

Temperature and Diffusivity in Frame-Invariant Scalar-Tensor Gravity

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- First-order thermodynamics of Brans–Dicke theories
- Frame-dependence of thermodynamical quantities
- Frame-agnostic formulation of scalar-tensor theories
- Frame-invariant diffusive equilibrium

- First-order thermodynamics of Brans–Dicke theories
 - ▶ **The established approach**
- Frame-dependence of thermodynamical quantities
 - ▶ **The problem (and why it's an actual problem)**
- Frame-agnostic formulation of scalar-tensor theories
 - ▶ **The right toolset for the job**
- Frame-invariant diffusive equilibrium
 - ▶ **The solution (and future work)**

- Gravity and thermodynamics have a deep relationship
 - laws of black hole thermodynamics can be derived from the Einstein field equations
 - Einstein field equations can be derived from thermodynamical considerations of entropy and horizon area ([Jacobson, 1995](#))
- Many motivations to study modified gravity, but scalar-tensor theories in particular are both elegant in their simplicity and rich in their features
- Effective energy-momentum tensor description of scalar-tensor theories provides a natural starting point for a thermodynamical description

- We examine the class of **Brans–Dicke theories**, specified by the action

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right] + S^{(m)}$$

- With the usual variational methods, the equations of motion are

$$G_{\mu\nu} = \frac{T_{\mu\nu}^{(m)}}{\phi} + \frac{\omega}{\phi^2} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{g_{\mu\nu}}{2} (\nabla\phi)^2 \right] + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square\phi) - \frac{g_{\mu\nu}}{\phi} \frac{V}{2}$$
$$\square\phi = \frac{1}{2\omega + 3} \left[T^{(m)} + \phi V' - 2V - \omega' (\nabla\phi)^2 \right]$$

- Complicated, but with the correct definition of the effective **energy-momentum tensor** they can be put in a very simple form

$$G_{\mu\nu} = \frac{T_{\mu\nu}^{(m)}}{\phi} + T_{\mu\nu}^{(\phi)}$$

- The effective energy-momentum tensor is

$$T_{\mu\nu}^{(\phi)} = \frac{\omega}{\phi^2} \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{g_{\mu\nu}}{2} (\nabla\phi)^2 \right] + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square\phi) - \frac{g_{\mu\nu}}{\phi} \frac{V}{2}$$

- This energy-momentum tensor can be *interpreted* as that of a **imperfect fluid** (only assumption necessary is timelike and future-oriented $\partial_\mu \phi$)
- Endows spacetime with **induced metric**

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu.$$

- Velocity is $u_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{-\nabla^\rho \phi \nabla_\rho \phi}}$
- Can define four-acceleration (with τ proper time defined along flow lines)

$$a^\mu \equiv \frac{du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu,$$

- Using projection operator $h_{\mu}{}^{\nu}$, we can define other kinematic quantities like the expansion scalar, the shear scalar, the vorticity tensor, and the symmetric traceless shear tensor
- Can also define mechanical quantities e.g. energy density and pressure: hence, we decompose the energy-momentum tensor in the usual **imperfect fluid** form

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p h_{\mu\nu} + q_{\mu} u_{\nu} + u_{\mu} q_{\nu} + \pi_{\mu\nu}$$

- We have the comoving effective energy density ρ , the isotropic pressure p , the heat flux q_{μ} , and the anisotropic stress $\pi_{\mu\nu}$
- But how to connect to thermodynamics?

- Eckart's first-order relativistic thermodynamics (though supplanted by the Israel–Stewart formalism) can be used to write down constitutive laws between dissipative quantities (which are responsible for entropy production) and mechanical/kinetic fluid quantities

$$\begin{aligned}p_{\text{visc}} &= -\zeta\theta, \\q_{\mu} &= -K(h_{\mu\nu}\nabla^{\nu}T + Ta_{\mu}), \\\pi_{\mu\nu} &= -2\eta\sigma_{\mu\nu}.\end{aligned}$$

- Properties of the medium are encoded in the bulk viscosity ζ , the thermal conductivity K , and the shear viscosity η , whereas T is the temperature
- *A priori*, there really is no reason why a scalar-tensor theory should follow a thermodynamical law
- However, if the constitutive equations are satisfied by the effective fluid, then we have a *thermodynamical interpretation* of the theory

