomy** and **topology** in 4-dimensional objects.

Einstein Manifolds

A Testing Ground for Conjectures:
For instance, the long-held Euclidean black hole uniqueness conjecture, which stated that only a few specific gravitational instantons existed. (**Eucledian Black Hole existence)**

It was proven false by the discovery of the Chen–Teo instanton in 2011, keeping the field active and full of open questions

- Gravitational instantons
  - What are they and where do they come from

Mathematicians use a rich variety of tools to classify and study these spaces.

The most prominent is ***asymptotic classification***,
which categorizes instantons by how their geometry behaves "at infinity."

Asymptotic Classification

Because they are non-compact, a key way to study them is by looking at their "shape" far away from any central region.

- Gravitational instantons
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Asymptotic Classification

Because they are non-compact, a key way to study them is by looking at their "shape" far away from any central region. The major asymptotic categories include:

ALE (Asymptotically Locally Euclidean): Approaches $\mathbb{R}^4/\Gamma$, a quotient of flat 4D space by a finite group Γ.

AF (Asymptotically Flat): Approaches standard flat $\mathbb{R}^4$. This is a special case of ALE.

ALF (Asymptotically Locally Flat): Approaches a circle bundle over flat 3D space $\mathbb{R}^3$. This is a key family that includes the Taub-NUT and Chen–Teo instantons.

ALG, ALG*, ALH, ALH*: These are more exotic families with even slower volume growth (quadratic and linear), representing more recent and complex research frontiers.

Meaning?

- Gravitational instantons
  - Where are they being used.

Testing Ground for:

Boundary Conditions

Exotic Smooth Structures: We gain insight into whether the theory constrains or permits unexpected differential structures on a given topological manifold.

Geometric Stability: They represent robust, robustly stable classical background geometries that resist small perturbations.

Study Parallel Transport Mechanisms and Dynamical Theories

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Gravitational instantons are defined precisely to avoid the pathologies of Lorentzian singularities.

To be an instanton, a Euclidean solution must be **free of conical and orbifold singularities.**

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First Pontryagin and Second Chern Numbers

$$c_2(A) = \frac{1}{2}p_1(A) = -\frac{1}{16\pi^2}\int_M Tr(F \wedge F)$$

Euler Characteristic

$$\chi(A) = \frac{1}{16\pi^2}\int_M Tr(\star F \wedge F) - \frac{1}{16\pi^2}\int_{\partial M} Q_{EC}(\theta, A)$$

Hirzerbruch Signature τ

$$\tau = \frac{1}{3}p_1 + \frac{1}{12\pi^2}\int Tr(\theta \wedge F) - \eta_s$$

- Spin(4)-Cartan-Khronon theory of spacetime
  - Palatini formulation and Cartan-Khronon

Relevance of Gravitational Instantons

- Spin(4)-Cartan-Khronon theory of spacetime
  - Palatini formulation and Cartan-Khronon

Palatini Formalism

Group G Algebra Generators J^i_j.

(signature Sij preserving)

G-Vector spaces (fibber) with basis $\{V_i\}$

Four-Dimensional smooth manifold $\{x^0, x^1, x^2, x^3\}$

g-Lie Algebra valued Connection $A = A^{ij}{}_\mu J_{ij} dx^\mu$

Soldering form co-frame, tetrad $e = e^i{}_\mu V_i dx^\mu$

Forming the line element $ds^2 = g_{\nu\mu} dx^\mu \otimes dx^\nu$

Using the metric tensor defined as: $g_{\mu\nu} = e^i_\mu e^j_\nu S_{ij}$,

A bundle isomorphism that identifies the abstract internal fibber with the tangent space of the base manifold.

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G-Vector spaces (fibber) with basis $\{V_i\}$

Four-Dimensional smooth manifold $\{x^0, x^1, x^2, x^3\}$

g-Lie Algebra valued Connection $A = A^{ij}{}_{\mu} J_{ij} dx^{\mu}$

Soldering form co-frame, tetrad $e = e^i{}_{\mu} V_i dx^{\mu}$

Forming the line element $ds^2 = g_{\nu\mu} dx^{\mu} \otimes dx^{\nu}$

Using the metric tensor defined as: $g_{\mu\nu} = e^i_{\mu} e^j_{\nu} S_{ij},$

A bundle isomorphism that identifies the abstract internal fibber with the tangent space of the base manifold.