- And they do! Constructing the normalized velocity out of the only vector as $u_\mu \equiv \frac{\nabla_\mu \phi}{\sqrt{-\nabla_\rho \phi \nabla^\rho \phi}}$, the acceleration and the heat flux are:

$$a_\mu = \frac{(\nabla^\nu \phi)}{(-\nabla^\rho \phi \nabla_\rho \phi)^2} [(-\nabla^\rho \phi \nabla_\rho \phi)(\nabla_\mu \nabla_\nu \phi) + (\nabla^\sigma \phi)(\nabla_\sigma \nabla_\nu \phi)(\nabla_\mu \phi)]$$

$$q_\mu = \frac{(\nabla^\rho \phi)(\nabla^\sigma \phi)}{\phi(-\nabla^\nu \phi \nabla_\nu \phi)^{3/2}} [(\nabla_\sigma \phi)(\nabla_\rho \nabla_\mu \phi) - (\nabla_\mu \phi)(\nabla_\rho \nabla_\sigma \phi)]$$

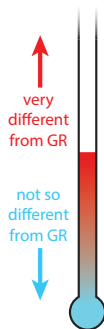
- Comparing to $q_\mu = -K(h_{\mu\nu} \nabla^\nu T + T a_\mu)$ and, since in the comoving frame the spatial temperature gradient vanishes, we have

$$KT = \frac{1}{\phi} \sqrt{-\nabla^\mu \phi \nabla_\mu \phi}$$

- A "hot" theory is therefore far away from GR, whereas a "cold" theory is closer to GR, and GR is at absolute zero
- For Brans–Dicke theories, GR emerges as a *thermal attractor* [Faraoni *et al.* '21]

$$\frac{d(KT)}{d\tau} = (KT)^2 - \theta KT + \frac{\square\phi}{\phi}$$

- A free theory $\square\phi = 0$ will therefore "cool down" or "warm up" depending on the expansion of spacetime: the critical line is $\theta = KT$, controlling whether the theory deviates from or converges to GR



- Scalar-tensor theories can be written in different **frames** through a conformal transformation followed by a field reparametrization

$$g_{\mu\nu} \mapsto \hat{g}_{\mu\nu} = \Omega(\phi)^2 g_{\mu\nu}$$
$$\phi \mapsto \hat{\phi} = f(\phi)$$

- Frame transformations do not affect the physics (at least classically): they essentially correspond to **local unit transformations** [Dicke, '62], leaving unchanged
 - inflationary observables (n_s , r , etc.)
 - solar system predictions from PPN parameters
 - black hole entropy
 - radiative corrections [Vilkovisky & De Witt '81, '84], [Pilaftsis & SK '17, '19]
- So we would hope that temperature, which is an intensive property, is uniquely defined

- We can go from one Brans–Dicke representation to another with $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega(\phi)^2 g_{\mu\nu}$ and $\hat{\phi} = \Omega(\phi)^{-2} \phi$

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \right] + \dots$$

$$S_{\text{BD}} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} \left[\hat{\phi} \hat{R} - \frac{\hat{\omega}(\hat{\phi})}{\hat{\phi}} \hat{g}^{\mu\nu} (\partial_\mu \hat{\phi})(\partial_\nu \hat{\phi}) \right] + \dots$$

- Temperature transforms as

$$KT \rightarrow K\hat{T} = \frac{\Omega(\phi) - 2\phi\Omega'(\phi)}{\Omega(\phi)^2} KT$$

- So we can tune the temperature arbitrarily by choosing the function $\Omega(\phi)$

The temperature frame problem

- Is this a problem? Does temperature have to be the same in all frames?
- Not necessarily: e.g. for Lorentz frames, mass is a useful concept, but we usually refer to the rest mass
- Similarly, to get useful information regarding temperature, we must then pick a frame to measure it in
- In a sense, the "rest frame" of a scalar-tensor theory is the Einstein frame, where equations of motion take on a simpler form, so we may be tempted to define the "proper temperature" of the theory in the Einstein frame

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- In a sense, the "rest frame" of a scalar-tensor theory is the Einstein frame, where equations of motion take on a simpler form, so we may be tempted to define the "proper temperature" of the theory in the Einstein frame
- However, we quickly find out that the temperature is identically zero in the Einstein frame with $\Omega^2 = \phi$:

$$K\hat{T} = \frac{\Omega(\phi) - 2\phi\Omega'(\phi)}{\Omega(\phi)^2}KT = \mathbf{0}$$

- Intuitive, since the effective fluid is now **perfect**, which means that all dissipative quantities vanish.

The temperature frame problem

- The proper temperature of a scalar-tensor theory makes about as much sense as the "proper velocity" of a test mass: it is unambiguously defined and invariant, but it's identically zero and offers no physical information
- In the Einstein frame, we can actually trade the temperature description for the diffusive description
- Instead of Fourier's law of thermal conduction and temperature, we have Fick's law of particle diffusion and the chemical potential μ

$$Q_\rho = -D(h_{\rho\sigma}\nabla^\sigma\mu + \mu a_\rho)$$

- This is the "correct" approach in the presence of minimal coupling, where the temperature just goes to zero

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- ...but it is a bit strange that our approach must depend on the representation of a theory. There must be some quantities that are *independent* of the frame the theory is expressed in
- **So how do we find them?**

The frame-invariant formalism: the why

- The different frames of a theory are none other than different **representations**: therefore, any quantity that parametrizes the physics of a theory should be **frame-invariant**
- A frame transformation can be interpreted as a local unit transformation: e.g. in both cases, all length scales are locally rescaled by a scale factor

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \rightarrow \Omega^2 g_{\mu\nu} dx^\mu dx^\nu = \Omega^2 ds^2$$

- Whether the physics is affected by the rescaling of some mass/length scale depends on whether dimensionless ratios π are affected:

$$\pi = \frac{Q}{M_P^D} \rightarrow \frac{\Omega^{-D} Q}{M_P^D} = \Omega^{-D} \pi \quad (\text{physics changes})$$

$$\pi = \frac{Q}{M_P^D} \rightarrow \frac{\Omega^{-D} Q}{(\Omega^{-1} M_P)^D} = \pi \quad (\text{physics doesn't change})$$

The frame-invariant formalism: the why

- Unit transformations are not physical precisely because they leave all dimensionless ratios π invariant, leading to the Buckingham- π theorem: all laws of physics can be expressed in terms of dimensionless ratios

$$f(\pi_1, \pi_2, \dots) = 0$$

- Thus if we "nondimensionalize" a scalar-tensor theory, i.e. write it in terms of **frame invariants**, we can easily extract quantities that are independent of its frame representation

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- **Conformal invariance** implies invariance of the action and is a statement about how the theory behaves at different scales
- **Frame invariance** implies form invariance of the action and is a statement about which features of a theory are independent of the choice of conformal frame

The frame-invariant formalism: the why

general covariance

trajectory
observer/coordinate system
equations of motion
coordinate transformation
relative quantities
absolute quantities
rest frame
moving frame

frame covariance

theory
frame
action
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frame-dependent quantities
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The frame-invariant formalism: the why

general covariance	frame covariance
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- Just like with general covariance, we don't *have* to work with a generally covariant formalism if we don't want to: it's perfectly fine to stick to coordinate-dependent descriptions...

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- Just like with general covariance, we don't *have* to work with a generally covariant formalism if we don't want to: it's perfectly fine to stick to coordinate-dependent descriptions...
- ...but a covariant formalism does help us figure out what properties of an action are artifacts of its representation and which are intrinsic to the theory it specifies

The frame-invariant formalism: how

- We consider a general scalar-tensor action, with three model functions

$$S_{ST} = \frac{1}{2} \int d^4x \sqrt{-g} [A(\phi)R - B(\phi)(\nabla\phi)^2 - 2V(\phi)]$$

- Given a frame transformation, we have the transformation rules:

$$\hat{A} = \Omega^{-2}A,$$

$$\hat{B} = \Omega^{-2} [f'^{-2}B - 6A(\ln \Omega)'^2 + 6A'(\ln \Omega)'],$$

$$\hat{V} = \Omega^{-4}V,$$

- Thus we have the frame-invariant metric, which can be used to compare physical lengths even between different frames

$$\mathfrak{g}_{\mu\nu} = Ag_{\mu\nu}$$

- We can also define the usual curvature tensors, now invariant:

$$F^{\rho}_{\mu\nu} \equiv \frac{g^{\rho\sigma}}{2} (D_{\mu}g_{\sigma\nu} + D_{\nu}g_{\mu\sigma} - D_{\sigma}g_{\mu\nu}).$$

$$\mathfrak{R}^{\rho}_{\sigma\mu\nu} = D_{\mu}F^{\rho}_{\nu\sigma} - D_{\nu}F^{\rho}_{\mu\sigma} + F^{\rho}_{\mu\lambda}F^{\lambda}_{\nu\sigma} - F^{\rho}_{\nu\lambda}F^{\lambda}_{\mu\sigma},$$

$$\mathfrak{R}_{\mu\nu} = \mathfrak{R}^{\rho}_{\mu\rho\nu},$$

$$\mathcal{R} = \mathfrak{g}^{\mu\nu}\mathfrak{R}_{\mu\nu}.$$

- Using the model functions, there exist two independent frame-invariant quantities: the invariant field and the invariant potential

$$\Phi \equiv \pm \int d\phi \sqrt{\frac{B(\phi)}{A(\phi)} + \frac{3}{2} \left[\frac{A'(\phi)}{A(\phi)} \right]^2} \quad \mathfrak{U}(\Phi) \equiv \frac{V(\phi(\Phi))}{A(\phi(\Phi))^2}$$

- Much like selecting a basis for a vector space, there are infinitely many choices of invariant quantities, but we can have at most two independent ones
- Along with invariant metric $g_{\mu\nu} \equiv A g_{\mu\nu}$ and the associated invariant scalar curvature \mathfrak{R} , we can write the **invariant action**

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [\mathfrak{R} - g^{\mu\nu} (\partial_\mu \Phi)(\partial_\nu \Phi) - 2\mathfrak{U}]$$

- Any quantities extracted from this action will also be invariant!