The Cartan-Khronon field

$$\phi = \phi(x)$$

It is a real scalar field.

It is given structure with respect to the group representation (inner space of the fibber).

$$\phi^i = (\phi^0, \phi^1, \phi^2, \phi^3)$$

The break of space-time symmetry

It "chooses" a direction in the abstract internal fibber space.

$$\phi^i = \delta^i_0 \phi = (\phi, 0, 0, 0)$$

Produce the soldering form.

$$e = e^i_{\mu} V_i dx^{\mu} = D\phi^i = d\phi^i + A^i{}_{j\mu} \phi^j$$

- Spin(4)-Cartan-Khronon theory of spacetime
  - Euclidean and Lorentzian descriptions – Wick Rotation

Euclidean and Lorentzian descriptions – Dynamical Equivalence

- Spin(4)-Cartan-Khronon theory of spacetime
  - Euclidean and Lorentzian descriptions

Through the symmetry breaking

$$\phi^i = \phi\delta^i_0$$

The Khronon and the group signature controls the foliation and fixes the time coefficient in

$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i \otimes dx^j$$

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$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^i \otimes dx^j$$

g_{00} **is not a free coefficient.**

It is fixed by the Khronon via:

$$g_{00} = D_0\phi^i D_0\phi^j S_{ii}$$

Hence Khronon and S of the group control the "sign of time"

$$ds^2 = d\tau d\tau + g_{ij}dx^i \otimes dx^j = -dtdt + g_{ij}dx^i \otimes dx^j,$$

$$\text{with } \tau, t \in \mathbb{R}.$$

Both time coordinates are real and yield a real line element.

The sign difference is in the interpretation

- of time unit
- the choice of internal metric S_{ij},

not in the field configuration itself.

Equivalence in the dynamic mechanisms of the two descriptions…..

- Spin(4)-Cartan-Khronon theory of spacetime
  - The group, the Algebra generators, the connection

- Spin(4)-Cartan-Khronon theory of spacetime
  - The group, the Algebra generators, the connection

The $spin(4)$ algebra generators are the $so(4)$ antisymmetric J_{ij} satisfying:

$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk}$$

More clearly, they separate to *rotations* $K_i = \frac{1}{2}\epsilon_i{}^{jk} J_{ij}$ and *translations* $T_i = J_{i4}$ {in the $SO(4)/SO(3)$}.

$$[T_i, T_j] = \epsilon_{ij}{}^{k} K_k$$

$$[K_i K_j] = \epsilon_{ij}{}^{k} K_k$$

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We define the *spin*(4) *Lie algebra* valued, one form connection on the spacetime manifold as:

$$A = \frac{1}{2}A_\mu^{ij}J_{ij}dx^\mu$$

Field Strength or *Curvature*:

$$F = dA + [A \wedge A],$$

Taking into account the permutation relations of the generators.

$$[A \wedge A] = \frac{1}{2}A^{kc}{}_\mu A_c{}^l{}_\nu J_{kl}dx^\mu \wedge dx^\nu$$

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Based on these structures we build the invariants of the theory, with variables the *Khronon field* and the *spin*(4) *connection*:

$$Tr(F \wedge F) \qquad Tr(F \wedge \star_G F) \qquad \epsilon_{ijkl} D\phi^i \wedge D\phi^j \wedge D\phi^k \wedge D\phi^l$$

$$D\phi \wedge D\phi \wedge \star_G F$$

with $\star_G = \frac{1}{2}\epsilon^{ij}{}_{kl}$ the *Hodge star operator* defined on the group algebra structure.