- Again we write out the kinematic quantities, except now they are frame-invariant, e.g. fluid velocity and acceleration:

$$\mathbf{u}^\mu \equiv \frac{\partial^\mu \Phi}{\sqrt{-\mathbf{g}^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi}} \equiv \frac{\partial^\mu \Phi}{\sqrt{2\mathfrak{X}}}$$
$$\mathbf{a}^\mu \equiv \mathbf{u}^\nu \nabla_\nu^{(\mathbf{g})} \mathbf{u}^\mu,$$

- Since the invariant action features minimal coupling, we can use the standard expressions for the mechanical quantities along with the particle density and the chemical potential

$$\varrho \equiv \mathfrak{X} + \mathfrak{U}$$

$$\mathfrak{p} \equiv \mathfrak{X} - \mathfrak{U}$$

$$\mathfrak{n} \equiv \sqrt{2\mathfrak{X}}$$

$$\mathfrak{M} \equiv \frac{\varrho + \mathfrak{p}}{\sqrt{2\mathfrak{X}}}$$

- But is the constitutive equation (Fick's law of diffusion) satisfied for this fluid, just like Fourier's law of thermal conduction is satisfied for the nonminimally coupled equations of motion?

$$\mathfrak{Q}^\mu = -D(\mathfrak{h}^{\mu\nu} \nabla_\nu^{(g)} \mathfrak{M} + \mathfrak{M} a^\mu),$$

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- Turns out... it is! In this case \mathfrak{Q}^μ vanishes identically (which is fine: it indicates the current is purely convective, not dissipative)
- As a result, we can again *read off* the (invariant, as promised!) chemical potential (which coincides with the particle number density)

$$\mathfrak{M} = n \equiv \sqrt{- \left[B(\phi) + \frac{3}{2} \frac{A'(\phi)^2}{A(\phi)} \right] g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi)}$$

<u>fluid quantity</u>	<u>frame-invariant quantity</u>
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energy density ρ	$\mathfrak{X} + \mathfrak{U}$
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pressure p	$\mathfrak{X} - \mathfrak{U}$
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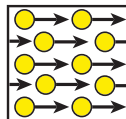
number density n	$\sqrt{2\mathfrak{X}}$
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chemical potential μ	$\sqrt{2\mathfrak{X}}$
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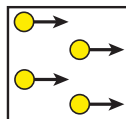
temperature T	0
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entropy density s	indeterminate
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- We now see that particle density/chemical potential controls whether a theory is "close to" or "far away from" GR
- Depending on the details of the theory, the theory will either "dilute" (towards GR) or "condense" (away from GR)



very different
from GR



closer to GR



GR

- The EoM (which is the frame-invariant generalized Klein–Gordon equation $\square^{(g)}\Phi + \mathfrak{L}_{,\Phi} = 0$) is simply a consequence of the invariant continuity equation (t now is the invariant proper time)

$$\frac{d\rho}{dt} + (\rho + p)\Theta = 0$$

- The field equation can therefore be written in a very condensed way

$$\frac{d^2\Phi}{dt^2} + \Theta \frac{d\Phi}{dt} + \mathfrak{L}_{,\Phi} = 0$$

- Differentiating, we finally find the "diffusive attractor" equation

$$\frac{d\mathfrak{M}}{dt} = -\mathfrak{M}\Theta + \square^{(g)}\Phi,$$

- For a free theory, a monotonically expanding or contracting universe implies that GR is either asymptotically recovered or that the theory diverges from GR, respectively

- State-of-the-art thermodynamical description of scalar-tensor theories features an **ambiguous** definition of temperature
- Frame-invariant approach **resolves** this ambiguity, and offers an alternative diffusive interpretation valid for **all** theories
- Questions:
 - What insights can the invariant diffusive approach offer when dealing with application of scalar-tensor theories (e.g. inflation, black holes, etc.)?
 - Is there a mapping between the thermodynamical and diffusive pictures? And how sensitive is it to the presence of matter?
 - Theories beyond scalar-tensor?

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- Stay tuned...