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Vector Space Representation

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Chiral Decomposition: We can decompose the $so(4)$ algebra to the usual $su(2)_+ \otimes su(2)_-$ by defining a new basis representation:

$$^{\pm}J_{ab} = \frac{1}{2}(J_{ab} \pm \star_G J_{ab}),$$

and set:

$$^{+}J^i = +^{+}J^{4i} - \epsilon^{i}{}_{jk}{}^{+}J^{jk}$$

$$^{-}J^i = -^{-}J^{4i} - \epsilon^{i}{}_{jk}{}^{-}J^{jk}$$

Now the commutation relations are taking the form:

$$[^{+}J^i, {}^{+}J^j] = \epsilon^{ij}{}_{k}{}^{+}J^k$$

$$[^{-}J^i, {}^{-}J^j] = \epsilon^{ij}{}_{k}{}^{-}J^k$$

$$[^{+}J^i, {}^{-}J^j] = 0$$

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Connection and Field Strength decomposition

$$A = {}^{+}A + {}^{-}A$$

$$F = {}^{+}F + {}^{-}F$$

$$^{\pm}F = d^{\pm}A + [^{\pm}A \wedge {}^{\pm}A]$$

- Spin(4)-Cartan-Khronon theory of spacetime
  - Action and Chiral decomposition

$$I_{(2)} = \frac{1}{2}\int \epsilon_{ijkl} D\phi^i \wedge D\phi^j \left(g_+ {}^+F^{kl} + g_- {}^-F^{kl}\right)$$

Decomposition to Self-Dual and Anti-Self Dual sectors.

Directly analogous to the Holst term used in Ashtekar variables.

Most gravitational instantons probe the different sectors.

Phase transition between chiral sectors.

- Spin(4)-Cartan-Khronon theory of spacetime
  - Instanton solutions predictions - Method

Solutions: Connection Components A and Khronon field configuration. (The larger picture)

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Solutions: Connection Components A and Khronon field configuration. (The larger picture)

The Riemannian metric to confirm is:

$$ds^2 = f_0^2(x)dt^2 + g_0^2(x)(dx^1)^2 + h_0^2(x)d\Omega^2 \qquad (1)$$

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Euclidean manifold M with: $x = \{x^1, x^2, x^3, x^4\} = \{x^1, x^2, x^3, t\}$

The Khronon scalar Field: $\phi^I = \kappa^{-1/2}\phi\delta_4^I$ (2) $\phi^i = \{0, 0, 0, \phi\}$

The $Spin(4)$ connection:

The Electric components

$$A^{14} = A^{14}(g, h, \phi),$$
$$A^{24} = A^{24}(g, h, \phi), \qquad (3)$$
$$A^{34} = A^{34}(g, h, \phi).$$

with f,g,h, scalar functions of spacetime x.

The Co-frame forms as:

$$e^I = D\phi^I = d\phi^I + A^I{}_J\phi^J, \quad e^I = (A^I{}_4\phi, d\phi)$$

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Set the time coordinate with respect to the Khronon:

$$d\phi = f(x)dt$$

The metric formed by this process is given by

$$g_{\mu\nu} = e^I{}_\mu e^J{}_\nu \delta_{IJ}$$

$$ds^2 = d\phi^2 + g^2(x)(dx^1)^2 + h^2(x)d\Omega^2 = f^2(x)dt^2 + g^2(x)(dx^1)^2 + h^2(x)d\Omega^2$$

(4)

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$$ds^2 = f_0^2(x)dt^2 + g_0^2(x)(dx^1)^2 + h_0^2(x)d\Omega^2$$
$$ds^2 = f^2(x)dt^2 + g^2(x)(dx^1)^2 + h^2(x)d\Omega^2 \quad (5)$$

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The Magnetic components

$$A^{12} = A^{12}(X),$$
$$A^{13} = A^{13}(Y), \quad (6)$$
$$A^{23} = A^{23}(Z).$$

with f,g,h, X, Y, Z scalar functions of spacetime x.

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Choosing Right-Hand Model with $g_- = 0$:

$$\delta S[{}^+A^{ij}, \phi^I] = \int_M \delta L({}^+A^{ij}, \phi^I) \qquad (7)$$

a simplified "toy model" to test the core ideas of the theory.

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Variation with respect to the connection constrain $X(x), Y(x), Z(x)$ with respect to $f(x), g(x), h(x)$.

$$T^i \wedge d\phi + \epsilon^i{}_{jk} T^j \wedge D\phi^i = 0$$

Variation with respect to the **Khronon field** ϕ^I yields *energy* and *momentum* constraints (8)

$${}^+F_i \wedge D\phi^i = 0, \qquad {}^+F^i \wedge D\phi - \epsilon^i{}_{jk} {}^+F^j \wedge D\phi^k = 0$$

Confirming these equations with the ansatz $(f,g,h) = (\pm f_0, \pm g_0, \pm h_0)$, shows that the corresponding Riemannian metric is reproduced by the theory.

with f,g,h, X, Y, Z scalar functions of spacetime x.

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Multiple Branches solution.
Multiple Connection Configurations
Multiple Sets of Topological Invariants.

- Gravitational instantons in the theory
  - Eguchi-Hanson

ALE metric
Self-Dual Rieman Curvature

$$ds^2 = \frac{1}{(1-\frac{a^4}{r^4})}dr^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) + r^2(1-\frac{a^4}{r^4})(\sigma^3)^2, \quad (9)$$

$$\sigma^1 = \frac{1}{2}(\sin\psi\mathbf{d}\theta - \cos\psi\sin\theta\mathbf{d}\phi), \quad \sigma^2 = -\frac{1}{2}(\cos\psi\mathbf{d}\theta - \sin\psi\sin\theta\mathbf{d}\phi), \quad \sigma^3 = \frac{1}{2}(\mathbf{d}\psi + \cos\theta\mathbf{d}\phi)$$

$$M_{EH} = \{(\psi,\theta,\phi,r)||r \in [a,+\infty)\ ,\ 0 \le \theta\ < \pi\ ,\ 0 \le \phi < 2\pi\ ,\ 0 \le \psi < 2\pi\}. \quad (10)$$

Chern & Ponrajagin numbers	*Euler Characteristic*	*Hirzebruch signature*
$c_2 = p_1/2 = -3/2$	$\chi = 2$	$\tau = -1$

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Choice for the space to be geodesically complete, avoiding what is known **as a conical singularity**.

$$\mathbb{R}^4 = \{(\psi,\theta,\phi,r)||r\in[0,+\infty)\ ,\ 0\le\theta\le\pi\ ,\ 0\le\phi\le 2\pi\ ,\ \boxed{0\le\psi\le 4\pi\}.}$$

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At infinity we have the metric behaving as:

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Locally, this is just the flat metric on $\mathbb{R}^4$. However, because of the angular identification needed for the smoothness at $r = a$, the sphere at infinity is not S^3, but $S^3/\mathbb{Z}_2$. This makes Eguchi-Hanson an asymptotically locally Euclidean space (ALE).

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The definition of ALE spaces includes complete non-compact Riemannian manifolds (4d) whose metric approaches the flat Euclidean metric on $\mathbb{R}^4/\Gamma$ at infinity, where $\Gamma \subset SO(4)$ is a finite group acting freely on S^3. (Also used in literature the projective space $S^3/\mathbb{Z}_2 \sim \mathbb{RP}^3$, brief description, if $p \in S^3$, then $S^3/\mathbb{Z}_2 := \{\{p, -p\} : p \in S^3\}$.

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At the bolt where $r \to a$, we have what appears to be a singularity. In fact it is a **coordinate singularity**.

By the coordinate transformation $\rho^2 = 4a\epsilon$, leading to $\frac{a}{4\epsilon}d\epsilon^2 = d\rho^2$ and the metric takes the form:

$$ds^2_{\rho\to 0} = \frac{1}{4}d\rho^2 + \rho^2(\sigma^3)^2 + a^2\Big((\sigma^2)^2 + (\sigma^2)^2\Big).$$

The collapsing part of the metric behaves like a closing plane to the origin, and the metric behaves as $\mathbb{R}^2 \times S$.
"Every (σ^1, σ^2) 2-sphere has a plane attached at each point."

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$$\phi^I = \kappa^{-1/2} R \delta^I_4$$

$$\mathbf{d}R \equiv f(r)\mathbf{d}r$$

$$ds^2 = f^2(r)dr^2 + r^2 g^2(r)(\sigma^3)^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \quad (11)$$

The $Spin(4)$ connection:

$$A^{14} = \frac{r(R)}{R}\sigma_1$$

$$A^{24} = \frac{r(R)}{R}\sigma_2$$

$$A^{34} = \frac{g(R)}{R}\sigma_3$$

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  - Eguchi-Hanson

ALE metric
Self-Dual Rieman Curvature

$$ds^2 = \frac{1}{(1-\frac{a^4}{r^4})}dr^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) + r^2(1-\frac{a^4}{r^4})(\sigma^3)^2, \quad (9)$$

The Khronon scalar Field:

$$\phi^I = \kappa^{-1/2} R \delta^I_4$$

$$\mathbf{d}R \equiv f(r)\mathbf{d}r$$

$$ds^2 = f^2(r)dr^2 + r^2 g^2(r)(\sigma^3)^2 + r^2\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \quad (11)$$

$$T^i \wedge D\phi^4 + \epsilon^i{}_{jk} T^j \wedge D\phi^k = 0.$$

The $Spin(4)$ connection:

$$A^{14} = \frac{r(R)}{R}\sigma_1 \qquad A^{12} = Z(r)\sigma^3,$$

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$$Y(r) = X(r) = \frac{1}{f} + g - \frac{r}{R(r)}$$

$$Z(r) = 2 - g\Big(g + \frac{r}{R(r)}\Big) + \frac{(g+g'r)}{f}$$

- Gravitational instantons in the theory
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$$^+F_i \wedge D\phi^i = 0$$

$$^+F^i \wedge D\phi^4 - \epsilon^i{}_{jk}\, {}^+F^j \wedge D\phi^k = 0.$$

$$Ansatz: \; f_0 = \left(1-\frac{a^4}{r^4}\right)^{-1/2} \quad g_0 = \left(1-\frac{a^4}{r^4}\right)^{1/2} \quad (12)$$

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4 Different Branches (f, g) confirming the constraints:

$$(+f_0, +g_0) \quad (+f_0, -g_0) \quad (-f_0, g_0) \quad (-f_0, -g_0)$$

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Overall, 2 different ${}^+A^i$ and 1 set of topological invariants (reproducing GR) ${}^-A^i$ with 4 divergent sets of topological invariants.

$${}^+A: \quad c_2^+ = -3/2 \quad \chi^+ = 2 \quad \tau = -1$$

$${}^-A: \quad c_2^- = \infty \quad \chi^- = \infty \quad \tau^- = \infty$$

- Gravitational instantons in the theory
  - Taub-NUT

ALF metric
Anti-Self-Dual Riemannian Curvature

The Taub-NUT space is a topological 4d manifold equipped with the Ricci-flat metric:

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}dr^2 + 4m^2\frac{r-m}{r+m}(\sigma^3)^2 + (r+m)(r-m)\Big((\sigma^1)^2 + (\sigma^2)^2\Big) \qquad (13)$$

Chern & Ponrajagin numbers	*Euler Characteristic*	*Hirzebruch signature*
$c_2 = \frac{1}{2}p_1 = 0$	$\chi = 1$	$\tau = 0$

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It is a non-compact 4-dimensional manifold, where the metric is smooth and that holds for the polar coordinates

$$M_{TN} = \{(\psi,\theta,\phi,r)||0 \le \psi < 4\pi \ , \ 0 \le \theta < 2\pi \ , \ 0 \le \phi < 2\pi \ , \ r \in [m,+\infty)\}$$

The g_{rr} component is smooth for $r \in (m,+\infty)$, but the smoothness of the line element can be extended to $r \in [m,+\infty)$.

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Near the isolated point, which can be set to operate as an origin after a coordinate transformation $d\rho = \frac{1}{2}\sqrt{\frac{r+m}{r-m}}dr$, we can see the line element behave as:

$$ds^2 = d\rho^2 + \rho^2\Big((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\Big),$$

which is the Euclidean $\mathbb{R}^4$.

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On the other hand, at infinity the metric behaves as:

$$ds^2 = \frac{1}{4}dr^2 + 4m^2(\sigma^3)^2 + r^3\Big((\sigma^1)^2 + (\sigma^2)^2\Big).$$

Instead of opening to a full $\mathbb{R}^4$ metric, the σ^3 circle direction approaches a constant circle of radius 2m, forming a **circle bundle** over a **base 2-sphere** at infinity. This is a typical behavior defining an ALF (Asymptotically Locally Flat) metric.

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$$Y = X = \frac{g}{h} - \frac{h}{R} + \frac{h'}{f}, \quad Z = 2 - \frac{g^2}{h^2} - \frac{g}{R} + \frac{g'}{f}.$$

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$$f_0(r) = \frac{1}{2}\sqrt{\frac{r+m}{r-m}}, \qquad g_0(r) = 2m\sqrt{\frac{r-m}{r+m}}, \quad h_0(r) = \sqrt{(r+m)(r-m)} \qquad (15)$$

- Gravitational instantons in the theory
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$$S_1(f,g,h)=(f_0,g_0,h_0),\quad S_2(f,g,h)=(f_0,g_0,-h_0),\quad S_3(f,g,h)=(f_0,-g_0-,h_0),\quad S_4(f,g,h)=(f_0,-g_0,h_0),$$
$$S_5(f,g,h)=(-f_0,-g_0,h_0),\quad S_6(f,g,h)=(-f_0,g_0,-h_0),\quad S_7(f,g,h)=(-f_0,-g_0,-h_0),\quad S_8(f,g,h)=(-f_0,-g_0,h_0).$$

Topological invariants of ^+A.

-Out of 8 cases, 4 different +A
2 sets of topological invariants

$$c_2=p_1/2=0 \quad \chi=2/3 \quad \tau=2/3$$
$$c_2=p_1/2=-1 \quad \chi=5/3 \quad \tau=1/3$$

Topological Invariants of ^-A.

-Out of 8 cases, 8 different -A
1 set of topological invariants

$$c_2=p_1/2=0 \quad \chi=1 \quad \tau=0$$

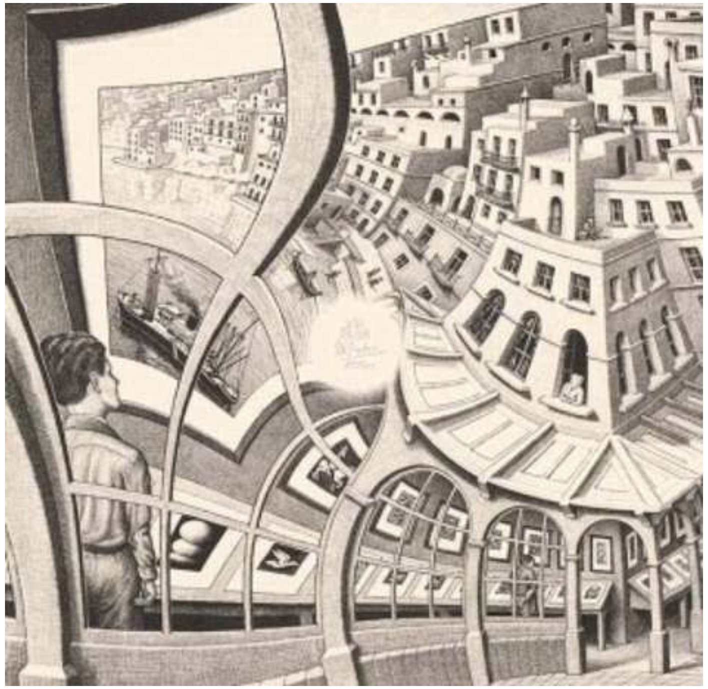
Exactly the GR topological Numbers

$$f_0(r)=\frac{1}{2}\sqrt{\frac{r+m}{r-m}},\qquad g_0(r)=2m\sqrt{\frac{r-m}{r+m}},\quad h_0(r)=\sqrt{(r+m)(r-m)} \qquad (15)$$

Other Solutions and future directions on instantons

- Gravitational instantons predicted by the theory.
- Interpretation of results – Lorentz Analogues
- More instanton solutions
- Gibbons-Hawking Parametrization (ALEs, ALFs)
- AF and time-periodic bolts
- Euclidean wormhole instanton solutions (Baby Universes)

References



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- Gravitational instantons
  - Where are they being used
  - Topological Invariants

Nieh-Yan Term

$$I_{NH} = \int_M (F_{IJ} \wedge F^{ik} \phi^j \phi^k - F_{IJ} \wedge D\phi^I \wedge D\phi^J) = \int_{\partial M} D\phi_I \wedge F^{IJ} \phi_J$$